18.03 Class 23, Apr 2

Laplace Transform: Second order equations; completing the square;
t-shift; step and delta signals.

Rules:

L is linear: af(t) + bg(t) ----> aF(s) + bG(s)

F(s) essentially determines f(t)

s-shift: e^{at}f(t) ----> F(s-a)

t-shift: f_a(t) ----> e^{-as} F(s)

s-derivative: t f(t) ----> - F'(s)

t-derivative: f'(t) ----> s F(s) - f(0+)

f''(t) ----> s^2 F(s) - s f(0+) - s f'(0+)

Computations:

1   ----> 1/s

e^{as}   ----> 1/(s-a)

cos(omega t) ----> s/(s^2+omega^2)

sin(omega t) ----> omega/(s^2+omega^2)

delta(t)   ----> 1

To handle second degree equations we'll need to know the LT of f''(t).
We'll compute it by regarding f''(t)   as the derivative of f'(t).
We'll employ a technique here that will get repeated several more times
today: pick a new function symbol and use it to name some function that

arises in the middle of a calculation.

The application of this principle here is to write $f'(t) = g(t)$. Then

$g(t) \longrightarrow G(s) = s F(s) - f(0+)$

Write down the t-derivative rule for $g(t)$:

$g'(t) \longrightarrow s G(t) - g(0+)$ ie

$f''(t) \longrightarrow s (s F(s) - f(0+)) - f'(0+) = s^2 F(s) - s f(0+) - f'(0+)$

Example: $x'' + 2 x' + 5 x = 5$, $x(0+) = 2$, $x'(0+) = 3$.

Note that this is EASY to solve using our old linear methods:
by inspection (or undetermined coefficients, or the Key Formula)
$x_p = 1/5$ is a solution; the general solution is this plus a homogeneous solution, which you choose to satisfy the initial conditions.
Nevertheless we have some technique to show you in working it out using LT.

Step 1: Apply LT: $(s^2 X - 2s - 3) + 2 (s X - 2) + 5 X = 5/s$

Step 2: Solve for X: $(s^2 + 2 s + 5) X = (2s + 7) + 5/s$

$(s^2 + 2 s + 5)$ is the characteristic polynomial $p(s)$! - this will always be the case. $(2s + 7)$ is data from the initial conditions; if we had used rest initial conditions this would have been zero. $1/s$ is the LT of the signal. From a certain perspective, the initial conditions are part of the signal.

So $X(s) = ((\text{data from initial conditions}) + (\text{LT of signal})) / p(s)$

In particular, with rest initial conditions, we find that the Laplace transform of the solution of $p(D)x = f(t)$ is

$X(s) = F(s) / p(s) = W(s) F(s)$

using the notation $W(s)$ for the "weight function" $1/p(s)$.
The s-domain is the world in which the operator $p(D)$ has the effect of multiplying by the function $p(s)$, and solving the equation in

effect simply divides by p(s).

In our example, X = (2s+7)/(s^2+2s+5) + 5/(s(s^2+3s+5)) .

By linearity, we can look at the terms separately.
Look first at the first term. To handle the quadratic denominator we use

Method: Complete the square: p(s) = s^2 + 2s + 5 = (s+1)^2 + 4
and then write the whole expression using (s+1):

(2s+7)/(s^2+2s+5) = ((2(s+1) + 5)/((s+1)^2 + 4)

The s-shift rule will provide us with the (s+1)'s. To apply it without
losing your way, I recommend using a new function name: write

F(s) = (2s+5)/(s^2+4)

so that (2s+7)/(s^2+2s+5) = F(s+1) . From the tables, the inverse LT of
F(s) is

f(t) = 2 cos(2t) + (5/2) sin(2t) .

The s-shift rule (with a = -1) gives:

e^{-t}(2 cos(2t) + (5/2) sin(2t)).

Now look at second term. We'll use partial fractions for it, too:
there are constants a, b, c, such that

5/(s((s+1)^2+4) = a/s + (b(s+1)+c)/((s+1)^2+4)

Note that I've completed the square and written the numerator using (s+1),
in anticipation that I'll need things in that form when it comes time to
recognize things from our table of Laplace Transforms.

