10.450 Process Dynamics, Operations, and Control
Lecture Notes - 3

Lesson 3. Math review.

3.0 Context
In the previous chapters, we solved a differential equation for different forcing functions. Here we will review this and other mathematical topics that we will need.

3.1 Quadratic equation
The roots of

$$\alpha s^2 + \beta s + 1 = 0 \tag{3.1.1}$$

are

$$s = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha}}{2\alpha} \tag{3.1.2}$$

If α is less than zero, the roots s will be real, of opposite sign, and of unequal magnitude. For α greater than zero, the roots may be real or complex; the real parts will have the same sign. The term under the radical, aptly called the discriminant, determines whether the roots are complex.

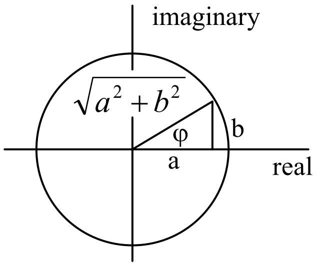
3.2 complex numbers
Consider the complex number

z = a + jb, where $j = \sqrt{-1}$, Re(z) = a, Im(z) = b (3.2.1)

The same number may be written in polar form

$z = |z|e^{j\varphi}$, where $|z| = \sqrt{a^2 + b^2}$ and $\varphi = \tan^{-1}\frac{b}{a}$ (3.2.2)

The diagram shows the complex plane, in which real numbers are confined to the horizontal axis. A complex number appears as a vector from the origin. The diagram relates the Cartesian and polar forms of the complex number. The phase angle φ is measured counterclockwise from the positive real axis.



If we let the magnitude |z| equal 1, we obtain Euler's formula relating the exponential and trigonometric functions.

$$e^{\pm j\varphi} = \cos\varphi \pm j\sin\varphi \tag{3.2.3}$$

Each complex number z has a *conjugate* z^*

$$z = a + jb = |z|e^{j\varphi} \tag{3.2.4}$$

$$z^* = a - jb = |z^*|e^{-j\varphi} \tag{3.2.5}$$

The conjugates have the same magnitude.

$$zz^* = |z|^2 = |z^*|^2 = a^2 + b^2 \tag{3.2.6}$$

Use the conjugate to eliminate j in the denominator of a ratio.

$$\frac{c}{a+jb}\frac{a-jb}{a-jb} = \frac{ca}{a^2+b^2} + j\frac{-cb}{a^2+b^2} \tag{3.2.7}$$

3.3 Oscillatory behavior

Suppose a variable y (a liquid level, a temperature, a chemical concentration, a flow rate) oscillates regularly in time:

$$y(t) = A\sin\left(2\pi\frac{t}{p} + \varphi\right) = A\sin(2\pi ft + \varphi) = A\sin(\omega t + \varphi) \tag{3.3.1}$$

where

A is the amplitude (dimension of y)
p is the period (dimension of time)
f is the cyclical frequency (dimension of time^{-1})
ω is the radian frequency (dimension of radians time^{-1})
φ is the phase angle (dimension of radians)

Inserting 2π into the argument is necessary because we are attempting to describe physical behavior (something varying in time) by an abstract math function that doesn't care what our time units are. View ω = 2π/p = 2πf as the conversion factor between time units and radians.

The phase angle φ represents an advance in the signal y(t) with respect to some other signal. That is, if

$$x(t) = B\sin(\omega t) \quad \text{and} \quad y(t) = C\sin(\omega t + \varphi) \qquad (3.3.2)$$

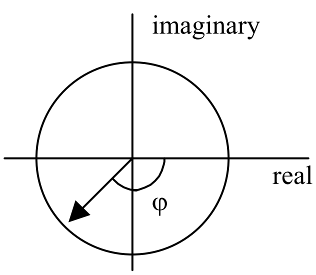
the oscillation y(t) is ahead of that of x(t) at any time t. However, we will most often encounter *phase lags*, so that the phase angle φ will have a negative value. If φ = 0, x(t) and y(t) are said to be *in phase*.

Representing an oscillation with a phase angle is quite useful, but on occasion it is helpful express the same signal in another fashion. This phase angle identity, derived from the trigonometric sum-of-angles formula, shows that the signal can be expressed as a combination of sine and cosine functions:

$$\sqrt{S^2 + C^2}\,\sin\left(\omega t + \tan^{-1}\left(\frac{C}{S}\right)\right) = S\sin\omega t + C\cos\omega t \qquad (3.3.3)$$

3.4 Arctangent

The arctangent is a multi-valued function, and thus must be treated carefully in calculations. For example, suppose we wish to express the complex number -1-j in polar form. From the figure we see that the phase angle φ should be designated as either 225° (3.927 radians) or -135° (-2.356 radians).

