18.03 Recitation Problems

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Laplace Transform and ODEs

Properties of the Laplace transform:

- Definition: $f(t) \rightsquigarrow F(s) = \int_0^\infty f(t)e^{-st}dt$ for $\text{Re}s >> 0$.
- Linearity: $af(t) + bg(t) \rightsquigarrow aF(s) + bG(s)$.
- $\mathcal{L}^{-1}$: $F(s)$ essentially determines $f(t)$.
- s-shift rule: $e^{at}f(t) \rightsquigarrow F(s-a)$.
- s-derivative rule: $tf(t) \rightsquigarrow -F'(s)$.
- t-derivative rule: $f'(t) \rightsquigarrow sF(s) - f(0+)$.

Formulas for the Laplace transform:

$$1 \rightsquigarrow 1/s \quad , \quad e^{at} \rightsquigarrow 1/(s-a)$$

$$\cos(\omega t) \rightsquigarrow s/(s^2+\omega^2) \quad , \quad \sin(\omega t) \rightsquigarrow \omega/(s^2+\omega^2)$$

$$t^n \rightsquigarrow n!/s^{n+1}$$

1. Compute the Laplace transform of $f(t) = 3te^{-2t} + t^2\sin(2t)$

2. Compute the inverse Laplace transform of $\frac{s+1}{s^2+4}$ and of $\frac{s}{s^2+3s+2}$.
Describe some other functions (not continuous) with the same Laplace transforms as these.

3. Solve $\dot{x} + 2x = e^{-t}$, $x(0) = 1$ using the Laplace transform. Then identify in your solution the transient and the particular solution x_p given by the Exponential Response Formula $x_p = e^{rt}/p(r)$ for a solution to $p(D)x = e^{rt}$.

4. Start with the formula for the Laplace transform of $\cos(\omega t)$ and verify the formula for the Laplace transform of $\sin(\omega t)$ using the t-derivative rule.