10.450 Process Dynamics, Operations, and Control
Lecture Notes - 4

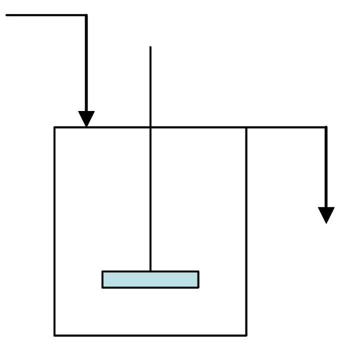
Lesson 4. Dynamic behavior of 'first-order' processes.

4.0 Context

System dynamics is an engineering science useful to mechanical, electrical, and chemical engineers, as well as others. This is because transient behavior, for all the variety of systems in nature and technology, can be described by a very few elements. This lesson concerns one of those elements.

4.1 The first order lag: mixed tank

Before we explain the term "first-order lag", we will work with an example of one: consider a tank equipped with a stirrer to mix the inlet stream into the contents of the tank. The composition of the liquid in the tank is uniform, and it is equal to the composition of the outlet stream. The tank is arranged with large overflow; thus the inlet and outlet flows are essentially equal at any time, and the volume is constant. This is a surge tank for smoothing concentration changes; contrast it with the flow surge tank of Lesson 1.



For simplicity, we will consider the flow to be constant, but the inlet concentration may vary with time; we wish to determine the effect on the outlet concentration.

Let's outline the procedure we will be following:

- write material and energy balances and other equations required to describe the process
- (for most process control applications) identify a steady state operating condition to serve as a reference
- substitute Taylor series approximations about the reference condition for nonlinear terms in the model
- solve the model for any required operating parameters at the reference condition
- subtract the steady state condition from the model equations to express all variables in deviation form
- arrange the equations in standard form, identifying dynamic parameters of known significance

- solve for the output variables as functions of the inputs
- introduce particular disturbances and calculate the responses

First, we write a component material balance on the solute.

$$\frac{d}{dt}VC_o = FC_i(t) - FC_o(t) \qquad C_o(0) = C_s \tag{4.1.1}$$

Because the flow F and volume V are constant, there are no nonlinear terms in the equation. We write (4.1.1) at steady state with reference inlet concentration C_s.

$$\left.\frac{dVC_o}{dt}\right|_s = 0 = FC_s - FC_{os} \tag{4.1.2}$$

From (4.1.2) we see that the outlet concentration at the reference condition is also C_s. Subtracting (4.1.2) from (4.1.1), we obtain the process model in terms of deviation variables, indicated by an asterisk superscript. These variables are zero when the process is at the reference condition; nonzero values indicate deviation from the reference.

$$\begin{aligned} &\frac{d}{dt}V\left(C_o(t) - C_s\right) = F\left(C_i(t) - C_s\right) - F\left(C_o(t) - C_s\right) \\ &\frac{d}{dt}VC_o^* = FC_i^*(t) - FC_o^*(t) \qquad C_o^*(0) = 0 \end{aligned} \tag{4.1.3}$$

This is a first-order ODE with constant coefficients. We rearrange it to standard form.

$$\frac{V}{F}\frac{dC_o^*}{dt} + C_o^*(t) = C_i^*(t) \tag{4.1.4}$$

In mathematical nomenclature, C_i^* is the forcing function and C_o^* the dependent variable. In our system nomenclature, C_i^* is the input and C_o^* the output. In standard form, the ratio of tank volume to flow rate clearly takes on the significance of a characteristic time, the *time constant* τ.

$$\tau\frac{dC_o^*}{dt} + C_o^* = C_i^*(t) \qquad C_o^*(0) = 0 \tag{4.1.5}$$

The solution (by (3.6.1)) is

$$C_o^*(t) = \frac{e^{-t/\tau}}{\tau}\int_0^t e^{t/\tau} C_i^*(t)dt \qquad (4.1.6)$$

Equation (4.1.6) is the process model for the mixing tank, showing how the outlet concentration behaves for arbitrary disturbances in the inlet. For example, if inlet concentration undergoes a step change ΔC at t_d,

$$C_o^* = \Delta C\left(1 - e^{-(t-t_d)/\tau}\right) \qquad (4.1.7)$$

As the disturbance is introduced, the outlet concentration begins to change; it gradually becomes equal to the inlet concentration. Notice that the tangent to the initial response reaches the final value in one time constant.

