10.450 Process Dynamics, Operations, and Control
Lecture Notes - 10

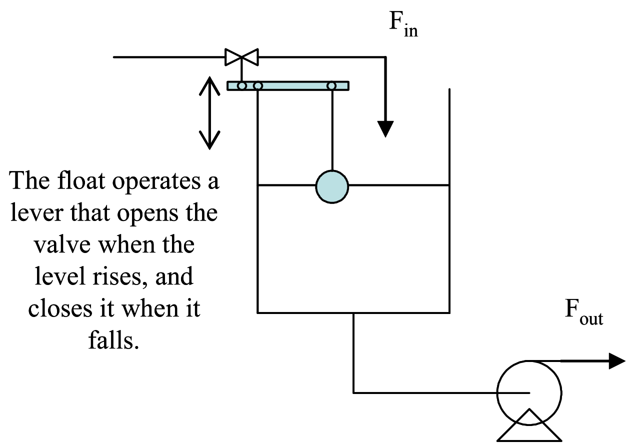
Lesson 10. Stability defined and related to the poles of the transfer function.

10.0 Context

In lesson 9 we found that a second-order process could be brought to instability by reducing the damping coefficient. Here we study criteria for stability in general. In a chemical process, instability may mean unsafe conditions, bad quality product, and loss of money.

10.1 An example, using a badly conceived process

Consider a tank with pumped output in which the input flow is directly proportional to the liquid level in the tank. Thus a rising level leads to increased inlet flow, and falling level cuts back on the inlet flow: each change makes matters worse.



Analyzing by material balance:

$$\begin{aligned} A\frac{dh}{dt} &= kh - F_{out}(t) \\ -\tau\frac{dh}{dt} + h &= \frac{1}{k}F_{out}(t) \qquad h(0) = h_s \end{aligned} \qquad (10.1.1)$$

Here, the outlet flow is an arbitrary function of time, and the initial condition is some tank level. Dividing by the inlet flow proportionality defines a characteristic time τ. Proceeding by integrating factor, we obtain the solution

$$h(t) = \frac{-1}{k\tau}e^{t/\tau}\int_0^t e^{-t/\tau}F_{out}(t)dt + h_s e^{t/\tau} \qquad (10.1.2)$$

Suppose F_{out} is constant; then

$$h(t) = \frac{F_{out}}{k}\left(1 - e^{t/\tau}\right) + h_s e^{t/\tau} = \frac{F_{out}}{k} + \left(h_s - \frac{F_{out}}{k}\right)e^{t/\tau} \qquad (10.1.3)$$

If F_{out} and h_s are in perfect balance, the level will be steady at h_s. However, the slightest change in F_{out} will lead to exponential variation in the liquid level. This is an unstable system, headed inevitably for overflow or draining.

10.2 Define stability

If we disturb a system, will it return to good operation, or will it get out of hand? This is asking whether the system is stable. We define stability as "bounded output for a bounded input". That means that

- a ramp disturbance is not fair – even stable systems can get into trouble if the input keeps rising.
- an impulse disturbance is fair – although it is briefly infinite, it soon passes.
- a stable system should also handle a step change in input, ultimately coming to some new steady state. (We must be realistic, however. If a system is so sensitive that a small input step leads to an unacceptably high, though steady, output, we might declare it unstable for practical purposes.)
- it should also handle a sine input; here the result is in general not steady state, because the output may oscillate. (Thus we distinguish between 'steady state' and 'long-term stability'.)

Stability depends on

- the system – certainly; we will discuss the characteristics of the system that determine stability
- the type of disturbance – yes, as discussed above. In addition, it is possible that a system is stable to some bounded inputs, but not others. For example, the first-order integrator (pumped outlet tank) is stable to sine disturbances but not to steps. No partial credit – we declare such a system unstable.
- the magnitude of disturbance – maybe. That is, we continue to represent real systems by LINEAR models. While the magnitude of the disturbance does not influence stability of the linear system, a sufficiently large disturbance to the real system may move it to a regime of operation that the linear approximation does not describe.

