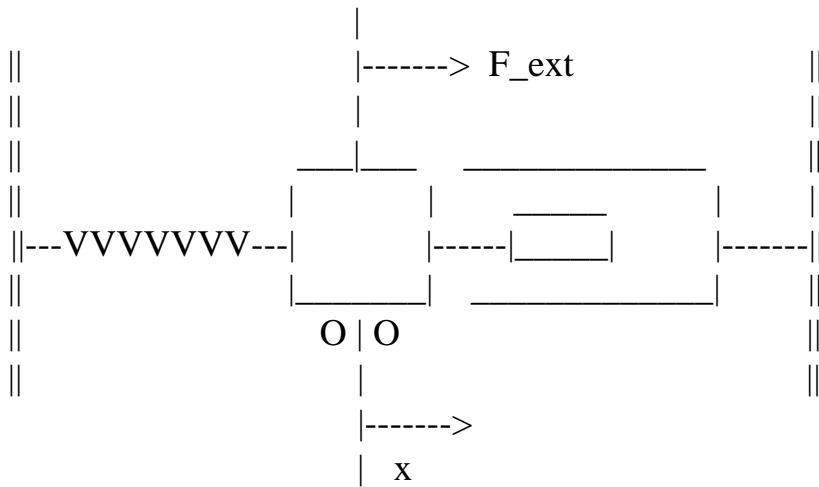


Second order equations: linear, constant coefficient, homogeneous.

Vocabulary:

1. $F = ma$ is the basic example. Take a spring attached to a wall,

spring mass dashpot



Set up the coordinate system so that at $x = 0$ the spring is relaxed.

The cart is influenced by three forces: the spring, the "dashpot" (which is a way to make friction explicit), and an external force:

$$mx'' = F_{\text{spr}} + F_{\text{dash}} + F_{\text{ext}}$$

The spring force is characterized by depending only on position: write $F_{\text{spr}}(x)$.

If $x > 0$, $F_{\text{spr}}(x) < 0$

if $x < 0$, $F_{\text{spr}}(x) > 0$

The simplest way to model this behavior (and one which is valid in general for small x , by the tangent line approximation) is

$$F_{\text{spr}}(x) = -kx \quad k > 0 \text{ the "spring constant."}$$

The dashpot force is frictional. This means that it depends only on the velocity.

Write $F_{\text{dash}}(x')$. Also it acts against the velocity:

If $x > 0$, $F_{\text{dash}}(x') < 0$

if $x < 0$, $F_{\text{dash}}(x') > 0$

The simplest way to model this behavior (and one which is valid in general for small x' , by the tangent line approximation) is

$F_{\text{dash}}(x) = -bx$ $b > 0$ the "damping constant."

So the equation is

$$mx'' + bx' + kx = F_{\text{ext}}$$

The left hand side represents the SYSTEM, the spring/mass/dashpot system. The right hand side represents the INPUT SIGNAL, an external force at work. The quantities m , b , k are the "coefficients." In general they may depend upon time: maybe the force is actually a rocket, and the fuel burns so m decreases. Maybe the spring gets softer as it ages. Maybe the honey in the dashpot gets stiffer with time. However, most of the time we will assume that the coefficients are CONSTANT.

2. Solutions. To get from x'' to x we must integrate twice, so you should expect two constants of integration in the general solution.

Physically, you can release the spring at $t = t_0$ from $x(t_0)$, but you also have to say what velocity you impart to it: $x'(t_0)$ is needed as part of the initial condition.

So: solutions can cross. There is no concept of direction field.

Still, initial conditions $(x(t_0), x'(t_0))$ determine the solution.

3. Today we'll find some solutions in the "homogeneous case." Homogeneous is pronounced "homogE'nEous" where E denotes a long e. It is not "homo'genous." It means $F_{\text{ext}} = 0$: the system is allowed to evolve on its own, without outside interference.

I displayed a rubber band with a weight on it. It bounced. I asked whether all solutions to systems like this bounce. Most people thought no. Let's see.

Constant coefficient, homogeneous: $mx'' + bx' + kx = 0$ (*)

This is a lot like $x' + kx = 0$, which has as solution $x = e^{-kt}$

(and more generally multiples of this). It makes sense to try for exponential solutions of (*): e^{rt} for some as yet undetermined constant r .

To see which r might work, plug $x = e^{rt}$ into (*). Organize the calculation: the k , b , m is a flag indicating that I should multiply the corresponding line by this number.

$$k) \quad x = e^{rt}$$

$$b) \quad x' = r e^{rt}$$

$$m) \quad x'' = r^2 e^{rt}$$

$$0 = mx'' + bx' + kx = (mr^2 + br + k) e^{rt}$$

An exponential is never zero, so we can cancel to see that e^{rt} is a solution to (*) exactly when r is a root of the "characteristic polynomial"

$$ms^2 + bs + k$$

Remark on the quadratic formula: it reads

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

It's often convenient to divide by m in advance. The effect is that we can take

$m = 1$: $x'' + bx' + kx$ with characteristic polynomial $s^2 + bs + k$.

The quadratic formula for the roots can be written

$$r = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - k}$$

Example A. $x'' + 5x' + 4x = 0$

The characteristic polynomial $s^2 + 5s + 4$ factors as $(s + 1)(s + 4)$

so the roots are $r = -1$ and $r = -4$. The corresponding exponential solutions

are e^{-t} and e^{-4t} . These ARE solutions, as you can check. You can also see that any constant multiple of each is again a solution.

They both damp down to zero; no oscillation here.

Example B. $x'' + 4x' + 5x = 0$

The characteristic polynomial $s^2 + 4s + 5$ has roots

$$r = -2 \pm \sqrt{4-5} = -2 \pm i$$

Our old friend $i = \sqrt{-1}$ appears, and we have exponential solutions

$$e^{(-2+i)t}, e^{(-2-i)t}$$

I guess we were expecting REAL solutions. For this we have:

Theorem: If x is a complex solution to $mx'' + bx' + kx = 0$, where m , b , and k are real, then the real and imaginary parts of x are also solutions.

Proof: Write $x = u + iv$ and build the table.

$$k) \quad x = u + iv$$

$$b) \quad x' = u' + iv'$$

$$m) \quad x'' = u'' + iv''$$

$$0 = (mu'' + bu' + ku) + i(mv'' + bv' + kv)$$

Both things in parentheses are real, so the only way this can happen is for both of them to be zero.

So in our situation,

$$e^{(-2+i)t} \text{ has real part } e^{-2t} \cos(t) \\ \text{and imaginary part } e^{-2t} \sin(t)$$

so we have those two solutions.

We have the other normal mode too. What does it lead to?

$$e^{(-2-i)t} \text{ has real part } e^{-2t} \cos(-t) = e^{-2t} \cos(t) \\ \text{and imaginary part } e^{-2t} \sin(t) = -e^{-2t} \sin(t)$$

Nothing new, since we already knew that multiples of solutions are solutions (of a homogeneous linear equation).