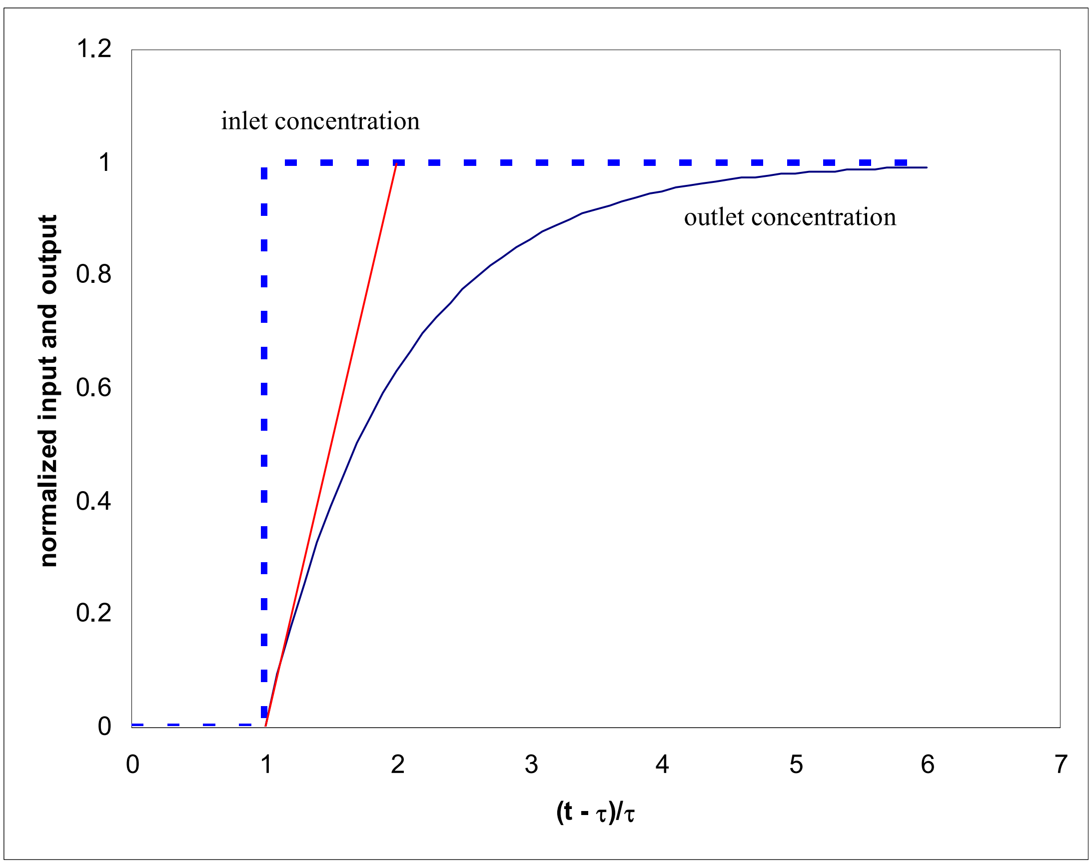


Fig 4-1. The ordinate has been normalized by the magnitude of the step change, and the abscissa by the time constant. Thus this non-dimensional plot is characteristic of all first order lag step responses. The time t_d at which the input occurs has been set for convenience to equal the time constant.

