

6. THE COMPLEX EXPONENTIAL

The exponential function is a basic building block for solutions of ODEs. Complex numbers expand the scope of the exponential function, and bring trigonometric functions under its sway.

6.1. Exponential solutions. The function e^t is *defined* to be the solution of the initial value problem $\dot{x} = x$, $x(0) = 1$. More generally, the chain rule implies the

Exponential Principle:

For any constant w , e^{wt} is the solution of $\dot{x} = wx$, $x(0) = 1$.

Now look at a more general constant coefficient homogeneous linear ODE, such as the second order equation

$$(1) \quad \ddot{x} + c\dot{x} + kx = 0.$$

It turns out that there is always a solution of (1) of the form $x = e^{rt}$, for an appropriate constant r .

To see what r should be, take $x = e^{rt}$ for an as yet to be determined constant r , substitute it into (1), and the Exponential Principle. We find

$$(r^2 + cr + k)e^{rt} = 0.$$

Cancel the exponential (which, conveniently, can never be zero); r must be a root of the polynomial $p(s) = s^2 + cs + k$. This is the characteristic polynomial of the equation. See Section 10 for more about this. The **characteristic polynomial** of the linear equation with constant coefficients

$$a_n \frac{d^n x}{dt^n} + \cdots + a_1 \dot{x} + a_0 x = 0$$

is

$$p(s) = a_n s^n + \cdots + a_1 s + a_0.$$

Its roots are the *characteristic roots* of the equation. We have discovered the

Characteristic Roots Principle:

(2) e^{rt} is a solution of a constant coefficient homogeneous linear equation exactly when r is a root of the characteristic polynomial.

Since most quadratic polynomials have two distinct roots, this normally gives us two linearly independent solutions, $e^{r_1 t}$ and $e^{r_2 t}$. The general solution is then the linear combination $c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

This is fine if the roots are real, but suppose we have the equation

$$(3) \quad \ddot{x} + 2\dot{x} + 2x = 0$$

for example. By the quadratic formula, the roots of the characteristic polynomial $s^2 + 2s + 2$ are the complex conjugate pair $-1 \pm i$. We had better figure out what is meant by $e^{(-1+i)t}$, for our use of exponentials as solutions to work.

6.2. The complex exponential. According to the Exponential Principle the function $z = e^{it}$ is the solution of the initial value problem

$$\dot{z} = iz, \quad z(0) = 1.$$

I can produce such a function:

$$z = \cos t + i \sin t.$$

For then $\dot{z} = -\sin t + i \cos t$, while $iz = i(\cos t + i \sin t) = i \cos t - \sin t$, using $i^2 = -1$.

We have proven

Euler's formula:

$$\boxed{e^{it} = \cos t + i \sin t}$$

In this formula, the left hand side is by definition the solution to $\dot{z} = iz$ such that $z(0) = 1$. The right hand side writes this function in more familiar terms.

We can reverse this process as well. Putting $z = e^{it}$ into the formulas (5.1), we find

$$(4) \quad \boxed{\cos t = \frac{e^{it} + e^{-it}}{2}, \quad \sin t = \frac{e^{it} - e^{-it}}{2i}}$$

We can express the solution to

$$\dot{z} = (a + bi)z, \quad z(0) = 1$$

in familiar terms as well: I leave it to you to check that it is

$$z = e^{at}(\cos(bt) + i \sin(bt)).$$

We have discovered what e^{wt} is for any complex constant $w = a + bi$:

$$(5) \quad \boxed{e^{(a+bi)t} = e^{at}(\cos bt + i \sin bt)}$$

Let's return to the example (3). The root $r_1 = -1 + i$ leads to

$$e^{(-1+i)t} = e^{-t}(\cos t + i \sin t)$$

and $r_2 = -1 - i$ leads to

$$e^{(-1-i)t} = e^{-t}(\cos t - i \sin t)$$

Exercise 6.2.1. Check that these are solutions.

We probably really wanted a *real* solution to (3), however. For this we have the

Reality Principle:

(6) If z is a solution to a homogeneous linear equation with real coefficients, then the real and imaginary parts of z are too.

We'll explain why this is in a minute, but first let's look at our example (3). The real part of $e^{(-1+i)t}$ is $e^{-t} \cos t$, and the imaginary part is $e^{-t} \sin t$. Both are solutions to (3).

In practice, you should just use the following consequence of what we've done:

Real solutions from complex roots:

If $r_1 = a + bi$ is a root of the characteristic polynomial of a homogeneous linear ODE whose coefficients are constant and real, then

$$e^{at} \cos(bt) \quad \text{and} \quad e^{at} \sin(bt)$$

are solutions. If $b \neq 0$, they are independent solutions.

To see why the Reality Principle holds, suppose z is a solution to a homogeneous linear equation with real coefficients, say

$$(7) \quad \ddot{z} + p\dot{z} + qz = 0$$

for example. Let's write x for the real part of z and y for the imaginary part of z , so $z = x + iy$. Since q is real,

$$\operatorname{Re}(qz) = qx \quad \text{and} \quad \operatorname{Im}(qz) = qy.$$

Derivatives are computed by differentiating real and imaginary parts separately, so (since p is also real)

$$\operatorname{Re}(p\dot{z}) = p\dot{x} \quad \text{and} \quad \operatorname{Im}(p\dot{z}) = p\dot{y}.$$

Finally,

$$\operatorname{Re} \ddot{z} = \ddot{x} \quad \text{and} \quad \operatorname{Im} \ddot{z} = \ddot{y}$$

so when we break down (7) into real and imaginary parts we get

$$\ddot{x} + p\dot{x} + qx = 0, \quad \ddot{y} + p\dot{y} + qy = 0$$

—that is, x and y are solutions of the same equation (7).