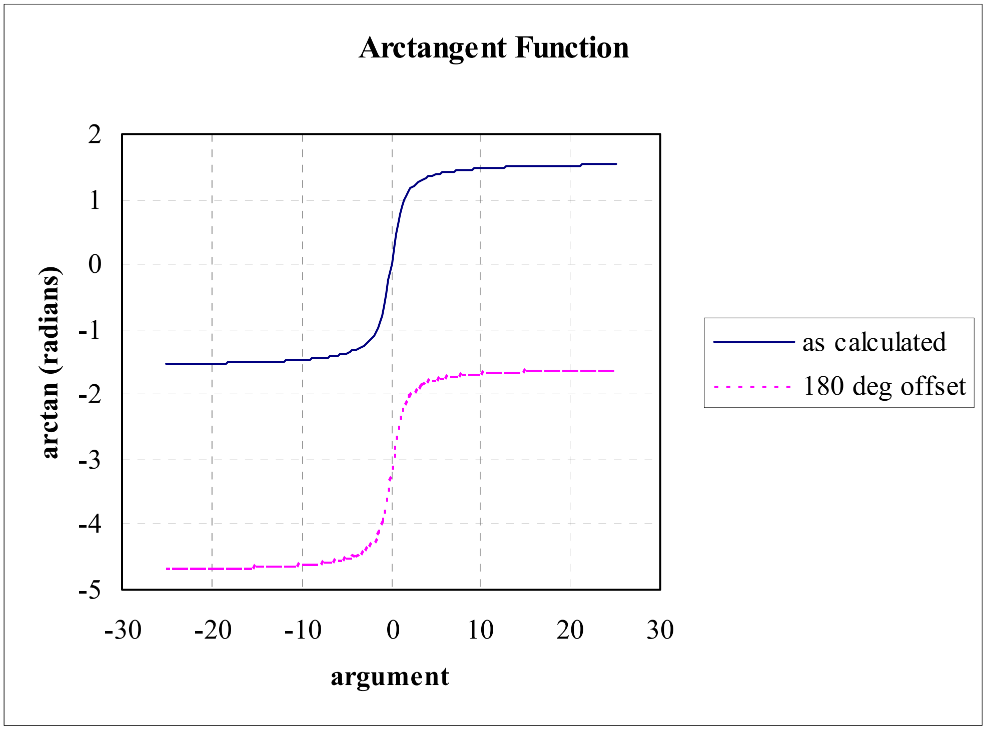


However, most calculators and spreadsheets will process (3.2.2) to give 45° (0.785 radians).

$$\varphi_{calculator} = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = 45^{\circ} \qquad (3.4.1)$$

Calculators and spreadsheets tend to work between -90° and +90°. Because we will be considering phase lags, we will instead tend to apply the arctangent between -180° and 0°. Hence angles in the upper right quadrant should be corrected.

$$\varphi = \varphi_{calculator} - 180^\circ, \qquad 0 \le \varphi_{calculator} \le 90^\circ \tag{3.4.2}$$



By the way, the conversion factor between degrees and radians is

$$1 = \frac{180}{\pi}\frac{^\circ}{\text{radians}} \tag{3.4.3}$$

3.5 First-order, linear, variable-coefficient ODE

We are addressing systems that vary in time, so our independent variable is always t.

$$a(t)\frac{dy}{dt} + y(t) = Kx(t) \qquad y(t_0) = known \tag{3.5.1}$$

In writing (3.5.1) we have arranged the coefficient functions to isolate the dependent variable y(t). By this means, a(t) must have dimensions of time t, and K has dimensions of y/x. We solve this equation by defining the integrating factor p(t)

$$p(t) = \exp\int \frac{dt}{a(t)} \tag{3.5.2}$$

and integrating to find

$$y(t) = \frac{K}{p(t)} \int_{t_0}^{t} \frac{p(t)x(t)}{a(t)} dt + \frac{p(t_0)y(t_0)}{p(t)} \qquad (3.5.3)$$

The solution y(t) comprises contributions from the forcing function Kx(t) and the initial condition $y(t_0)$. These are known as the particular (depends on the right-hand side) and homogeneous (as if the right-hand side were zero) solutions. In the language of dynamic systems, we can think of y(t) as the *response* of the system to input disturbances Kx(t) and $y(t_0)$.