Systems described by (4.1.5), with solution (4.1.6), are called first-order lags. "First order" refers to the order of the governing differential equation (4.1.5). "Lag" refers to the way in which the output lags behind the input. The lag occurs because the system has storage capacity, and that capacity takes time to fill or deplete when conditions change. In this problem, the system stores the dissolved component.

First order lags always feature a time constant τ that indicates the speed of response, because time is normalized by, or scaled to, the time constant. From the properties of the exponential function, we see that the step is 95% complete when time equal to three time constants has elapsed. If the tank time constant is large (large volume, low flow) this time will be large. If the time constant is smaller (small volume, large flow) the outlet concentration will respond more quickly. This is consistent with intuition and experience.

In addition to speed of response, we are also interested in the degree to which a dynamic system amplifies or attenuates the input signal. This is often expressed by the steady-state gain, which is the ratio of steady output change to input change following a step disturbance.

$$gain = \frac{C_o^*(\infty) - C_o^*(0)}{C_i^*(\infty) - C_i^*(0)} = \frac{\Delta C}{\Delta C} = 1 \tag{4.1.8}$$

For the mixing tank, the gain is 1, showing that permanent disturbances are merely passed through the system. Both time constant and gain are independent of the size of the disturbance ΔC.

4.2 Integrator: pumped outlet tank

The pumped outlet tank of Lessons 1 and 2 is an example of a first order integrator.

$$\rho A \frac{dh}{dt} = w_i - w_o \qquad h(0) = h_s \tag{4.2.1}$$

All terms in the equation are linear. We define a steady state reference condition in which the liquid level is h_s, and the inlet and outlet flows are equal to w_s. In deviation variables, (4.2.1) becomes

$$\rho A \frac{dh^*}{dt} = w_i^* - w_o^* \qquad h^*(0) = 0 \tag{4.2.2}$$

To be strict about placing (4.2.2) in standard form, we should define a gain and a time constant. Gain always has dimensions of output/input, or in this case, length divided by mass flow. Hence we multiply (4.2.2) by the height of the tank, and divide by w_s.

$$\rho A \frac{h_T}{w_s}\frac{dh^*}{dt} = \frac{h_T}{w_s}(w_i^* - w_o^*) \qquad h^*(0) = 0 \tag{4.2.3}$$

Collecting terms, we find

$$\tau \frac{dh^*}{dt} = K(w_i^* - w_o^*) \qquad h^*(0) = 0 \tag{4.2.4}$$

Equation (4.2.4) is separable; its solution is

$$h^* = K(w_i^* - w_i^*)\frac{t}{\tau} \tag{4.2.5}$$

and the response to a step change of magnitude Δw in inlet flow at time t_d is

$$h^* = K\Delta w \frac{(t - t_d)}{\tau} \tag{4.2.6}$$

The integrator has no steady state response to a step disturbance, so K cannot be viewed as a steady-state gain. The time constant is the residence time for a full tank.

