10.450 Process Dynamics, Operations, and Control
Lecture Notes - 11

Lesson 11. Frequency response of dynamic systems.

11.0 Context

We have worked with step, pulse, and sine disturbances. Of course, there are many sine disturbances, because the applied frequency may vary. Surely the system response should differ for low- and high-frequency inputs. In this lesson we will explore the frequency response of dynamic systems.

11.1 Response to sine input

Recall a first order lag, such as a stirred tank reactor, mixer, or heater. The transfer function between inlet and outlet is

$$G(s) = \frac{C_o^*(s)}{C_i^*(s)} = \frac{K}{\tau s + 1} \tag{11.1.1}$$

Let's disturb the inlet in a way that remains bounded, as with a step disturbance, but prevents approach to a steady state:

$$C_i^*(t) = A\sin(\omega t) \tag{11.1.2}$$

If we take the Laplace transform of this inlet disturbance, the transfer function gives us

$$\begin{aligned} C_o^*(s) &= G(s)C_i^*(s) \\ &= \frac{K}{\tau s + 1}\frac{A\omega}{s^2 + \omega^2} \end{aligned} \tag{11.1.3}$$

From a transform-pair table, we find

$$\begin{aligned} C_o^*(t) &= \frac{KA\omega\tau}{1+\omega^2\tau^2} e^{-t/\tau} + \frac{K}{\sqrt{1+\omega^2\tau^2}} A\sin(\omega t + \phi) \\ \phi &= \tan^{-1}(-\omega\tau) \end{aligned} \tag{11.1.4}$$

The first term dies out – it represents the system adjusting to the onset of the disturbance. The second term is an oscillation – the system's continuing response to the prevailing input. It looks like the disturbance (11.1.2), except that the inlet amplitude is modified and the response lags the input by a phase angle ϕ (ϕ will be negative). Both amplitude and phase angle depend on properties of the system, parameters in the transfer function (11.1.1).

11.2 Frequency response

We began by talking about a sine wave disturbance. We have derived (11.1.4), which shows that the long-term response is to multiply the amplitude by a factor and delay the signal. Hence, we define the long-term response in terms of an amplitude ratio (AR) and a phase angle ϕ. We call it a *frequency response*, because both AR and ϕ depend on the disturbance frequency ω: the system will react differently according to how it is shaken.

$$\text{AR} = \frac{\left|C_o^*(t)\right|}{\left|C_i^*(t)\right|} = \frac{K}{\sqrt{1+\omega^2\tau^2}} \qquad (11.2.1)$$

$$\phi = \tan^{-1}(-\omega\tau)$$

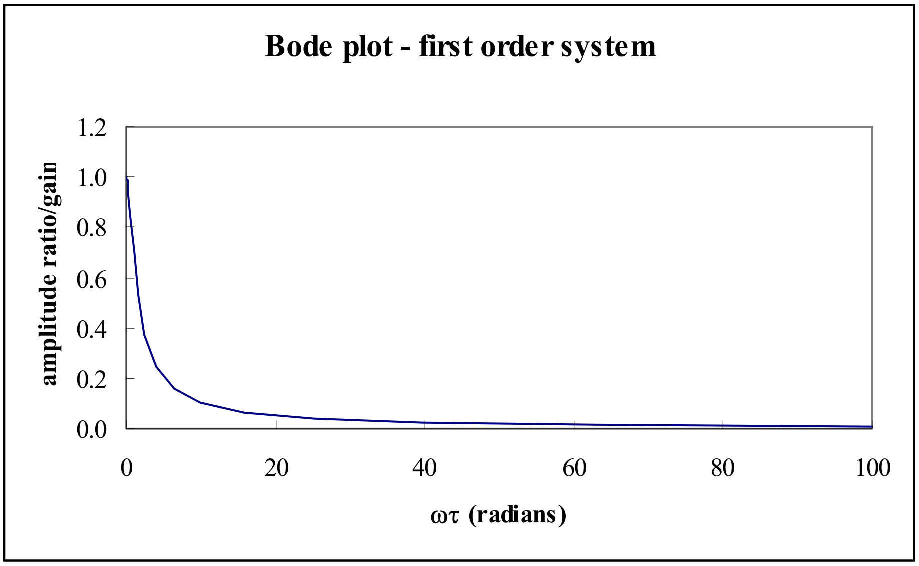
Consider the mixing tank. Make the inlet concentration vary slowly, so slowly that the tank has plenty of time to mix each incremental change in concentration and deliver it to the outlet. Then the outlet will follow the inlet concentration closely. For small ω, (11.2.1) shows that the amplitude ratio is the same as the tank gain K, and the phase angle is zero. The trace of the outlet signal lies over that of the inlet.

