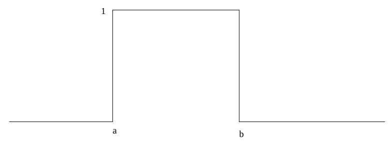
Solutions to Recitation 14 Problems

1. This is a function which is 1 between a and b, and zero otherwise. From the definition of the Laplace transform,



$$F(s) = \int_a^b e^{-st}\,dt = \frac{1}{s}(e^{-as} - e^{-bs}).$$

2. Let's denote by $g_b(t) = \frac{1}{b}(u(t) - u(t-b))$ to denote the dependence on b. Then using the formula for #1 with $a = 0$, we see that the corresponding laplace transform is

$$\frac{1}{b} \cdot \frac{1}{s}(1 - e^{-bs}).$$

Using l'Hôpital's rule (taking derivative with respect to b),

$$\lim_{b \to 0} \frac{1 - e^{-bs}}{bs} = \lim_{b \to 0} \frac{se^{-bs}}{s} = 1.$$

Alternatively, note that the derivative of the function e^{-st} at $t = 0$ is computed by

$$\lim_{b \to 0} \frac{e^{-bs} - 1}{b}$$

so what we want is negative of this derivative times $\frac{1}{s}$. Since the derivative is $-s$, we get to the same conclusion.

3. Because $e^{at}\cos(\omega t) = \frac{e^{at+\omega ti} + e^{at-\omega ti}}{2}$, by linearity, the Laplace transform of this function is

$$\frac{1}{2}\left(\frac{1}{s-a-\omega i} + \frac{1}{s-a+\omega i}\right) = \frac{s-a}{(s-a)^2 + \omega^2}.$$

Similarly, $e^{at}\sin(\omega t) = \frac{e^{at+\omega ti} - e^{at-\omega ti}}{2i}$, the Laplace transform is

$$\frac{1}{2i}\left(\frac{1}{s-a-\omega i} - \frac{1}{s-a+\omega i}\right) = \frac{\omega}{(s-a)^2 + \omega^2}.$$

The poles are complex values s for which the denominator of $F(s)$ is zero. So in both the cos and sin cases, the poles are located at $s = a \pm \omega i$. Following are the graphs and pole diagrams for the three functions. Recall that theoretically (see Prof. Miller's notes section 19) we expect the rightmost poles of $F(s)$ to determine the general behavior of $f(t)$ for large t: that is, if the rightmost pole is located at $\alpha + \beta i$, then $f(t)$ behaves like $e^{(\alpha+\beta i)t}$ for large t. Since we are looking at the functions exactly of this form in this problem, we get the exact match.

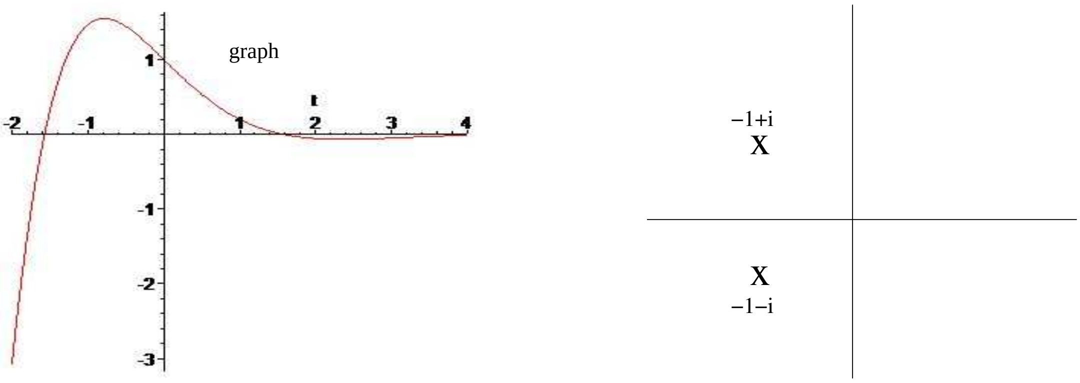


Figure 1: $e^{-t}\cos(t)$

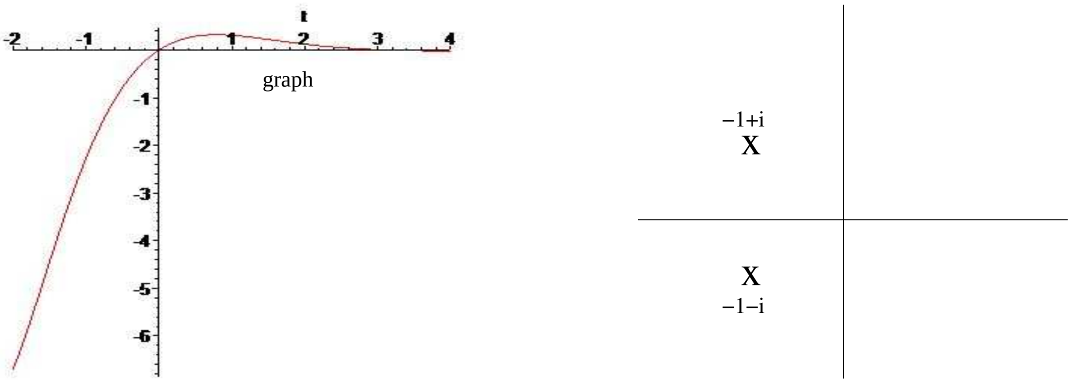


Figure 2: $e^{-t}\sin(t)$

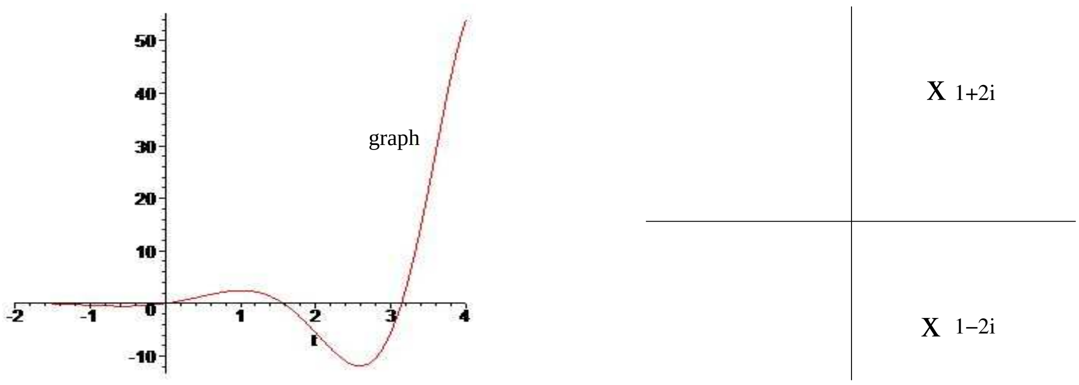


Figure 3: $e^{t}\sin(2t)$