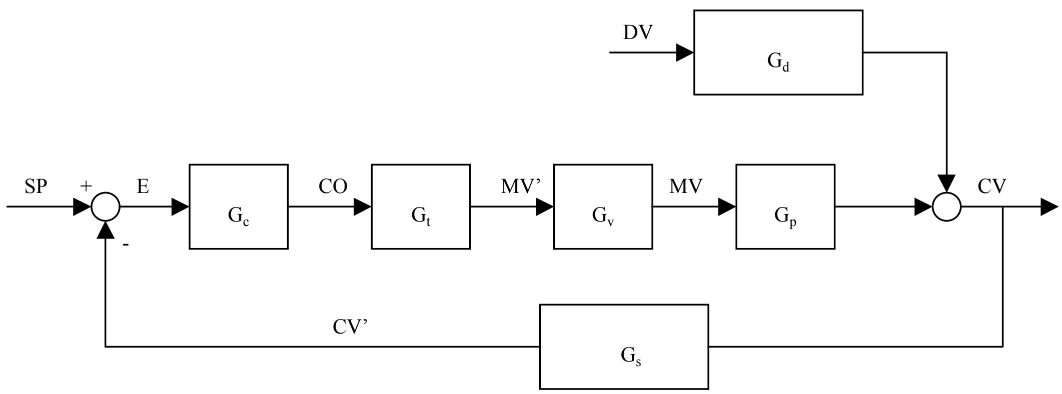
10.450 Process Dynamics, Operations, and Control
Lecture Notes - 15b

Lesson 15b. Equations for the closed loop - Laplace domain.

15b.0 Context
We again analyze the closed loop of Lesson 15a, this time from the Laplace transform point of view.

15b.1 Block diagram of the loop
NOW do the same thing again from the block diagram with transfer functions…



The transfer functions are

$$
\begin{aligned}
G_p(s) &= \frac{CV^*(s)}{MV^*(s)} = \frac{h^*(s)}{F_m^*(s)} = \frac{K_p}{\tau_p s+1} \\
G_d(s) &= \frac{CV^*(s)}{DV^*(s)} = \frac{h^*(s)}{F_d^*(s)} = \frac{K_d}{\tau_p s+1} \\
G_s(s) &= \frac{CV'^*(s)}{CV^*(s)} = K_s \\
G_c(s) &= \frac{CO^*(s)}{E^*(s)} = K_c + \frac{K_c}{sT_i} \\
G_t(s) &= \frac{MV'^*(s)}{CO^*(s)} = K_t \\
G_v(s) &= \frac{MV^*(s)}{MV'^*(s)} = \frac{F_m^*(s)}{MV'^*(s)} = K_v
\end{aligned}
\qquad (15b.1.1)
$$

15b.2 Closed loop transfer function
Derive the equation consistent with this diagram and these definitions.

$$h^*(s) = G_d(s)F_d^*(s) + G_p(s)F_m^*(s)$$
$$= G_d(s)F_d^*(s) + G_p(s)G_v(s)G_t(s)G_c(s)\left(SP^*(s) - G_s(s)h^*(s)\right) \quad (15b.2.1)$$
$$h^*(s) = \frac{G_d(s)F_d^*(s) + G_p(s)G_v(s)G_t(s)G_c(s)SP^*(s)}{1 + G_p(s)G_v(s)G_t(s)G_c(s)G_s(s)}$$

Substituting the transfer functions

$$h^*(s) = \frac{\dfrac{K_d}{\tau_p s + 1}F_d^*(s) + \dfrac{K_p K_v K_t K_c}{\tau_p s + 1}\left(1 + \dfrac{1}{T_i s}\right)SP^*(s)}{1 + \dfrac{K_p K_v K_t K_c K_s}{\tau_p s + 1}\left(1 + \dfrac{1}{T_i s}\right)}$$
$$= \frac{K_d T_i s F_d^*(s) + K_p K_v K_t K_c (T_i s + 1) SP^*(s)}{T_i s(\tau_p s + 1) + K_p K_v K_t K_c K_s (T_i s + 1)} \quad (15b.2.2)$$
$$= \frac{\dfrac{K_d T_i}{K_p K_v K_t K_c K_s} s F_d^*(s) + \dfrac{1}{K_s}(T_i s + 1) SP^*(s)}{\dfrac{T_i \tau_p}{K_p K_v K_t K_c K_s} s^2 + \dfrac{T_i}{K_p K_v K_t K_c K_s} s + T_i s + 1}$$

Thus the first-order process under PI control behaves as a second-order dynamic system in which a time constant and damping factor may be identified.

$$\tau = \sqrt{\frac{T_i \tau_p}{K_p K_v K_t K_c K_s}}$$
$$\xi = \frac{1}{2}\sqrt{\frac{T_i}{\tau_p}}\frac{1 + K_p K_v K_t K_c K_s}{\sqrt{K_p K_v K_t K_c K_s}} \quad (15b.2.3)$$

The characteristic time τ depends on the intrinsic process characteristic time τ_p, but also upon the controller settings K_c and T_i. More rapid response (smaller τ) can be achieved by increasing the gain and decreasing the integral time. This represents more aggressive control. Decreasing the integral time tends to make the system underdamped.

15b.3 Step response

Input a step in the disturbance flow of ΔF at time t_d. No change in set point.

$$h^*(s) = \frac{\frac{K_d \tau^2}{\tau_p} \Delta F e^{-t_d s}}{\tau^2 s^2 + 2\xi \tau s + 1} \tag{15b.3.1}$$

Inverting by table of transform pairs,

$$h^*(t) = \frac{K_d \Delta F \tau}{\tau_p \sqrt{1-\xi^2}} e^{-\xi(t-t_d)/\tau} \sin \frac{\sqrt{1-\xi^2}}{\tau}(t-t_d) \tag{15b.3.2}$$

This form is most convenient if $\xi < 1$; otherwise it would need to be manipulated using Euler's relations to remove the imaginary terms. It shows an oscillatory response that decays to zero according to the system time constant τ. Thus the integral control mode removes offset in a step response.

15b.4 What is the purpose of all this analysis?

Integral mode also modifies a first order system, capable only of asymptotic responses, making it second-order, and thus capable of oscillatory responses, as well. Given the analytic expression for the closed loop transfer function, it would be a relatively straightforward calculation to choose numerical values of K_c and T_i to best manage the process. However, such a transfer function is unlikely to be reliably known for a realistic process. Hence, we must find other means of selecting controller parameters. This is the topic of *controller tuning*.