18.03 Class 21, Mar 29

Laplace Transform: basic properties; functions of a complex variable; poles diagrams; s-shift law.

The Laplace transform connects two worlds:

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   ---------------------------------------------------------------------
| The "t-world"                                                        |
|                                                                      |
| t  is real and positive                                              |
|                                                                      |
| functions  f(t)  are signals and solutions, perhaps nasty            |
| (with discontinuities) or even generalized (with delta functions)|
| (possibly complex valued)                                            |
|                                                                      |
| and ODEs relating them                                               |
|                                                                      |
| convolution                                                          |
|                                                                      |
   ---------------------------------------------------------------------
                          |      ^
                          | L    | L^{-1}
                          V      |
   ---------------------------------------------------------------------
| The "s-world"                                                        |
|                                                                      |
| s  is complex                                                        |
|                                                                      |
| beautiful functions  F(s) ,  often rational = poly/poly              |
|                                                                      |
| and algebraic equations relating them                                |
|                                                                      |
| ordinary multiplication of functions                                 |
|                                                                      |
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The use in ODEs will be to apply L to an ODE, solve the resulting very simple algebraic equation in the s world, and then return to reality using the "inverse Laplace transform" L^{-1}.

The definition can be motivated but it is more efficient to simply give it and come to the motivation later. Here it is.

$$F(s) = L(f(t);s) = \int_0^\infty e^{-st} f(t)\, dt \qquad (\text{for } s \text{ large}).$$

This is like a hologram, in that each value $F(s)$ contains information about ALL values of $f(t)$.

Example:

$$L(1;s) = \int_0^\infty e^{-st}$$

$$= \lim_{T \to \infty} e^{-st}/(-s) |^T_0$$

$$= (-1/s) (\lim_{T \to \infty} e^{-st} - 1).$$

To compute this limit, write $s = a + bi$ so

$$e^{-sT} = e^{-aT} (\cos(-bT) + i \sin(-bT))$$

The second factor lies on the unit circle, so $|e^{-sT}| = e^{-aT}$.
This goes to infinity with T if $a < 0$ and to zero if $a > 0$.
When $a = 0$, this term is 1, but the other factor oscillates and the limit doesn't exist then either. Thus:

$$L(1;s) = 1/s \quad \text{for} \quad Re(s) > 0$$

and the improper integral fails to converge for $Re(s)$ not positive.

This is typical behavior: the integral converges to the right of some vertical line in the complex plane C, and diverges to the left.
So in the definition I really meant:

"for $Re(s)$ large."

The expression obtained by means of the integration makes sense everywhere in C except for a few points - like $s = 0$ here, and this is how we define the Laplace transform for values of s with small real part.

Let's understand the complex function $1/s$. You can't graph it per se, because the input requires 2 dimensions and so does the output. So let's graph its absolute value |1/s| = 1/|s| instead. So lay the complex plane down horizontally. The graph is a surface lying over it. In this case it's a surface of revolution, with one branch of a hyperbola as contour. It sweeps up to infinity over the origin. It looks like a circus tent, suspended by a pole over the origin. The complex number 0 itself is known as a "pole" of the function 1/s.

This computation can be exploited using general properties of the Laplace Transform. We'll develop quite a few of these rules, and in fact normally you will not be using the integral definition to compute Laplace transforms.

Rule 1 (Linearity): af(t) + bg(t) ----> aF(s) + bG(s).

This is clear, and has the usual benefits.

Rule 2 (s-shift): e^{at}f(t) ----> F(s-a).

Here's the calculation:

e^{at}f(t) ----> integral_0^infinity e^{at} f(t) e^{-st} dt

              = integral_0^infinity f(t) e^{-(s-a)t} dt

              = F(s-a).

Using f(t) = 1 and our calculation of L(1;s) we find

e^{at} ----> 1/(s-a).   (*)

This function has a pole at s = a instead of s = 0. It's a general fact that the growth of f(t) as t ---> infinity can be read off from the position of the poles of F(s). If the right-most pole has real part a, then the growth is roughly like that of e^{at}.

This calculation (*) is more powerful than you may imagine at first, since a may be complex. Using linearity and

cos(omega t) = (e^{i omega t} + e^{-i omega t})/2

we find

cos(omega t) ----> (1/(s - i omega) + 1/(s + i omega))/2

This function has two poles, namely s = ib and s = -ib . It's a general fact that if f(t) is real then the poles of F(s) are either real or come in complex conjugate pairs. Cross multiplying, we can rewrite

cos(omega t) ----> s/(s^2 + omega^2)

Using

sin(omega t) = (e^{i omega} - e^{-i omega})/(2i)

we find

sin(omega t) ----> b/(s^2 + omega^2).