Now make the variations faster, so fast that the tank concentration scarcely begins to rise before the inlet concentration falls again. Such rapid fluctuations at the inlet will never propagate to the outlet, but be lost in the 'capacitance' of the tank. For sufficiently large ω, (11.2.1) shows that the amplitude ratio goes to zero, and the phase angle to -90°.

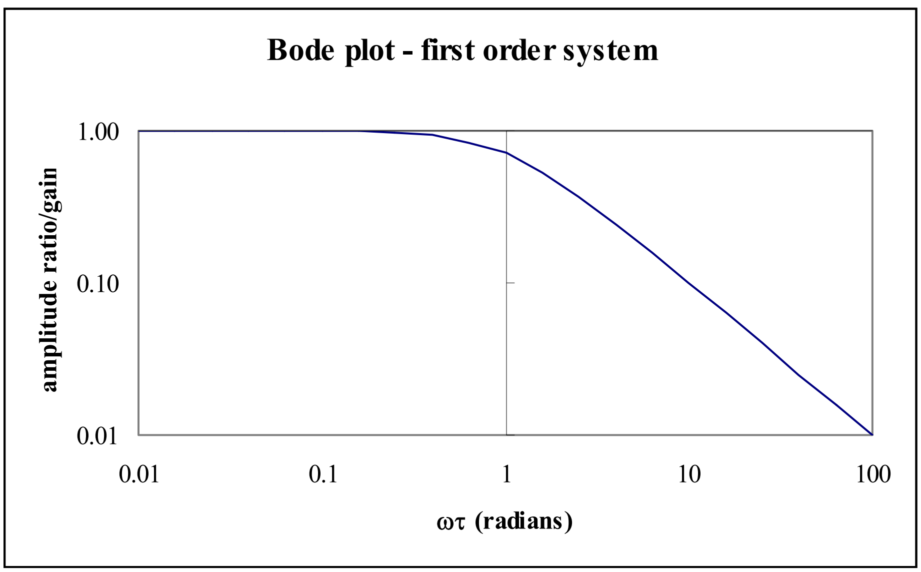
If you are responsible for a process that is disturbed by fluctuations in the input stream, you might specify a surge tank to damp out those fluctuations. Understanding frequency response will assist you in sizing the tank (thereby specifying its time constant τ) to do an effective job on the disturbance frequencies of interest.

11.3 The Bode plot

Frequency response information is commonly presented on a Bode (bo'-dee) plot.



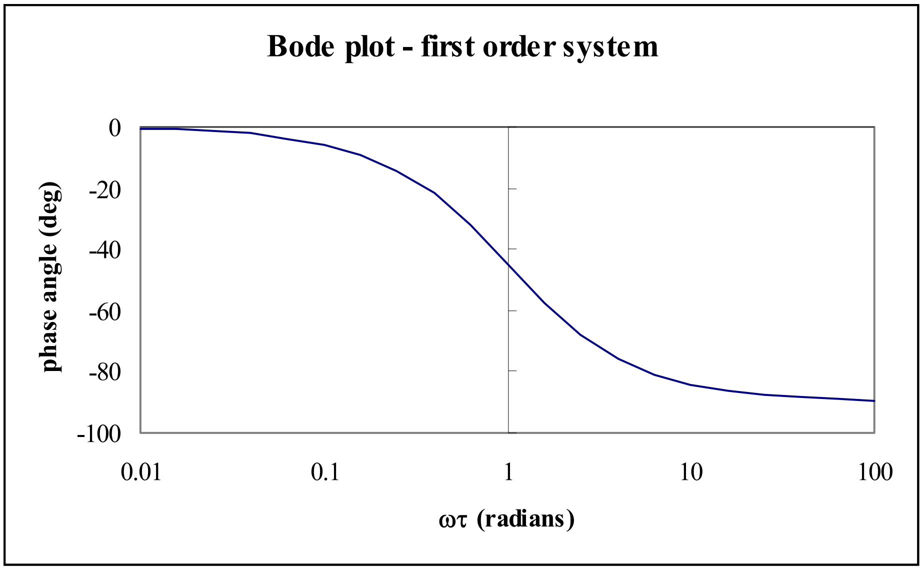
Because frequency and amplitude may meaningfully vary over several orders of magnitude, Bode plots use logarithmic axes.



