18.03 Recitation Problems 14

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Laplace Transform; Poles

Laplace transform:

$$f(t) \rightsquigarrow F(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

s-shift law: $e^{at}f(t) \rightsquigarrow F(s-a)$.

$$1 \rightsquigarrow \frac{1}{s}, \quad e^{at} \rightsquigarrow \frac{1}{s-a}, \quad \cos(\omega t) \rightsquigarrow \frac{s}{s^2+\omega^2}, \quad \sin(\omega t) \rightsquigarrow \frac{\omega}{s^2+\omega^2}$$

A "pole" of a complex function $F(s)$ is a complex number z at which the function value becomes infinite.

1. Sketch a graph of the "window" or "bump" function $f(t) = u(t-a) - u(t-b)$, $0 \leq a < b$, and compute its Laplace transform $F(s)$ using the integral definition.

2. Using fact that the Laplace transform is linear to deduce from **1.** what the Laplace transform of $g(t) = (1/b)(u(t) - u(t-b))$ is. The "limit" of these functions $g(t)$ as $b \to 0$ is the delta function. What is the limit of their Laplace transforms? (Hint: use l'Hôpital's rule, or, better, the definition of the derivative.)

3. Using the fact that $e^{wt} \rightsquigarrow 1/(s-w)$ for any complex number w, together with the expressions

$$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i},$$

to compute the Laplace transforms of $e^{at}\cos(\omega t)$ and $e^{at}\sin(\omega t)$.

Where are the poles of these functions of s?

Sketch a graph of $e^{-t}\cos(t)$ and of the pole diagram of its Laplace transform. Do the same for $e^{-t}\sin(t)$ and $e^{t}\sin(2t)$.