21. The Laplace transform and generalized functions

21.1. What the Laplace transform doesn’t tell us. What do we mean, in Section 19.3, when we say that \( F(s) \) “essentially determines” \( f(t) \)?

The Laplace transform is defined by means of an integral. We don’t need complete information about a function to determine its integral, so knowing its integral or integrals of products of it with exponentials won’t be enough to completely determine it.

For example, if we can integrate a function \( g(t) \) then we can also integrate any function which agrees with \( g(t) \) except at one value of \( t \), or even except at a finite number of values, and the integral of the new function is the same as the integral of \( g \). Changing a few values doesn’t change the “area under the graph.”

Thus if \( f(t) \sim F(s) \), and \( g(t) \) coincides with \( f(t) \) except at a few values of \( t \), then also \( g(t) \sim F(s) \). We can’t hope to recover every value of \( f(t) \) from \( F(s) \) unless we put some side conditions on \( f(t) \), such as requiring that it should be continuous.

Therefore, in working with functions via Laplace transform, when we talk about a function \( f(t) \) it is often not meaningful to speak of the value of \( f \) at any specific point \( t = a \). It does make sense to talk about \( f(a−) \) and \( f(a+) \), however. Recall that these are defined as

\[
  f(a−) = \lim_{t \to a^-} f(t), \quad f(a+) = \lim_{t \to a^+} f(t).
\]

This means that \( f(a−) \) is the limiting value of \( f(t) \) as \( t \) increases towards \( a \) from below, and \( f(a+) \) is the limiting value of \( f(t) \) as \( t \) decreases towards \( a \) from above. In both cases, the limit polls infinitely many values of \( f \) near \( a \), and isn’t changed by altering any finite number of them or by altering \( f(a) \) itself; in fact \( f \) does not even need to be defined at \( a \) for us to speak of \( f(a±) \). The best policy is to speak of \( f(a) \) only in case both \( f(a−) \) and \( f(a+) \) are defined and are equal to each other. In this case we can define \( f(a) \) to be this common value, and then \( f(t) \) is continuous at \( t = a \).

The uniqueness theorem for the inverse Laplace transform asserts that if \( f \) and \( g \) have the same Laplace transform, then \( f(a−) = g(a−) \) and \( f(a+) = g(a+) \) for all \( a \). If \( f(t) \) and \( g(t) \) are both continuous at \( a \), so that \( f(a−) = f(a+) = f(a) \) and \( g(a−) = g(a+) = g(a) \), then it follows that \( f(a) = g(a) \).
Part of the strength of the theory of the Laplace transform is its ability to deal smoothly with things like the delta function. In fact, we can form the Laplace transform of a generalized function as described in Section 16, assuming that it is of exponential type. The Laplace transform $F(s)$ determines the singular part of $f(t)$: if $F(s) = G(s)$ then $f_s(t) = g_s(t)$.

21.2. **Worrying about $t = 0$.** When we consider the Laplace transform of $f(t)$ in this course, we always make the assumption that $f(t) = 0$ for $t < 0$. Thus

$$f(0-) = 0.$$  

What happens at $t = 0$ needs special attention, and the definition of the Laplace transform offered in Edwards and Penney (and most other ODE textbooks) is not consistent with the properties they assert.

They define

$$F(s) = \int_{0}^{\infty} e^{-st} f(t) \, dt.$$ 

Suppose we let $f(t) = \delta(t)$, so that $F(s)$ should be the constant function with value 1. The integrand is $\delta(t)$ again, since $e^{-st}|_{t=0} = 1$. The indefinite integral of $\delta(t)$ is the step function $u(t)$, which does not have a well-defined value at $t = 0$. Thus the value they assign to the definite integral with lower limit 0 is ambiguous. We want the answer to be 1. This indicates that we should really define the Laplace transform of $f(t)$ as the integral

$$F(s) = \int_{0-}^{\infty} e^{-st} f(t) \, dt.$$ 

The integral is defined by taking the limit as the lower limit increases to zero from below. It coincides with the integral with lower limit $-\infty$ since $f(t) = 0$ for $t < 0$:

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t) \, dt.$$ 

With this definition, $\delta(t) \sim 1$, as desired.

Let’s now check some basic properties of the Laplace transform, using this definition.

21.3. **The $t$-derivative rule.** Integration by parts gives us

$$\int_{0-}^{\infty} e^{-st} f'(t) \, dt = f(0-) - (-s) \int_{0-}^{\infty} e^{-st} f(t) \, dt.$$
Since \( f(0-) = 0 \), we find that if \( f(t) \sim F(s) \) then
\[
(2) \quad f'(t) \sim sF(s).
\]

Here I am using the generalized derivative described in Section 16. Including the delta terms in the derivative is important even if \( f(t) \) is continuous for \( t > 0 \) because the jump from \( f(0-) = 0 \) to \( f(0+) \) contributes \( f(0+)\delta(t) \) to \( f'(t) \). Let’s assume that \( f(t) \) is continuous for \( t > 0 \). Then \( f'(t) = f(0+)\delta(t) + f'_r(t) \), where \( f'_r(t) \) is the regular part of the generalized derivative, that is, the ordinary derivative of the function \( f(t) \). Substituting this into (2) and using \( \delta(t) \sim 1 \) and linearity gives us the familiar formula
\[
(3) \quad f'_r(t) \sim sF(s) - f(0+).
\]

Formula (16) on p. 281 of Edwards and Penney is an awkward formulation of (2).

