18.03 Class 22, March 31

Laplace Transform II: inverse transform, t-derivative rule,
use in solving ODEs; partial fractions: cover-up method; s-derivative rule.

Rules:

L is linear: L(af+bg) + aF + bG

F(s) essentially determines f(t)

s-shift: e^{at}f(t) ----> F(s-a)

Computations:

```
1              ----> 1/s

e^{as}         ----> 1/(s-a)

cos(omega t) ----> s/(s^2+omega^2)

sin(omega t) ----> omega/(s^2+omega^2)
```

In summary the use of Laplace transform in solving ODEs goes like this:

```
                    L
IVP for  x(t)   --------->  Alg equation for  X(s)
                                     |
                                     | solve
                                     |
                   L^{-1}            V
x(t) = ...      <---------      X(s) = ...
```

For this to work we have to recover information about f(t) from F(s).
There isn't a formula for L^{-1}; what one does is look for parts of F(s) in our table of computations. It's an art, like integration.

We can't expect to recover f(t) exactly, if f(t) isn't required to be continuous, since F(s) is defined by an integral, which is left unchanged if we alter any individual value of f(t) . What we have is:

Theorem: If f(t) and g(t) are generalized functions with the same Laplace transform, then f(a+) = g(a+), f(a-) = g(a-) , and any occurances of delta functions are the same in f(t) as in g(t).

So if f(t) and g(t) are continuous at t = a, then f(a) = g(a).

We'll also need:

The t-derivative rule: Using the integral definition, for Re(s) >> 0,

f'(t) ----> integral_0^infinite e^{-st} f'(t) dt

This demands integration by parts: integral u dv = u v - integral v du

u = e^{-st}, dv = f'(t) dt
du = -s e^{-st} dt, v = f(t)

so ... = [e^{-st} f(t)]_0^infty - integral_0^infinity f(t) (-s e^{-st} dt)

We have assumed that s has large real part, so as t gets large e^{-st} gets very small; small enough to cancel out anything f(t) might be doing (for f(t) of "exponential type," as the book says): so the upper evaluation of the first term is zero:

... = - f(0) + s integral_0^infinity f(t) e^{-st} dt

= - f(0) + s F(s) .

[To be more precise, we want to use f(0+): we always assume f(t) = 0 for t < 0 in this game, so f(0-) = 0 no matter what f(t) starts out doing for t > 0. Also, for this to be right one should use the "generalized derivative"; see the Supplementary Notes.]

t-derivative rule: f'(t) ----> s F(s) - f(0)

Example: Solve x' + 3x = e^{-t}, x(0) = 5.

Step 1: Apply L : (sX - 5) + 3X = 1/(s+1) , using linearity, the s-shift rule, and the t-derivative rule.

Step 2: Solve for X: (s+3)X = 5 + 1/(s+1)

so X = 5/(s+3) + 1/((s+1)(s+3))

Step 3: Massage the result into a linear combination of recognized forms.

Here the method is:

Partial Fractions: 1/((s+1)(s+3)) = a/(s+1) + b/(s+3) .

Old method: cross multiply and identify coefficients.

This works fine, but for excitement let me offer:

The Cover-up Method: Step 1: multiply through by (s+1) :

1/(s+3) = a + (s+1)(a/(s+3))

Step 2: Set s+1 = 0, or s = -1:

$1/(3-1) = a + 0$: $a = 1/2$.

This process "covers up" occurances of the factor $(s+1)$, and also all unwanted unknown coefficients. It gives b too:

$1/(-3+1) = 0 + b$: $b = -1/2$.

So $X = (1/2)/(s+1) + (9/2)/(s+3)$

Step 4: Apply L^{-1}: we can now recognize both terms:

$x = (1/2) e^{-t} + (9/2) e^{-3t}$.

You have to be somewhat crazy to like this method. This problem is completely straightforward using our old methods: the Key Formula gives the particular solution $x_p = (1/2) e^{-t}$; the basic homogeneous solution is e^{-3t}, and the transient needed to produce the initial condition $x(0) = 5$ is $(9/2) e^{-3t}$. I don't show you this to advertise it as a good way to solve this sort of problem, but rather to illustrate by a simple example how the method works.

Clearly, the more signals the better. We'll get more computations using the:

s-derivative rule:

$$F'(s) = (d/ds) \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty (-t e^{-st}) f(t)\, dt$$

which is the Laplace transform of $- t f(t)$. Thus:

$t f(t) \longrightarrow - F'(s)$

Sample use: start with $f(t) = 1 \longrightarrow 1/s = s^{-1}$

$t f(t) = t \longrightarrow - (d/ds) s^{-1} = s^{-2}$

Now take $f(t) = t$, so $t f(t) = t^2 \longrightarrow - (d/ds) s^{-2} = 2 s^{-3}$

and then $f(t) = t^2$ so $t f(t) = t^3 \longrightarrow - (d/ds) s^{-3} = (2 \times 3) s^{-4}$

The general picture is $t^n \longrightarrow n! / s^{(n+1)}$