10.450 Process Dynamics, Operations, and Control
Lecture Notes - 7

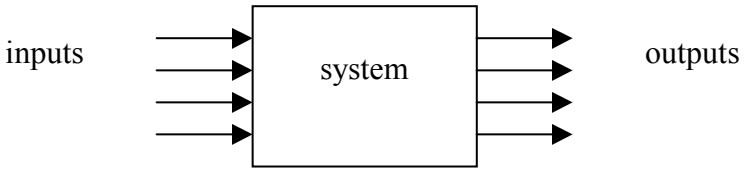
Lesson 7. Transfer functions and block diagrams

7.0 Context

In Lesson 5, we introduced Laplace transforms as a method of solving the linearized equations of system dynamics. However, a more important reason is that so many of the concepts of dynamics and control theory are expressed in the LT language, even if the methods are not required for solution. In this lesson we present the transfer function, a concise description of a dynamic system that is based on Laplace transforms. We also present block diagrams, a convenient way to represent the structure of dynamic systems.

7.1 Dynamics of systems

Process control deals with systems that change in time. In Lesson 2, we asserted that systems are characterized by input disturbances (causes), and output responses (effects).



We have claimed that a variety of physical systems can be satisfactorily described by relatively few mathematical models. We have dwelt on the first-order lag as a prime example:

$$\tau \frac{dy^*}{dt} + y^* = Kx^*(t) \qquad y^*(0) = 0 \tag{7.1.1}$$

The system model is the ordinary differential equation (7.1.1), relating input x^* and output y^* as they vary in time. In (7.1.1), $x^*(t)$ is the mathematical forcing function, and $y^*(t)$ the dependent variable. After taking Laplace transforms, we can relate input and output by an algebraic equation:

$$y^*(s) = \frac{K}{\tau s + 1} x^*(s) \tag{7.1.2}$$

The ratio in (7.1.2) contains all the information about the ODE (7.1.1). When multiplying $x^*(s)$, the transform of disturbance $x^*(t)$, the ratio converts it into $y^*(s)$, the transform of the response $y^*(t)$. We call this ratio the *transfer function*.

7.2 Transfer functions

Let's take a larger view of getting transfer functions from differential equations: we have a lot more to learn about dynamic systems, but it's not too early to speculate a bit from what we already know. Perhaps systems more complicated than our first-order lag may be described by higher-order equations. If we linearize such equations, and express them in deviation variables, they must look like

$$a_n \frac{d^n y^*}{dt^n} + a_{n-1} \frac{d^{n-1} y^*}{dt^{n-1}} + \ldots + a_1 \frac{dy^*}{dt} + y^* = f(x^*) \qquad (7.2.1)$$

It's not outlandish to speculate that a complicated dynamic system might depend not only on the disturbance x^*, but its rate of change, as well. For that matter, it may depend on higher derivatives of x^*, leading us to write (7.2.1) as

$$a_n \frac{d^n y^*}{dt^n} + a_{n-1} \frac{d^{n-1} y^*}{dt^{n-1}} + \ldots + a_1 \frac{dy^*}{dt} + y^* = b_l \frac{d^l x^*}{dt^l} + \ldots + b_1 \frac{dx^*}{dt} + b_0 x^*(t) \qquad (7.2.2)$$

We have already encountered systems with multiple disturbances in Section 4.4. Hence we may expand our speculative model further.

$$\begin{aligned} a_n \frac{d^n y^*}{dt^n} + a_{n-1} \frac{d^{n-1} y^*}{dt^{n-1}} + \ldots + a_1 \frac{dy^*}{dt} + y^* = {} & b_l \frac{d^l x_1^*}{dt^l} + \ldots + b_1 \frac{dx_1^*}{dt} + b_0 x_1^*(t) \\ & + c_m \frac{d^m x_2^*}{dt^m} + \ldots + c_1 \frac{dx_2^*}{dt} + c_0 x_2^*(t) \\ & + \ldots \end{aligned} \qquad (7.2.3)$$

As is usual in process control, we presume all initial conditions are zero, which describes a system expressed in deviation variables initially at steady state. Taking Laplace transforms of (7.2.3) leads to

$$\begin{aligned} y^*(s) = {} & \frac{\left(b_l s^l + b_{l-1} s^{l-1} + \ldots + b_1 s + b_0\right)}{\left(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + 1\right)} x_1^*(s) \\ & + \frac{\left(c_m s^m + c_{m-1} s^{m-1} + \ldots + c_1 s + c_0\right)}{\left(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + 1\right)} x_2^*(s) \\ & + \ldots \end{aligned} \qquad (7.2.4)$$

If we set all the (7.2.4) coefficients except b_0 and a_1 to zero, we recover the particular example of (7.1.2). The ratios of polynomials in (7.2.4), like

the ratio in (7.1.2), are transfer functions. We will represent a general transfer function by G(s). Thus (7.2.4) becomes

$$y^*(s) = G_1(s)x_1^*(s) + G_2(s)x_2^*(s) + \ldots \qquad (7.2.5)$$

$G_1(s)$ is the transfer function that relates $y^*(s)$ to $x_1^*(s)$. $G_2(s)$ similarly relates $y^*(s)$ to $x_2^*(s)$. Notice that

- The Laplace transforms of the disturbances, when substituted for the $x_i(s)$ variables, will not change the polynomial nature of the $G_i(s)$ terms in (7.2.4). Thus polynomial ratios in the Laplace variable s will always result from the linear, constant-coefficient ordinary differential equations of process control.
- It is the nature of the linear ODE that the effects of the inputs are additive. Each disturbance $x_i^*(s)$, when processed through its particular transfer function, contributes to the overall response of $y^*(s)$.
- As we learned from the partial fraction expansion of Lesson 6, the time-domain response will finally be a sum of exponential and trigonometric terms. The various time constants, frequencies, and phase lags in these terms are determined by the coefficients in the transfer functions, and thus the original differential equation.