21.4. The initial singularity formula. If \( f(t) \) is a generalized function with singular part at zero given by \( b\delta(t) \), then
\[
\lim_{s \to \infty \cdot 1} F(s) = b.
\]

The notation means that we look at values of \( F(s) \) for large real values of \( s \).

To see this, break \( f(t) \) into regular and singular parts. We have a standing assumption that the regular part \( f_r(t) \) is of exponential order, and for it the result is given in Edwards and Penney, formula (25) on p. 271.

Each delta function \( \delta(t - a) \) in \( f(t) \) contributes a term \( e^{-as} \) to \( F(s) \), and as long as \( a > 0 \), these all decay to zero as \( s \to \infty \cdot 1 \) as well. Only \( a = 0 \) is left, and we know that \( b\delta(t) \sim b \). This finishes the proof.

When \( b = 0 \)—that is, when \( f(t) \) is nonsingular at \( t = 0 \)—the result is that
\[
\lim_{s \to \infty \cdot 1} F(s) = 0.
\]

21.5. The initial value formula. If \( f(t) \) is a piecewise differentiable generalized function, then
\[
\lim_{s \to \infty \cdot 1} sF(s) = f(0+).
\]

To see this, let \( f'(t) \) again denote the generalized derivative. The jump from \( f(0-) = 0 \) to \( f(0+) \) contributes the term \( f(0+)\delta(t) \) to \( f'(t) \).
The initial value formula results directly from the initial singularity formula applied to \( f'(t) \).

For example, if \( f(t) = \cos t \) then \( F(s) = \frac{s}{s^2 + 1} \), and

\[
\lim_{s \to \infty} \frac{s^2}{s^2 + 1} = 1
\]

which also happens to be \( \cos(0) \). With \( f(t) = \sin t \), on the other hand, \( F(s) = \frac{1}{s^2 + 1} \), and \( \lim_{s \to \infty} \frac{s}{s^2 + 1} = 0 \) in agreement with \( \sin 0 = 0 \).

21.6. Final value formula. Suppose that \( f(t) \) has only finitely many points of discontinuity and that its singular part has only finitely many delta functions in it. If all poles of \( F(s) \) are to the left of the imaginary axis, then

\[
\lim_{t \to \infty} f(t) = 0.
\]

If there is a simple pole at \( s = 0 \) and all other poles are to the left of the imaginary axis, then

\[
\lim_{t \to \infty} f(t) = \text{res}_{s=0} F(s).
\]

In other situations, \( \lim_{t \to \infty} f(t) \) doesn’t exist; either \( f(t) \) oscillates without decaying, or \( |f(t)| \) grows without bound.

We will not attempt to justify this in detail, but notice that \( c/s \) has a pole at \( s = 0 \) with residue \( c \), and its inverse Laplace transform is the constant function \( c \), which certainly has “final value” \( c \).

The final value formula implies a description of the behavior of \( f(t) \) as \( t \to \infty \) for a much broader class of functions \( f(t) \). Suppose that the rightmost pole of \( F(s) \) has real part strictly less than \( a \). By the \( s \)-shift rule, the Laplace transform of \( e^{-at} f(t) \) is \( F(s + a) \). The pole diagram of \( F(s + a) \) is the same as the pole diagram of \( F(s) \) but shifted to the left by \( a \) units in the complex plane. Thus all its poles are to the left of the imaginary axis, and so \( \lim_{t \to \infty} e^{-at} f(t) = 0 \). This says that \( f(t) \) grows more slowly than \( e^{at} \), or (if \( a < 0 \)) decays faster than \( e^{-at} \).

Similarly, if \( F(s) \) has one pole whose real part is larger than those of all its other poles, and that pole is real—say it’s \( a \)—then

\[
\lim_{t \to \infty} e^{at} f(t) = \text{res}_{s=a} F(s).
\]

If the pole at \( s = a \) is simple, so the residue is finite, then this says that \( f(t) \) grows or decays like a constant multiple of the exponential function \( e^{-at} \).
If on the other hand there is a rightmost pole which is not real (so if \( f(t) \) is to be real there is at least a complex conjugate pair of rightmost poles) then the function oscillates. If there is one conjugate pair of rightmost poles, \( a \pm i\omega \), then \( f(t) \) oscillates with an overall approximate circular frequency of \( \omega \), while growing or decaying approximately as fast as \( e^{-at} \). This can be seen in the examples

\[
e^{at} \cos(\omega t) \sim \frac{s - a}{(s - a)^2 + \omega^2}, \quad e^{at} \sin(\omega t) \sim \frac{\omega}{(s - a)^2 + \omega^2}.
\]

In these cases the poles of \( F(s) \) are at \( a \pm i\omega \).