10.3 Develop a criterion for stability of linear systems

The key is the poles of the LINEAR characteristic equation. If any pole has a zero or positive real part, the system will be unstable to bounded disturbances. This happens because of the structure of the linear, constant-coefficient, ordinary differential equations that we are using – the solutions are exponential terms. Here's the general form

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = K_1 x_1(t) + K_2 x_2(t) + \cdots \qquad (10.3.1)$$

To obtain the homogeneous solution, we substitute a candidate solution y = e^{rt} and set the right-hand side to zero. We find that the parameter r is constrained to be a root of the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0 = 0 \qquad (10.3.2)$$

These n roots will in general be complex numbers.

$$r_i = \alpha_i + j\beta_i \qquad (10.3.3)$$

The homogeneous solution of (10.3.1) will be a sum of terms, each containing a factor $e^{\alpha_i t}$. Any term in which α_i is positive will grow without bound, and thus render the entire solution unstable. Such instability results from the very structure of the system itself (i.e., the values of the coefficients a_i in (10.3.1)), and not from the particular disturbances on the right-hand side.

Notice that (10.1.1) resembles our well-known first-order lag system, but differs in a crucial respect:

$$\tau \frac{dy}{dt} + y = Kx(t) \qquad \text{first-order lag}$$

$$-\tau \frac{dy}{dt} + y = Kx(t) \qquad \text{unstable system from (10.1.1)}$$

In the first-order lag, the dependent variable opposes change in itself: a higher value of y forces a smaller rate of change. The opposite effect is seen in (10.1.1), and this mathematical structure leads to the positive root of the characteristic equation.

Considering the homogeneous solution has led us to regard positive real parts of roots as unstable. However, we must become even more restrictive, as shown by considering the disturbances. Taking the Laplace transform of (10.3.1),

$$y(s) = \frac{K_1}{a_n s^n + \cdots + a_0} x_1(s) + \frac{K_2}{a_n s^n + \cdots + a_0} x_2(s) + \cdots \qquad (10.3.4)$$

Equation (10.3.2), the characteristic equation of (10.3.1), is the same as the denominator of the transfer functions in (10.3.4). Of course, they both represent the left-hand side of the differential equation, and the poles of the transfer function are the same as the roots of the characteristic equation. They have the same significance for the solution (as they must) in that partial fraction expansion of the transfer function in terms of the poles leads to exponential terms in the solution when the transform is inverted.

Consider now a transfer function that has a zero pole.

$$y(s) = \frac{K_1}{s(s+b)} x_1(s) \qquad (10.3.5)$$

The root -b is a typical stable root, and more denominator terms of the form $(s+b_i)$ would not change the outcome. Now suppose that $x_1(t)$ is a step of magnitude A. Then

$$y(s) = \frac{K_1 A}{s^2(s+b)} = K_1 A\left(\frac{1}{bs^2} - \frac{1}{b^2 s} + \frac{1}{b^2(s+b)}\right) \qquad (10.3.6)$$

Upon taking inverse transforms,

$$y(t) = K_1 A\left(\frac{t}{b} - \frac{1}{b^2} + \frac{1}{b^2} e^{-bt}\right) \qquad (10.3.7)$$

Thus the step disturbance, by *repeating the existing zero pole*, elicits a term that grows steadily with time. The zero pole would give a bounded output for a sine disturbance, but its behavior for a persistent disturbance of one sign is grounds for declaring it categorically unstable.

A system is unstable if any root of its characteristic equation (i.e., pole of its transfer function) has a real part of zero or greater.

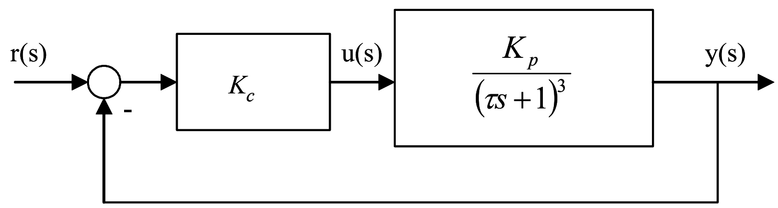
By the way, the system of (10.3.5) is an example of an integrator process, in which the dependent variable has no influence on its time derivatives. Its model equation, reasoning backward from (10.3.5), is

$$\frac{d^2 y}{dt^2} + b\frac{dy}{dt} = K_1 x_1(t)$$

We saw a first-order integrator in the pumped outlet tank of Lesson 1.

10.4 Stability of a recycle structure

Consider a non-oscillating third-order system in a recycle structure. For example, this might be three mixed tanks in series under feedback control, in which the controlled variable is compared to a set point and used to drive a manipulated input variable, as we discussed in Lesson 1. The block diagram for the system is



The value of signal y is subtracted from the set point r. The difference, which constitutes error, is fed to the controller. The controller operates upon the error by a proportional gain to produce a signal u. Input u acts upon output y according to the transfer function shown in the block. The controller gain K_c is variable; process gain K_p and characteristic time τ are fixed.