3.6 First-order ODE, special case for process control applications

If a(t) is constant in (3.5.1), we call it the *time constant* τ. In this context, K is called the *gain*. By its magnitude and sign, the gain influences how severely y responds to x. The solution (3.5.3) becomes

$$y(t) = \frac{K}{\tau} e^{-t/\tau} \int_{t_0}^{t} e^{t/\tau} x(t) dt + y(t_0) e^{-(t-t_0)/\tau} \qquad (3.6.1)$$

or for initial time at zero,

$$y(t) = \frac{K}{\tau} e^{-t/\tau} \int_{0}^{t} e^{t/\tau} x(t) dt + y(0) e^{-t/\tau} \qquad (3.6.2)$$

3.7 First-order ODE, special case for no disturbance (forcing function)

If K = 0, the system response depends only on the initial conditions. This can be obtained from the general solution (3.5.3), of course, but can also be obtained by directly integrating equation (3.5.1), which has become *separable*.

$$\begin{aligned} \frac{dy}{y} &= \frac{-dt}{a(t)} \\ \ln \frac{y}{y(0)} &= -\int_0^t \frac{dt}{a(t)} \end{aligned} \qquad (3.7.1)$$

3.8 First-order ODE, special case for missing dependent variable

If the y(t) term is removed (for example, if K and a(t) are both very large), another separable equation results.

$$dy = \frac{Kx(t)}{a(t)}dt$$
$$y = y(0) + K\int_0^t \frac{x(t)}{a(t)}dt \qquad (3.8.1)$$

Separable equations are convenient to solve by straightforward function integration. The surge tank of Chapters 1 and 2 was described by a separable equation of form (3.8.1).

3.9 First-order ODE, delayed disturbance

Let the forcing function be delayed; suppose x(t) is a unit step at time $t_d > 0$. Then from (3.5.3)

$$y(t) = \frac{K}{p(t)}\left[\int_0^{t_d} \frac{p(t)(0)}{a(t)}dt + \int_{t_d}^{t} \frac{p(t)(1)}{a(t)}dt\right] + \frac{p(0)y(0)}{p(t)}$$
$$y(t) = \frac{K}{p(t)}\int_{t_d}^{t} \frac{p(t)}{a(t)}dt + \frac{p(0)y(0)}{p(t)} \qquad (3.9.1)$$

3.10 Second-order, linear, constant-coefficient ODE

$$\alpha\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + y(t) = Kx(t) \qquad y(0), \left.\frac{dy}{dt}\right|_0 = known \qquad (3.10.1)$$

The coefficient a_2 has the dimension of time squared. The solution to (3.10.1) is the sum of two terms:

$$y(t) = y_h(t) + y_p(t) \qquad (3.10.2)$$

The homogeneous solution y_h, which depends only on the left-hand-side of (3.10.1), is itself the sum of two linearly independent exponential functions

$$y_h(t) = C_1e^{r_1t} + C_2e^{r_2t} \qquad (3.10.3)$$

where r_1 and r_2 are the roots of the characteristic equation of (3.10.1).

$$r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha}}{2\alpha} \qquad (3.10.4)$$

The value of the discriminant determines three distinct forms of the solution.

<u>real, unequal roots for r</u>

$$y_h(t) = e^{-\beta t/2\alpha}\left(C_1 e^{\sqrt{\beta^2 - 4\alpha}\, t/2\alpha} + C_2 e^{-\sqrt{\beta^2 - 4\alpha}\, t/2\alpha} \right) \qquad (3.10.5)$$

In process control, we prefer stable systems, those in which disturbances do not grow with time. We observe that (3.10.5) decays if α and β have the same sign.

<u>real, equal roots for r</u>

$$y_h(t) = e^{-\beta t/2\alpha}\left(C_1 + C_2 t \right) \qquad (3.10.6)$$

The solution will decay if α and β have the same sign.

<u>complex roots for r</u>

$$y_h(t) = e^{-\beta t/2\alpha}\left(C_1 \cos \frac{\sqrt{4\alpha - \beta^2}}{2\alpha} t + C_2 \sin \frac{\sqrt{4\alpha - \beta^2}}{2\alpha} t \right) \qquad (3.10.7)$$

Once again, the solution will decay if α and β have the same sign. The coefficient of t in the trigonometric functions is the radian frequency of the oscillation.

The particular solution y_p for any disturbance x(t) may be determined by the 'method of undetermined coefficients', or the 'method of variation of parameters'. The initial conditions are then applied to the solution y(t) to determine coefficients C_1 and C_2.

The response of the system (3.10.1) then depends on

- the character of the system itself (through the left-hand-side coefficients, affecting the exponential and trigonometric terms in the homogeneous solution)
- the initial conditions (affecting coefficients C_1 and C_2 in the homogeneous solution)
- the nature of the disturbance (through the particular solution, as well as C_1 and C_2, if the disturbance is initially non-zero)

In a later lesson, we will introduce Laplace transforms as an alternative method for determining the solution.