6.3. Polar coordinates. The expression

$$e^{it} = \cos t + i \sin t$$

parametrizes the unit circle in the complex plane. As t increases from 0 to 2π , the complex number $\cos t + i \sin t$ moves once counterclockwise around the circle. The parameter t is just the radian measure counterclockwise from the positive real axis.

More generally,

$$z(t) = e^{(a+bi)t} = e^{at}(\cos(bt) + i \sin(bt)).$$

parametrizes a curve in the complex plane. What is it?

Begin by looking at some values of t . When $t = 0$ we get $z(0) = 1$ no matter what a and b are. When $t = 1$ we get

$$(8) \quad e^{a+bi} = e^a(\cos b + i \sin b).$$

The numbers a and b determine the polar coordinates of this point in the complex plane. The absolute value (=magnitude) of $\cos(b) + i \sin(b)$ is 1, so (since $|wz| = |w||z|$ and $e^{at} > 0$)

$$|e^{a+bi}| = e^a.$$

This is the radial distance from the origin.

The polar angle θ , that is the angle measured counterclockwise from the positive x axis, is called the **argument** of the complex number z and is written $\arg z$. According to (8), the argument of e^{a+bi} is simply b . As usual, the argument of a complex number is only well defined up to adding multiples of 2π .

Any complex number except for zero can be expressed as e^{a+bi} for some a, b . You just need to know a polar expression for the point in the plane.

Exercise 6.3.1. Find expressions of $1, i, 1 + i, (1 + \sqrt{3}i)/2$ as complex exponentials.

For general t ,

$$(9) \quad e^{(a+bi)t} = e^{at}(\cos(bt) + i \sin(bt))$$

parametrizes a spiral (at least when $b \neq 0$). If $a > 0$, it runs away from the origin, exponentially, while winding around the origin (counterclockwise if $b > 0$, clockwise if $b < 0$). If $a < 0$, it decays exponentially towards the origin, while winding around the origin. Figure 3 shows a picture of the curve parametrized by $e^{(1+2\pi i)t}$.

In particular,

$$(13) \quad e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)}.$$

In terms of polar coordinates, this says that

$$(14) \quad \textbf{Angles Add:} \quad \arg(wz) = \arg(w) + \arg(z).$$

Exercise 6.4.1. Compute $((1+\sqrt{3}i)/2)^3$ and $(1+i)^4$ afresh using these polar considerations.

Exercise 6.4.2. Derive the addition laws for cosine and sine from Euler's formula and (13). Understand this exercise and you'll never have to remember those formulas again.

6.5. Roots of unity and other numbers. The polar expression of multiplication is useful in finding roots of complex numbers. Begin with the sixth roots of 1, for example. We are looking for complex numbers z such that $z^6 = 1$. Since *moduli multiply*, $|z|^6 = |z^6| = |1| = 1$, and since moduli are nonnegative this forces $|z| = 1$: all the sixth roots of 1 are on the unit circle. *Arguments add*, so the argument of a sixth root of 1 is an angle θ so that 6θ is a multiple of 2π (which are the angles giving 1). Up to addition of multiples of 2π there are six such angles: $0, \pi/3, 2\pi/3, \pi, 4\pi/3$, and $5\pi/3$. The resulting points on the unit circle divide it into six equal arcs. From this and some geometry or trigonometry it's easy to write down the roots as $a + bi$: ± 1 and $(\pm 1 \pm \sqrt{3}i)/2$. In general, the n th roots of 1 break the circle evenly into n parts.

Exercise 6.5.1. Write down the eighth roots of 1 in the form $a + bi$.

Now let's take roots of numbers other than 1. Start by finding a single root z of the complex number $w = re^{i\theta}$ (where r is a positive real number). Since magnitudes multiply, $|z| = \sqrt[n]{r}$. Since angles add, one choice for the argument of z is θ/n : one n th of the way up from the positive real axis. Thus for example one square root of $4i$ is the complex number with magnitude 2 and argument $\pi/4$, which is $\sqrt{2}(1+i)$. To get all the n th roots of w notice that you can multiply one by any n th root of 1 and get another n th root of w . Angles add and magnitudes multiply, so the effect of this is just to add a multiple of $2\pi/n$ to the angle of the first root we found. There are n distinct n th roots of any nonzero complex number $|w|$, and they divide the circle with center 0 and radius $\sqrt[n]{r}$ evenly into n arcs.

Exercise 6.5.2. Find all the cube roots of -8 . Find all the sixth roots of $-i/64$.

We can use our ability to find complex roots to solve more general polynomial equations.

Exercise 6.5.3. Find all the roots of the polynomials $x^3 + 1$, $ix^2 + x + (1 + i)$, and $x^4 - 2x^2 + 1$.