We derive the transfer function for the system by block diagram rules.

$$\begin{aligned} y(s) &= \frac{K_p}{(\tau s+1)^3}u(s) \\ &= \frac{K_pK_c}{(\tau s+1)^3}\left(r(s)-y(s)\right) \\ &= \frac{\dfrac{K_pK_c}{(\tau s+1)^3}}{1+\dfrac{K_pK_c}{(\tau s+1)^3}}r(s) \\ &= \frac{K_pK_c}{(\tau s+1)^3+K_pK_c}r(s) \end{aligned} \qquad (10.4.1)$$

We emphasize that transfer function (10.4.1) does not describe the third-order process, but rather that process under feedback control by a proportional controller. The transfer function shows how the controlled variable y responds to changes in set point r. The poles of (10.4.1) are most easily found by numerical methods. In Figure 10-1 we have plotted the poles as a function of the value of K_pK_c, assuming $\tau = 1$. The real part of the complex conjugate roots becomes zero at $K_pK_c = 8$.

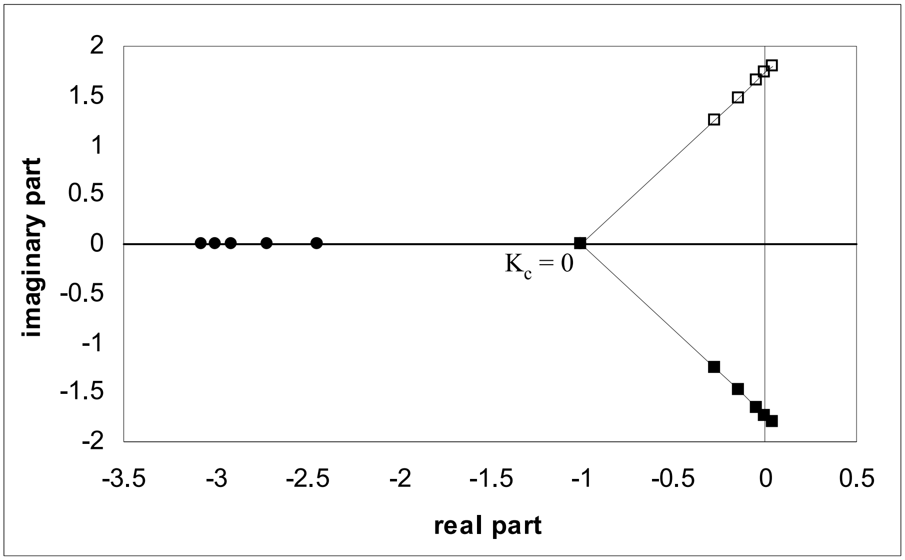
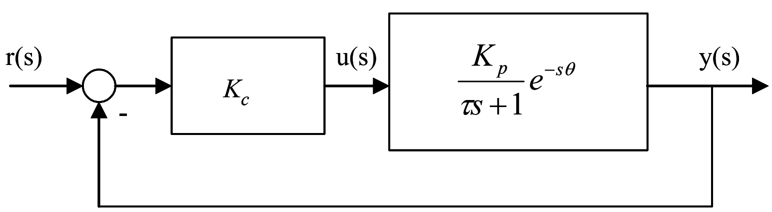


Figure 10-1. Poles of transfer function (10.4.1). With gain of zero, the repeated root is -1. As gain increases (KpKc = 3, 5, 7, 8, 9) the complex conjugate poles approach the imaginary axis with increasing imaginary parts.

We can calculate the step response as we have before by substituting a step for r(s) and inverting the transform y(s) in (10.4.1).

10.5 Stability of systems with dead time

In Lesson 8 we introduced the dead time. Consider feedback control of the FODT process.



By the same means as in Section 10.4, we derive the transfer function for the system.

$$y(s) = \frac{K_p K_c e^{-s\theta}}{\tau s + 1 + K_p K_c e^{-s\theta}} r(s) \tag{10.5.1}$$

This departs from the polynomial equations we used as a basis for the stability criterion! We might salvage that method by approximating the exponential term by an infinite polynomial series. We might also develop another method – that's for another lesson.

10.6 Stability and control

Many chemical processes, left alone, are stable. Of course we want to maintain the operation at specific conditions, so we add control. As indicated in Section 10.4, however, we will find that it is possible, via poorly applied control, to produce instability in a process that, without control, would be stable.