3.11 Representing functions by Taylor series

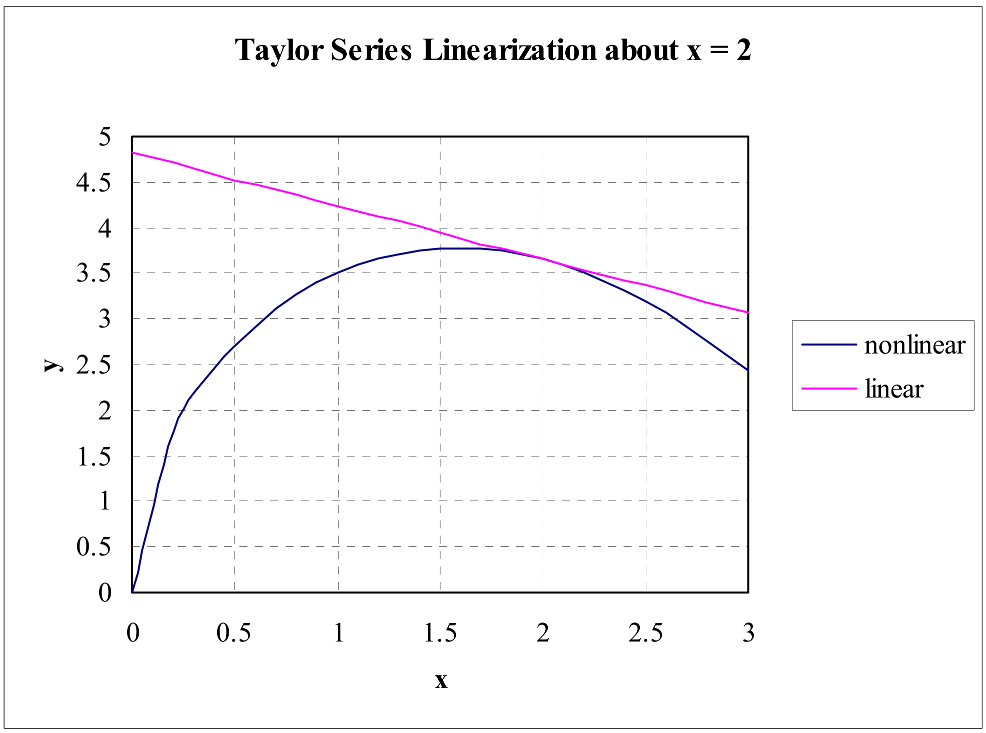
We specify some reference value of the independent variable, and represent the function in the neighborhood of that reference as a series of terms. For a function of one variable:

$$f(x) = f(x_s) + \left.\frac{df}{dx}\right|_{x_s} (x - x_s) + O\left((x - x_s)^2\right) \qquad (3.11.1)$$

For a function of more than one variable:

$$f(x, y, \ldots) = f(x_s, y_s, \ldots) + \left.\frac{\partial f}{\partial x}\right|_{x_s, y_s, \ldots} (x - x_s) + \left.\frac{\partial f}{\partial y}\right|_{x_s, y_s, \ldots} (y - y_s) + \ldots + O\left((x - x_s)^2, (y - y_s)^2, \ldots\right) \qquad (3.11.2)$$

By retaining only linear terms, we obtain a linear approximation. The derivatives are evaluated at the reference point. Of course, the approximation is exact at the reference, and it is often satisfactory in some region about the reference value. As the figure indicates, however, extrapolation to x = 0 would be erroneous.



3.12 Chain rule for differentiation

$$\frac{d}{ds} g(f(s)) = \frac{dg}{df} \frac{df}{ds} \qquad (3.12.1)$$

The functions g and f are said to be nested. For example, let g be the exponential and f the square root.

$$\frac{d}{ds}e^{\sqrt{s}} = \left(e^{\sqrt{s}}\right)\left(\frac{1}{2}s^{-1/2}\right) \qquad (3.12.2)$$

The chain rule applies to the special cases of a product

$$\frac{d}{ds}f(s)g(s) = f(s)\frac{dg}{ds} + \frac{df}{ds}g(s) \qquad (3.12.3)$$

or a quotient

$$\frac{d}{ds}\frac{N(s)}{D(s)} = \frac{D(s)\dfrac{dN}{ds} - N(s)\dfrac{dD}{ds}}{D(s)^2} \qquad (3.12.4)$$

3.13 Must we?

The math topics collected here will be used during the course. Please review any that seem unfamiliar.