AR begins at 1.0 at low frequency and drops off at high frequency with a slope of -1. The high frequency region can be extrapolated to AR=1 at the 'corner frequency' ω_c.

$$\omega_c = \frac{1}{\tau} \qquad (11.3.1)$$

The phase angle varies asymptotically between 0 and -90°.



The Bode plot was once used for calculations in control system design. Now computers have replaced that function, but it is still useful as an illustrative tool.

11.4 Calculate frequency response for a system

In 11.1 we obtained the frequency response by calculating the solution to the system equation when disturbed by a sine input. That gave us, in addition, the short-term transient response.

We can get frequency response directly from the Laplace domain transfer function if we're willing to replace the transform-pair table with a bit of complex variable algebra. First an example, and then the rule…

The first order transfer function is

$$G(s) = \frac{C_o^*(s)}{C_i^*(s)} = \frac{K}{\tau s + 1} \qquad (11.4.1)$$

Substitute $j\omega$ for s in (11.4.1).

$$G(j\omega) = \frac{K}{j\tau\omega + 1} \qquad (11.4.2)$$

Multiply by the complex conjugate to remove j from the denominator.

$$
\begin{aligned}
G(j\omega) &= \frac{K}{1+j\tau\omega}\frac{1-j\tau\omega}{1-j\tau\omega} \\
&= K\frac{1-j\tau\omega}{1+\tau^2\omega^2}
\end{aligned}
\tag{11.4.3}
$$

This is a complex number whose real and imaginary parts are

$$
\begin{aligned}
\mathrm{Re}[G(j\omega)] &= \frac{K}{1+\tau^2\omega^2} \\
\mathrm{Im}[G(j\omega)] &= \frac{-K\tau\omega}{1+\tau^2\omega^2}
\end{aligned}
\tag{11.4.4}
$$

The complex number can be written in the alternate polar form. The amplitude is

$$
\begin{aligned}
|G(j\omega)| &= \sqrt{\left(\frac{K}{1+\tau^2\omega^2}\right)^2 + \left(\frac{-K\tau\omega}{1+\tau^2\omega^2}\right)^2} \\
&= \frac{K\sqrt{1+\tau^2\omega^2}}{1+\tau^2\omega^2} \\
&= \frac{K}{\sqrt{1+\tau^2\omega^2}}
\end{aligned}
\tag{11.4.5}
$$

and the angle is

$$
\begin{aligned}
\phi = \angle G(j\omega) &= \tan^{-1}\left(\frac{\dfrac{-K\tau\omega}{1+\tau^2\omega^2}}{\dfrac{K}{1+\tau^2\omega^2}}\right) \\
&= \tan^{-1}(-\tau\omega)
\end{aligned}
\tag{11.4.6}
$$

You probably saw this coming. When $j\omega$ is substituted for s in the transfer function Eqn (1), the amplitude and angle of the resulting complex number are the amplitude ratio and phase angle of the frequency response of the transfer function. We demonstrated for first order, but it's true in general. To summarize, for $x(t) = A\sin\omega t$ and $G(s) = \dfrac{y(s)}{x(s)}$

$$y_{f.r.}(t) = A\left|G(j\omega)\right|\sin\left(\omega t + \phi\right)$$

$$\text{AR} = \frac{\left|y_{f.r.}(t)\right|}{A} = \left|G(j\omega)\right| = \sqrt{\text{Re}\left[G(j\omega)\right]^2 + \text{Im}\left[G(j\omega)\right]^2} \qquad (11.4.7)$$

$$\phi = \angle G(j\omega) = \tan^{-1}\left(\frac{\text{Im}\left[G(j\omega)\right]}{\text{Re}\left[G(j\omega)\right]}\right)$$

Notice that the mean value of both x and y is zero. This means that we have implicitly assumed deviation variables in defining the frequency response.

