18.03 Class 3, Feb 9, 2004

First order linear equations

Coupling constant, system, signal, system response, standard form, integrating factor

I began with two examples of linear equations:

(1) the bank account model $x' = Ix + q$ provided $I = I(t)$ not $I(t,x)$

(2) temperature, or diffusion: $T(t)$ = temperature inside my cooler; $Te(t)$ outside. If $Te > T$ you expect $T' > 0$; if $Te < T$ you expect $T' < 0$. The easiest way to make this happen is

$$T' = Te - T$$

but that can't be quite right because the units don't match. Better:

$$T' = k(Te - T)$$

where the "coupling constant" k has units 1/hrs. $k = 0$ means perfect insulation; k large means poor insulation.

A "linear ODE" is one that can be put in the form

$$n(t)x' + p(t)x = q(t)$$

(1) $x' - I x = q(t)$ so $p = -I$

(2) $T' + k T = k Te(t)$ so $p = k$

This way of writing the ODE puts the "system" on the left and the "signal" on the right. The solution is the "system response."

[By the way, some would say that Te rather than the product $k Te$ is the input signal. In this course we will just use $k Te$.]

Question 1. Which of the following are linear ODE's?
(a) $\dot x+x^2=t$
(b) $\dot x=(t^2+1)(x-1)$
(c) $\dot x+x=t^2$

1. None
2. (a) only
3. (b) only
4. (c) only}
5. All
6. All but (a)
7. All but (b)
8. All but (c)}

In both classes, 10% answered 4, 10% 5, 80% 6.

Algorithm for solving a first order linear ODE:--

I'll teach this to you by means of a sequence of examples:

(a) x' + 2x = 1
(b) tx' + x = 2t
(c) tx' - x = t
(d) x' + 2x = t

Step 0: Is the ODE separable? if so, solve it this way.

(a) is separable, so we omit it. The others are not.

(b) has an interesting LHS: tx' + x = (tx)'. So the equation is

(tx)' = 2t which we can integrate to get tx = t^2 + c and hence

x = t + c/t. This is excellent; we solves a non-separable equation!

Aside on "solutions": With c = 0 this is the straight line x = t. When c = 1 we add the hyperbola 1/t to this. The part with t > 0 is one solution; the part with t < 0 is another solution. We must agree to this if we want the uniqueness theorem to hold, since

x = t + 1/t for t > 0
  = t + 2/t for t < 0

is another solution to the same ODE, with the same initial condition (1,2) as
the solution

x = t + 1/t for all t not 0.

The single formula x = t + 1/t actually describes two solutions. End aside.

(c) is different; the left hand side is NOT (vx)' for any v. Nevertheless,
maybe we can find u such that after we multiply the equation through by u
we do see a (ux)' on the left hand side.

To make things simpler, the first thing to do now is to put the equation in
"standard form,"

x' + p(t) x = q(t)

by dividing through by n(t) if necessary. Then multiply through by u:

ux' + pux = uq   (*)

If the left hand side is to be (ux)' = ux' + u'x , we must have u' = pu .

This is separable, and

u = exp(int p(t) dt)

With this function for u, (*) is (ux)' = uq and we can integrate this to get

x = u^{-1} int uq dt

Example (c): put this in standard form: x' - t^{-1}x = 1 : p(t) = - t^{-1}

so int p(t) dt = - ln t and u = exp int p(t) dt = t^{-1} .

Multiply through by u = t^{-t}: t^{-1} x' - t^{-2} x = t^{-1}

Check: the LHS really is (ux)' : so we have (t^{-1} x)' = t^{-1}

Integrate to get t^{-1} x = ln|t| + c and so

x = t ln|t| + ct , the general solution

(but really you have to restrict t to be either positive or negative, as noted in the aside above).

Here's the algorithm:

(1) Put the equation in standard form x' + p(t) x = q(t)

(2) Compute the integrating factor u = exp(int p(t) dt)

(3) Solve (ux)' = uq ; that is, x = u^{-1} int uq dt

Example (d): this has constant coefficient. In general if p(t) is constant,

u = exp(int p dt) = e^{pt}

This is useful to remember.

(d) is not separable and is already in standard form. With u = e^{2t},

(e^{2t} x)' = e^{2t}t

Integrate e^{2t} t dt by parts:

int e^{2t} t dt = (1/2) e^{2t} t - int (1/2) e^{2t} dt

            = ((t/2) - (1/4)) e^{2t} + c

General solution: x = ((t/2 - (1/4)) + c e^{-2t}

Lessons about solutions to x' + p(t) x = q(t) (**)

(1) If $p(t)$ and $q(t)$ are defined everywhere between a and b, then
any solution to (**) is too.

- for the integral exists in this way.

(2) Any two solutions to (**) differ by a multiple of u^{-1}

$x = x_p + c u^{-1}$: the constant of integration appears very simply.