Solutions to Recitation 15 Problems

1. We have $\mathscr{L}\{t\} = \frac{1}{s^2}$, so by the exponential shift rule, we have
$$\mathscr{L}\{3te^{-2t}\} = \frac{3}{(s+2)^2}.$$
For the second term, again using the exponential shift, we have
$$\mathscr{L}\{t^2\sin(2t)\} = \mathscr{L}\{\frac{t^2e^{2it} - t^2e^{-2it}}{2i}\} = \frac{1}{2i}\cdot\left(\frac{2}{(s-2i)^3} - \frac{2}{(s+2i)^3}\right)$$
$$= \frac{1}{2i}\cdot\frac{2(s+2i)^3 - 2(s-2i)^3}{(s^2+4)^3} = \frac{12s^2-16}{(s^2+4)^3}.$$
So in conclusion, we have
$$\mathscr{L}\{f(t)\} = \frac{3}{(s+2)^2} + \frac{12s^2-16}{(s^2+4)^3}.$$
For the second term, we could have used the s-derivative rule, but the computations are a bit messier.

2. From the formula sheet, we have $\mathscr{L}\{\cos(2t)\} = \frac{s}{s^2+4}$ and $\mathscr{L}\{\sin(2t)\} = \frac{2}{s^2+4}$, so
$$\mathscr{L}^{-1}\{\frac{s+1}{s^2+4}\} = \cos(2t) + \frac{1}{2}\sin(2t).$$
For the second one,
$$\frac{s}{s^2+3s+2} = \frac{s}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1},$$
then we have $s = A(s+1) + B(s+2)$. Plugging in $s=-2$, we see $A=2$; plugging in $s=-1$, we see $B=-1$. Therefore,
$$\mathscr{L}^{-1}\{\frac{s}{s^2+3s+2}\} = 2e^{-2t} - e^{-t}.$$
Because the Laplace transform of $f(t)$ is computed as an integral, having a different value at say finitely many points will not produce anything different. So for example, in the first case, a function which is 100 million at $t = 1, 4, 17, \pi^{40}$ and which is $\cos(2t) + \frac{1}{2}\sin(2t)$ at all other t will have the same Laplace transform.

3. Let $\mathscr{L}\{x(t)\} = X(s)$. Then Laplace transforming the entire equation, we obtain
$$sX(s) - x(0) + 2X(s) = \frac{1}{s+1},$$
so we have
$$X(s) = \left(\frac{1}{s+1} + 1\right) \div (s+2) = \frac{1}{s+1}.$$
Therefore, taking Laplace inverse, $x(t) = e^{-t}$. Note that this is exactly the particular solution x_p obtained by the exponential response formula (with $p(r) = r+2$), and the transient part is zero.

4. Let $f(t) = \cos(\omega t)$ and $F(s) = \mathscr{L}\{\cos(\omega t)\}$. Then
$$\mathscr{L}\{-\omega\sin(\omega t)\} = \mathscr{L}\{f'(t)\} = sF(s) - \cos(\omega\cdot 0) = \frac{s^2}{s^2+\omega^2} - 1.$$
Therefore,
$$\mathscr{L}\{\sin(\omega t)\} = \frac{-\omega^2}{s^2+\omega^2} \div (-\omega) = \frac{\omega}{s^2+\omega^2}.$$