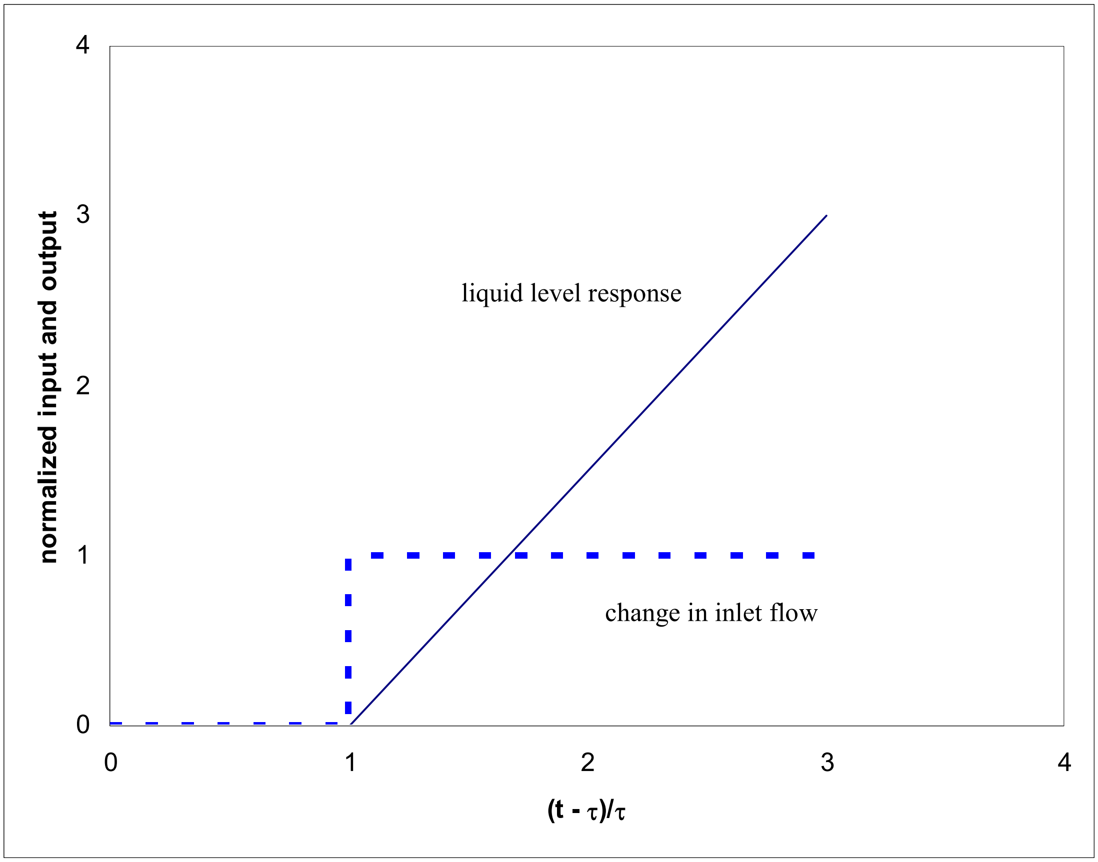


Fig 4-2. The ordinate has been normalized by the product of the gain and step disturbance, and the abscissa by the time constant. The time t_d at which the input occurs has been set for convenience to equal the time constant.

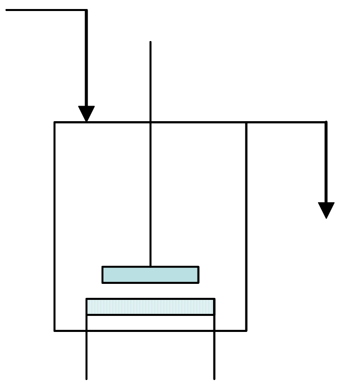
4.3 Summary of disturbance responses

Initial condition is zero; disturbance is introduced at time t_d.

system	first order lag	first order integrator
model	$\tau \frac{dy}{dt} + y(t) = Kx(t)$	$\tau \frac{dy}{dt} = Kx(t)$
step: $x = Au(t - t_d)$	$AK\left(1 - e^{-(t-t_d)/\tau}\right)$ steady-state gain = K	$\frac{AK(t - t_d)}{\tau}$
impulse: $x = A\delta(t - t_d)$	$\frac{AK}{\tau} e^{-(t-t_d)/\tau}$	$\frac{AK}{\tau}$
sine: $x = A\sin(\omega t - \omega t_d)$	$\frac{AK\omega\tau e^{-(t-t_d)/\tau}}{1+\omega^2\tau^2} + \frac{AK\sin(\omega(t-t_d)+\phi)}{\sqrt{1+\omega^2\tau^2}}$ $\phi = \tan^{-1}(-\omega\tau)$	$\frac{AK}{\tau\omega}\left(1+\sin\left(\omega(t-t_d)-\frac{\pi}{2}\right)\right)$

4.4 Multiple system inputs from multiple inlet streams

The first order lag equation in Section 4.3 is the mathematical form that results from applying material and energy balances to well-mixed volumes, as illustrated by the mixing tank in Section 4.1. Of course, a tank may have more than one inlet stream. If so, it will usually be added to the right-hand side of the equation. In illustration, consider a well-stirred tank heated by an electric resistance element of output power Q.