11.5 Frequency response by combining components

We can further simplify our task of determining frequency response if the transfer function is assembled as a product of known components.

$$G(s) = G_1(s)G_2(s)... \qquad (11.5.1)$$

After substituting jω for s, the complex numbers can be expressed in polar form

$$\begin{aligned} G(j\omega) &= G_1(j\omega)G_2(j\omega)... \\ &= \left|G_1(j\omega)\right|e^{j\phi_1}\left|G_1(j\omega)\right|e^{j\phi_2}... \\ \text{AR}e^{j\Phi} &= \left|G_1(j\omega)\right|\left|G_1(j\omega)\right|...\exp\left(j\sum\phi_i\right) \end{aligned} \qquad (11.5.2)$$

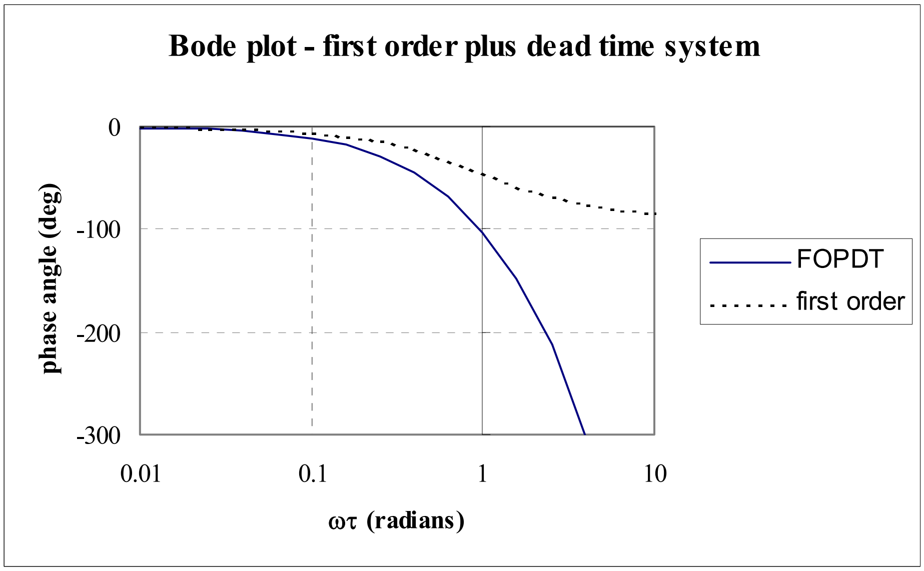
so that

$$\text{AR} = \prod_i \left|G_i(j\omega)\right| \qquad \Phi = \sum_i \phi_i \qquad (11.5.3)$$

For example, the important case of first-order-plus-dead-time (FOPDT):

$$\begin{aligned} G(s) &= \frac{Ke^{-\theta s}}{\tau s + 1} = K \cdot e^{-\theta s} \cdot \frac{1}{\tau s + 1} \\ AR &= \left|G(j\omega)\right| = K \cdot 1 \cdot \frac{1}{\sqrt{\tau^2\omega^2 + 1}} \\ &= \frac{K}{\sqrt{\tau^2\omega^2 + 1}} \\ \phi &= \angle G(j\omega) = 0 + (-\theta\omega) + \tan^{-1}(-\tau\omega) \end{aligned} \qquad (11.5.4)$$

The amplitude ratio is unchanged from that of the first order system. However the phase delay is greatly increased at high frequencies. The plot is for the case in which $\theta = \tau$.



Graphical combination of transfer functions is convenient on Bode plots. However, this technique is not so commonly used when computers are available.