

18.03 Class 32, April 26, 2004

First order systems: Introduction

Vocabulary: System of equations. Solution, trajectory, phase plane.

Linear, constant coefficient, homogeneous.

Matrix, matrix product, column vector.

1. Example: My house has two rooms, connected by a door. The coefficient of conductivity of the outside walls is  $k_1$  and of the wall between the rooms is  $k$ . The temperature of the first room is  $x(t)$  and of the second room is  $y(t)$ . If the ambient temperature outside is  $T(t)$ , we can model the two temperatures by the "system"

$$x' = k(y-x) + k_1(T-x)$$

$$y' = k(x-y) + k_1(T-y)$$

This is a first order system of equations. It looks like it will be hopeless to solve it: to know about  $x'$  you need to know  $y$ ; to know about  $y'$  you need to know  $x$ .

This equation is linear, because of the simple way  $x$  and  $y$  enter.

If  $k$  and  $k_1$  are constant, it is a "constant coefficient equation.

If the outside wall is a perfect insulator,  $k_1 = 0$  and so the equations are

$$x' = k(y-x)$$

$$y' = k(x-y)$$

which is a homogeneous linear system.

[I did not do the following but should have:

Let's take  $k = 1$  and solve this system:

$$x' = y - x$$

$$y' = x - y$$

Add the two equations to see:  $x' + y' = 0$ : so  $x + y$  is constant; say  $a$ .

Subtract the second equation from the first:

$$(x-y)' = -2(x-y)$$

If we write  $z = x - y$ , this says  $z' = -2z$  which has general solution  $z = c e^{-2t}$ .

So  $x + y = a$   
 $x - y = c e^{-2t}$

Adding,  $x = (a/2) + (c/2)e^{-2t}$   
 $y = -(a/2) + (c/2)e^{-2t}$  ]

The general constant coefficient linear homogeneous equation in two variables:

$$\begin{aligned}x' &= ax + by & (*) \\y' &= cx + dy\end{aligned}$$

2. Analysis of linear equations by matrices.

We can represent linear equations using matrices. The matrix of coefficients of (\*) is the array of numbers (enclosed by brackets)

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

In these notes I will use Matlab notation and write this array as

$$[ a \ b ; c \ d ]$$

There is another matrix in sight, the "column vector"

$$\begin{vmatrix} x \\ y \end{vmatrix}$$

or  $[ x ; y ]$  with entries  $x$  and  $y$ .

Matrix multiplication is set up so that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} ax + by \\ cx + dy \end{vmatrix}$$

$$\text{or } [ a \ b ; c \ d ] [ x ; y ] = [ ax+by ; cx+dy ]$$

You run along a row of the first matrix and a column of the second, multiplying and adding up. Another way to look at matrix multiplication comes from

rewriting the right hand side as

$$\begin{aligned} | ax + by | &= x | a | + y | b | \\ | cx + dy | &= x | c | + y | d | \end{aligned}$$

$$\text{or } [ ax+by ; cx + dy ] = x [ a ; c ] + y [ b ; d ]$$

The ODE (\*) can thus be written as

$$[ x' ; y' ] = [ a \ b ; c \ d ][ x ; y ]$$

If we write  $u$  for the column vector  $[ x ; y ]$  then  $u' = [ x' ; y' ]$ , and

$$u' = Au$$

This compact expression is exactly equivalent to (\*).

### 3. Companion matrices.

Here's an important source of systems of equations.

Suppose we have a second order homogeneous linear equation, say

$$x'' - (1/2)x' + (17/16)x = 0$$

We can derive a first order linear system from this, by the trick of defining

$$y = x'$$

so then

$$y' = x'' = -(17/16)x + (1/2)x' = -(17/16)x + (1/2)y$$

Together we have

$$\begin{aligned} x' &= y \\ y' &= -(17/16)x + (1/2)y \end{aligned}$$

This is a first order constant coefficient homogeneous linear system whose matrix

$$A = [ 0 \ 1 ; -17/16 \ 1/2 ]$$

is the "companion matrix" of the original second order equation.