Our previous first-order systems have stored mass; this one stores energy. The energy balance is

$$\frac{d}{dt}\left(\rho C_p V\left(T_o - T_{ref}\right)\right) = \rho C_p F\left(T_i - T_{ref}\right) - \rho C_p F\left(T_o - T_{ref}\right) + Q \qquad (4.4.1)$$

where T_{ref} is a reference temperature for computing the enthalpy of the flowing stream. Presuming that flow F is constant and the physical properties are not a function of temperature, we see that (4.4.1) is a first-order linear ODE with constant coefficients. The energy balance at steady conditions is

$$\frac{d}{dt}\left(\rho C_p V\left(T_{os} - T_{ref}\right)\right) = 0 = \rho C_p F\left(T_{is} - T_{ref}\right) - \rho C_p F\left(T_{os} - T_{ref}\right) + Q_s \qquad (4.4.2)$$

Subtracting (4.4.2) from (4.4.1) and introducing deviation variables

$$\begin{aligned} &\frac{d}{dt}\left(\rho C_p V\left(T_o - T_{os}\right)\right) = \rho C_p F\left(T_i - T_{is}\right) - \rho C_p F\left(T_o - T_{os}\right) + Q - Q_s \\ &\frac{d}{dt}\left(\rho C_p V T_o^*\right) = \rho C_p F T_i^* - \rho C_p F T_o^* + Q^* \end{aligned} \qquad (4.4.3)$$

We rearrange (4.4.3) to standard form and consider the case of initial steady state.

$$\frac{V}{F}\frac{dT_o^*}{dt} + T_o^* = T_i^* + \frac{1}{\rho C_p F}Q^* \qquad T_o^*(0) = 0 \tag{4.4.4}$$

The time constant for temperature change is the tank residence time, equal to the tank volume divided by the volumetric flow rate. We find that the outlet temperature response depends on two inputs: the inlet temperature and the heater power; either can act as a disturbance to the first-order system. The gain for inlet temperature disturbances is unity; thus a step change in temperature would ultimately propagate through the tank. The gain for power disturbances converts dimensions of power to dimensions of temperature. This gain is a function of the flow rate, so that, for example, power disturbances have less effect on T_o when the flow F is large.

Equation (4.4.4) is linear first-order, and can be solved by (3.6.1), just as we did in Section 4.1.

$$T_o^*(t) = \frac{e^{-t/\tau}}{\tau}\int_0^t e^{t/\tau}\left(T_i^* + \frac{Q^*}{\rho C_p F}\right)dt \tag{4.4.5}$$

In a linear model, the effect of the disturbances is additive. That is, each affects the response independently of the other, and the effects are simply added. Consider a step increase in inlet temperature at time τ/2, followed at 2τ by a compensating step decrease in heater power. The outlet temperature first rises and then falls in response.