This is an important principle: the general behavior of \( f(t) \) as \( t \to \infty \) is controlled by the pole diagram of its Laplace transform \( F(s) \); in fact, its main features are controlled by the rightmost poles in the pole diagram.

21.7. Laplace transform of the unit impulse response. Suppose we have an LTI system represented by the operator \( L = p(D) \). We have seen (and for this see also Edwards and Penney, pp. 320–321) that the system response (from rest initial conditions) to a general signal \( f(t) \) can be expressed in terms of the unit impulse response (or “weight function”) \( w(t) \), as the convolution:

\[
x(t) = f(t) \ast w(t).
\]

The weight function thus determines the system parameters, and it is reasonable to ask how the characteristic polynomial \( p(s) \) can be recovered from the weight function. To find out, apply the Laplace transform to the equation \( p(D)w(t) = \delta(t) \). The fact that \( w(t) \) is assumed to satisfy rest initial conditions implies that the side terms in the \( t \)-derivative formula vanish. We find that if \( w(t) \sim W(s) \) then

\[
p(D)w(t) \sim p(s)W(s).
\]

On the other hand

\[
p(D)w(t) = \delta(t) \sim 1,
\]

so

\[
W(s) = \frac{1}{p(s)}:
\]

The Laplace transform of the unit impulse response is the reciprocal of the characteristic polynomial.

The Laplace transform \( W(s) \) of the weight function \( w(t) \) is called the “transfer function.”
We may then ask what the system response is to an exponential signal. Much of our work has dealt with that case, and revolved around the fact that a solution to \( p(D)x = e^{st} \) (for \( s \) a (possibly complex) constant) is given by \( x_p(t) = e^{st}/p(s) \). (This assumes \( p(s) \neq 0 \). If \( p(s) = 0 \) then the system is in resonance with the signal, and no constant multiple of \( e^{st} \) is a solution.) We can express this in terms of the transfer function:

\[ x_p(t) = W(s)e^{st}. \]

That is:

The transfer function describes what (possibly complex) multiple of \( e^{st} \) occurs as a system response to the signal \( e^{st} \).

We are then very interested in the magnitude of \( W(s) \)—the gain—and the argument of \( W(s) \)—the phase shift. Moreover, the poles of \( W(s) \) record the resonant modes of the system.

21.8. Initial conditions. Let’s return to the first order bank account model, \( \dot{x} + px = q(t) \). When we come to specify initial conditions, at \( t = 0 \), the following two procedures are clearly equivalent: (1) Fix \( x(0+) = x_0 \) and proceed; and (2) Fix \( x(0-) = 0 \)—rest initial conditions—but at \( t = 0 \) deposit a lump sum of \( x_0 \) dollars. The second can be modeled by altering the signal, replacing the rate of deposit \( q(t) \) by \( q(t) + x_0 \delta(t) \). Thus if you are willing to accept a generalized function for a signal, you can get always away with rest initial conditions.

Let’s see how this works out in terms of the Laplace transform. In the first scenario, the \( t \)-derivative theorem gives us

\[ (sX - x_0) + qX = Q. \]

In the second scenario, the fact that \( \delta(t) \sim 1 \) gives us

\[ sX + qX = Q + x_0, \]

an equivalent expression.

This example points out how one can absorb certain non-rest initial conditions into the signal. The general picture is that the top derivative you specify as part of the initial data can be imposed using a delta function.

The mechanical model of the second degree case helps us understand this. We have an equation

\[ m\ddot{x} + b\dot{x} + kx = q(t). \]
Suppose the initial position is $x(0) = 0$. Again, we have two alternatives: (1) Fix the initial velocity at $\dot{x}(0+) = v_0$ and proceed; or (2) Fix $\dot{x}(0-) = 0$—so, together with $x(0-) = 0$ we have rest initial conditions—and then at $t = 0$ give the system an impulse, by adding to the signal the term $mv_0\delta(t)$. The factor of $m$ is necessary, because we want to use this impulse to jack the velocity up to the value $v_0$, and the force required to do this will depend upon $m$.

**Exercise 21.8.1.** As above, check that this equivalence is consistent with the Laplace transform.

For a higher order example, consider the LTI operator $L = (D^3 + 2D^2 - 2I)$ with transfer function $W(s) = 1/(s^3 + s^2 - 2)$. In the section on the Laplace Transform in the complex plane we computed the corresponding weight function: $w(t) = e^t/5 + (e^{-t}/5)(-\cos t - 2\sin t)$ (for $t > 0$). This is a solution to the ODE $Lx = \delta(t)$ with rest initial conditions. This is equivalent (for $t > 0$) to the IVP $Lx = 0$ with initial conditions $x(0) = \dot{x}(0) = 0$ and $\ddot{x}(0) = 1$, and indeed $w(t)$ satisfies all this.

In order to capture initial conditions of lower derivatives using impulses and such in the signal, one must consider not just the delta function but also its derivatives. The desire to do this is a good motivation for extending the notion of generalized functions, but not something we will pursue in this course.