#### 4. Visualization of autonomous systems using vector fields.

The general system of equations (in two variables) looks like

$$\begin{aligned}x' &= f(x,y,t) \\ y' &= g(x,y,t)\end{aligned}$$

If  $t$  doesn't occur explicitly, so that

$$\begin{aligned}x' &= f(x,y) \\ y' &= g(x,y)\end{aligned}$$

the equation is called "autonomous." For example, constant coefficient homogeneous linear equations are autonomous.

Think of  $x, y$  as the coordinates of a point in the plane. In fact it is good practice to denote a point in the plane by a column vector  $\begin{bmatrix} x \\ y \end{bmatrix}$ .

The right hand side of an autonomous equation is a "vector field": To each point  $(x,y)$  it associates a vector

$$f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$$

A solution to the equation is a parametrized curve  $\mathbf{u}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$  with the property that when it passes through a point  $\begin{bmatrix} x \\ y \end{bmatrix}$ , its velocity vector  $\begin{bmatrix} x' \\ y' \end{bmatrix}$  is given by the vector  $f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$ .

For example, we have the system above:  $x' = y-x$ ,  $y' = x-y$ . I showed a picture of the resulting vector field. You can see that the solutions are restricted to lines where  $x + y$  is constant, and they converge to the diagonal, where the solutions are constant.

I also showed a slide of the direction field of the second example,  $A = \begin{bmatrix} 0 & 1 \\ -17/16 & 1/2 \end{bmatrix}$

In it, the solutions travel along spirals.

The paths of the solutions are called "trajectories." The diagram consisting of the plane with some trajectories drawn on it is the "phase plane."

We can see more precisely what the trajectories are in this case, by solving the original equation

$$x'' - (1/2)x' + (17/16)x = 0$$

Its characteristic polynomial is  $p(s) = s^2 - (1/2)s + (17/16)$

You can find its roots using the quadratic formula, or by completing the square:

$$p(s) = (s - (1/4))^2 + 1$$

so a root  $r$  satisfies  $(r - (1/4))^2 = -1$  or  $r = (1/4) \pm i$

The general solution is thus

$$x = A e^{t/4} \cos(t - \phi)$$

These oscillate under an exponentially growing envelope. The derivative does the same, but is off phase. The result is that the trajectory traced out by  $(x,y)$  is an expanding spiral.

5. It turns out that the same system models the relationship between Romeo and Juliet. The MIT Humanities Department has analyzed the plot of Shakespeare's play and found the following. If  $R$  denotes Romeo's love for Juliet, and  $J$  denotes Juliet's love for Romeo, then

$$R' = J$$

$$J' = -(17/16)R + (1/2)J$$

Romeo is a puppy dog. He has little self-awareness; the change in his feeling towards Juliet has nothing to do with how he himself feels at the moment; it is completely dependent on how she feels about him. Juliet is more complex. She has a healthy self-awareness; if she loves him, that very fact causes her to love him more. On the other hand, if he seems to love her, she gets frightened and starts to love him less.

Let's start the action at  $(1,0)$ . So Romeo is fond of Juliet but she is neutral towards him. However, she does notice that he is fond of her, and this makes her somewhat hostile. As she becomes more distant, his affection wanes. Eventually he is neutral and she really doesn't like him. This continues; presently he stays away from her, and this very fact makes her more interested. She warms to him, he notices and his rate of increase of disinterest starts

to ameliorate. Eventually she is neutral and just as he bottoms out. He then starts to feel better towards her, but still stays away, and now both his attitude and hers cause her to feel progressively more well disposed towards him. This causes him to continue to warm to her.

Following this around, you wind up at  $J = 0$  again, but now  $R$  has increased. This is a cyclical relationship, but with each cycle the intensity increases. We all know the sad outcome.