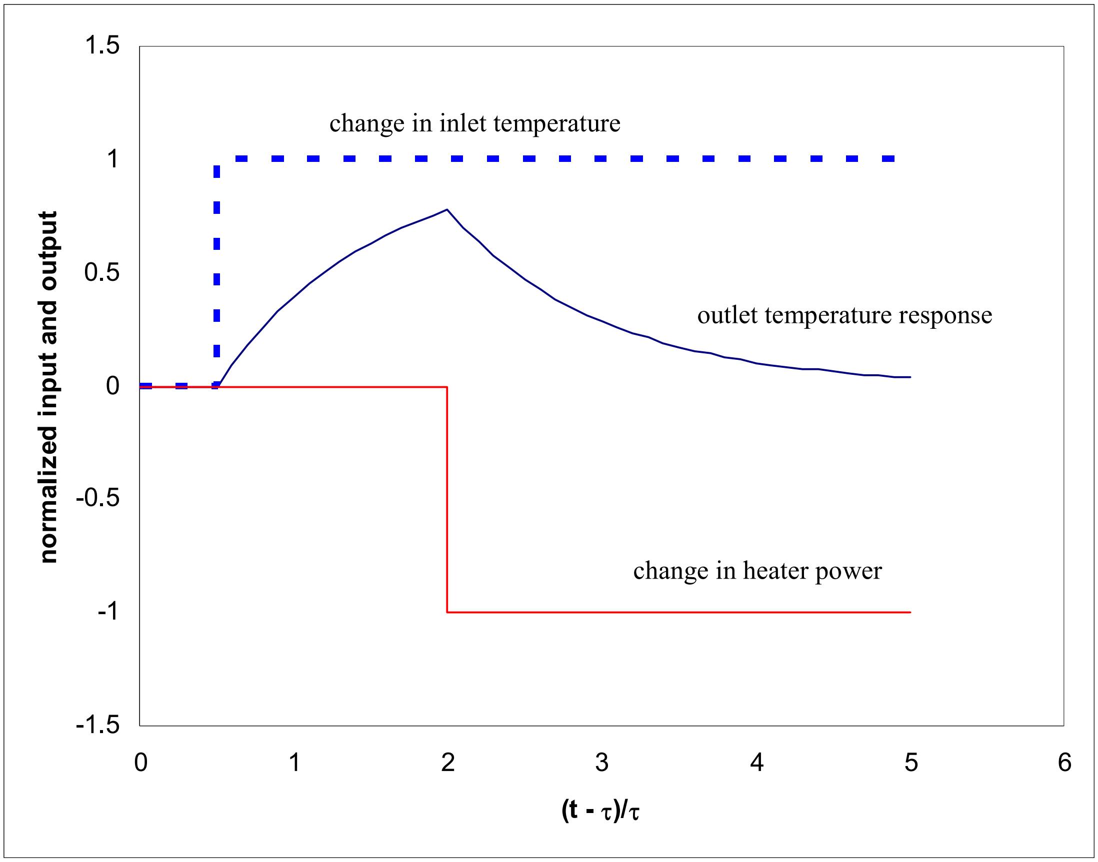


Figure 4-3 The disturbances to linear equations produce additive responses.

4.5 Multiple Outlet Streams

A tank may have multiple outlet streams, as well. In modeling first order systems, we often find that these streams depend on the response variable. In this case, the effect of additional outlet streams is to alter the time constant and gain of the system. For example, suppose that a mixed overflow tank is cooled by convective heat transfer to a condensing vapor. Thus there are two outlet streams: the enthalpy carried out with the outlet flow, and the heat transferred to the cooling coil. The energy balance is

$$\frac{d}{dt}\left(\rho C_p V\left(T_o - T_{ref}\right)\right) = \rho C_p F\left(T_i - T_{ref}\right) - \rho C_p F\left(T_o - T_{ref}\right) - UA\left(T_o - T_c\right) \tag{4.5.1}$$