You can find a, b, c, by cross multiplying and equating coefficients.
Or you can use the coverup method. To find a multiply through by s
and then set s = 0:

5/(1+4) = a or a = 1.

To get b and c, use the "complex coverup": multiply through by

((s+1)^2+4) and then set s equal to a root of this quadratic.
Roots: (s+1)^2 = -4 so s+1 = +-2i or s = -1 +- 2i. We can pick either one, say s = -1 + 2i or s+1 = 2i. We get:

5/(-1+2i) = b(2i) + c.

Notice how useful it was to have s+1 at hand here. We can use this to solve for b and c, which are supposed to be real. Rationalizing the denominator,

(1-2i) = 5(1-2i)/(1+4) = 2bi + c so b = -1 and c = -1.

5/(s((s+1)^2+4) = 1/s - ((s+1)+1)/((s+1)^2+4)

The first term is LT of 1 , and using complex coverup and the tables again we see that this is the LT of

1 - e^{-t}(cos(2t) + (1/2) sin(2t))

Adding this to the

x = 1 + e^{-t}(cos(2t) + 2 sin(2t)) .

You have to be crazy to like this method of solving

x'' + 2 x' + 5 x = 5, x(0+) = 2, x'(0+) = 3.

After all, xp = 1 ; the roots of the characteristic polynomial are
-1 +- 2i (a fact that we used in the complex coverup), so the general homogeneous solution is

xh = e^{-t} (c1 cos(2t) + c2 sin(2t))

and x = 1 + xh for suitable choice of c1 and c2 , which can be found by substituting in the initial conditions. I showed you this to illustrate LT technique, not to advertise it as a good way to solve such ODEs.

One new rule and two new signals:

The t-shift rule: recall the notation

f_a(t) = u(t-a)f(t-a)

The graph of f_a(t) is the same as the graph of f(t) but shifted to the right by a units. For t < a , f_a(t) = 0 .

f_a(t) ----> integral_a^infinity f(t-a) e^{st} dt

The lower limit is a because for t < a ,   f_a(t) = 0 .
The method of wishful thinking suggests inventing a new letter for the quantity t - a :   u = t - a ; t = u + a ; du = dt ; so

f_a(t) ----> integral_0^infinity f(u) e^{s(t+a)} du

The lower limit here is u = 0 , because this is what u is when t = a.
By the exponential law e^{s(t+a)} = e^{st} e^{as} , and e^{as} is constant, so

f_a(t) ----> e^{as} integral_0^infinity f(u) e^{-su} du

This integral is precisely F(s) ; the choice of variable name inside the integral (u here instead of t) makes no difference to the value of the integral. Thus:

f_a(t) ----> e^{as} F(s)

For example, if we take f(t) to be the step function u(t) , we find

u(t-a) ----> e^{as}/s

We've found LT of a new signal, a discontinuous one, one whose definition comes in two parts (t > a, t < a) . The LT is perfectly fine, though, with just a single part to its definition.

Finally I want to find the LT of the delta signal.

Here is a property of the delta function. Remember that we can approximate delta(t-a) by a skyscraper function with value 1/Delta t between a and a + Delta t and value 0 elsewhere. If we multiply this by a continuous function f(t) you get something well approximated by f(a) times the same skyscraper function. Thus for any A < a < B,

integral_A^b delta(t-a) f(t) dt = f(a)

Integrating against delta(t-a) just picks out the value of f(t) at t=a.

In particular, for a > 0

delta(t-a) ----> integral_0^infinity delta(t-a) e^{-st} dt = e^{-as}.

This is a new and valuable calculation. Note that it bears out my claim, that the Laplace transform takes jumpy functions (like delta(t-a)) to nice smooth beauties like e^{-as}.

In particular we can take s = 0 : delta(t) ----> 1.

[Actually taking a = 0 requires some special care, for which see the homework and the Supplementary Notes.]

This is quite amazing: the delta function is highly singular, not even a proper function at all; it is entirely concentrated at t = 0 .
But its Laplace transform is totally smooth - the constant function 1 , evenly spread out over the whole of the complex plane.