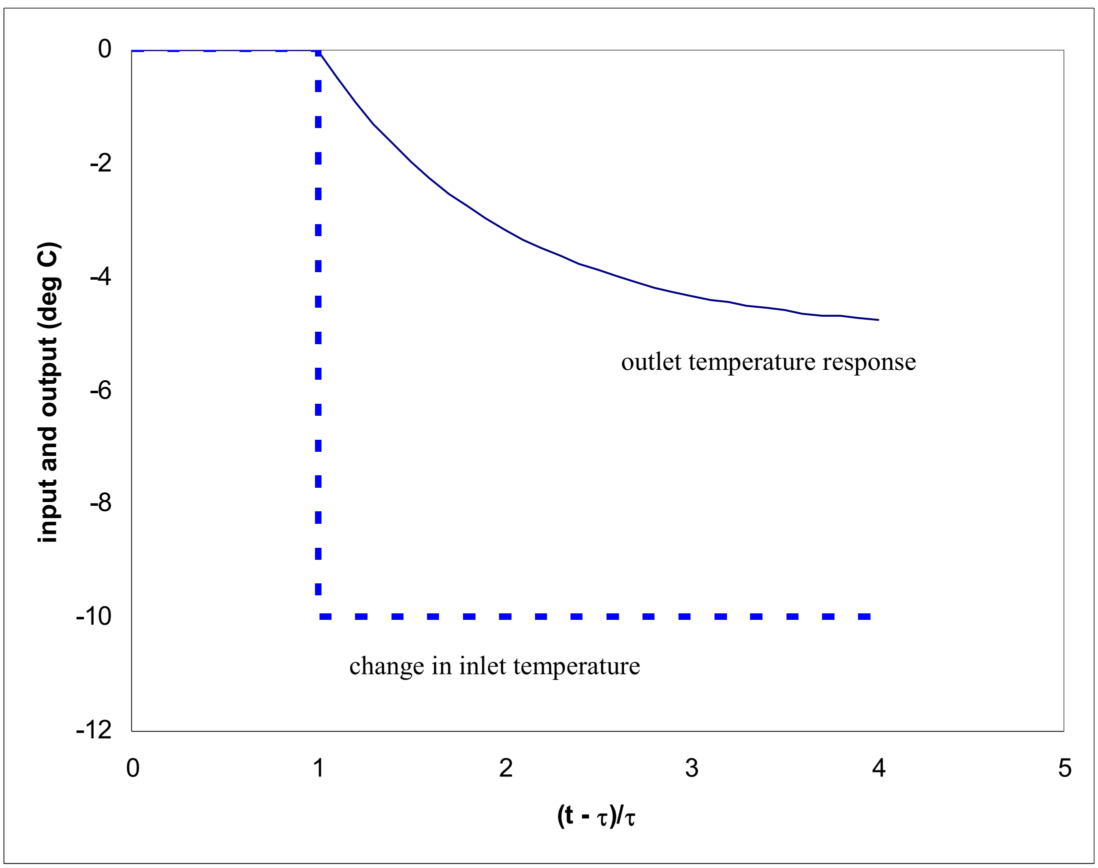
where the overall heat transfer coefficient is U and the coolant condenses at temperature T_c. Writing (4.5.1) at steady state and subtracting this result from (4.5.1) gives

$$\frac{d}{dt}\left(\rho C_p V T_o^*\right) = \rho C_p F T_i^* - \rho C_p F T_o^* - UAT_o^* + UAT_c^* \tag{4.5.2}$$

where we have two terms that depend on the response variable T_o^*.
Writing (4.5.2) in standard form gives

$$\frac{\rho C_p V}{\rho C_p F + UA}\frac{dT_o^*}{dt} + T_o^* = \frac{\rho C_p F}{\rho C_p F + UA}T_i^* + \frac{UA}{\rho C_p F + UA}T_c^* \qquad (4.5.3)$$

Because there are two paths out for enthalpy - flow and heat transfer - the time constant for temperature change is less than the tank residence time, in contrast to (4.4.4). Equation (4.5.3) also features two enthalpy inputs: inlet flow and the coolant. In fact, if the coolant temperature T_c exceeds T_i, (4.5.3) will describe heating of the tank. Notice that the gain for an inlet temperature disturbance is less than unity: because there are two paths out, the outlet response will not grow to equal a permanent inlet disturbance. Equation (4.5.3) is solved as before, and the response is shown in the figure for a gain of 0.5 (that is, UA = $\rho C_p F$) and a step decrease in T_i at time τ.



4.6 Summing up

First order systems arise from material and energy balances on perfectly mixed volumes. The system output or response variable is a measure of the storage in the system - for example, liquid level or concentration for mass and temperature for energy. Two parameters, the time constant and the gain, characterize the response of the output variable to disturbances.