The dynamic response calculated from Equation (7.2.5) may be complicated indeed, but the essential concept - a dynamic system approximated by a linear equation and expressed in terms of transfer functions - is no different from what we have already studied in the simpler first order system of (7.1.2).

7.3 Using the transfer function

A transfer function represents a differential equation. Just as we classify differential equations into recognizable types, we will classify transfer functions, learn their characteristics, and use them as a concise representation of particular behaviors. For example, the transfer function in (7.1.2) represents a first-order lag; it contains the same information as the ODE of (7.1.1).

Given the transfer function for a system, therefore, we can predict some features of its behavior without actually calculating its response to particular disturbances. Consider these terms:

order – the highest power of s in the denominator. Equivalent to the order of the differential equation describing the system.
The first-order lag is described by a first-order differential equation; its transfer function has a single s in the denominator.

pole – root of the denominator. In later lessons, we will learn that poles with negative real parts result in output signals that decay in time,

so that the system will be stable. If there exist poles with imaginary parts, the system may oscillate, even without oscillatory disturbances.

The first order system has a pole at $-\tau^{-1}$; this negative, real value indicates a stable response with no oscillation.

zero – root of the numerator. These generally have no influence on stability, but can influence the rate and character of the dynamic response.

The first-order system has no zeroes.

steady state gain – the ratio of long-term output change to input step change. The gain is a measure of how sensitive the system is to disturbances. If the system is a chemical process, we would like a low value of gain, so that disturbances would have little effect on the output variable. In a sound system, we would like a large gain, so that tiny input signals from the source (tape, vinyl, CD) are amplified to audibility. The gain is found by setting s = 0 in the transfer function.

Here we summarize the first-order lag and integrator with respect to these properties

type	equation	transfer function	poles	steady state gain
lag	$\tau\frac{dy}{dt}+y(t)=Kx(t)$	$\frac{K}{\tau s+1}$	$-\tau^{-1}$	K
integrator	$\tau\frac{dy}{dt}=Kx(t)$	$\frac{K}{\tau s}$	0	none; increases without bound

7.4 Transfer function for the stirred reactor

Let's combine our knowledge of modeling first-order systems from Lesson 4, Laplace transforms from Lessons 5 and 6, and notions of the transfer function. In Section 4.1 we modeled a stirred overflow tank containing a dissolved substance A. Let's now assume that A disappears by first order chemical reaction.

$$r_A \equiv \frac{1}{V}\frac{dN_A}{dt} = -kC_A \qquad (7.4.1)$$

where the negative sign shows that A is consumed in the reaction. The component material balance is written assuming that the volumetric flow rate F is constant.

$$V\frac{dC_{Ao}}{dt} = F(C_{Ai}-C_{Ao}) - VkC_{Ao} \qquad C_{Ao}(0)=C_{As} \qquad (7.4.2)$$

There are no nonlinear terms; subtracting the steady state from (7.4.2) leaves deviation variables.

$$V\frac{dC_{Ao}^{\ *}}{dt} = F(C_{Ai}^{\ *} - C_{Ao}^{\ *}) - VkC_{Ao}^{\ *} \qquad C_{Ao}^{\ *}(0) = 0 \tag{7.4.3}$$

Equation (7.4.3) is then put into standard form:

$$\begin{aligned} \frac{V}{F+Vk}\frac{dC_{Ao}^{\ *}}{dt} + C_{Ao}^{\ *} &= \frac{F}{F+Vk}C_{Ai}^{\ *} \\ \tau\frac{dC_{Ao}^{\ *}}{dt} + C_{Ao}^{\ *} &= KC_{Ai}^{\ *} \qquad C_{Ao}^{\ *}(0) = 0 \end{aligned} \tag{7.4.4}$$

The time constant is smaller than that for the mixing tank in Equation (4.1.4) -- in a manner analogous to the multiple outlet streams in Section 4.5, the combination of outflow and chemical consumption in (7.4.4) reduces the time response of the outlet concentration. Similarly the gain is less than unity -- a disturbance in inlet concentration is only partly transmitted to the outlet stream

Taking Laplace transforms of (7.4.4) gives

$$C_{Ao}^{*}(s) = \frac{K}{\tau s+1}C_{Ai}^{*}(s) \tag{7.4.5}$$

Guided by (7.1.2), we identify the transfer function of the mixing tank.

$$G(s) = \frac{K}{\tau s+1} \qquad \left(= \frac{C_{Ao}^{*}(s)}{C_{Ai}^{*}(s)}\right) \tag{7.4.6}$$

As we remarked in Section 7.2, the transfer function depends only on the geometry and operating conditions of the tank itself, not on the disturbance. The particular nature of the inlet disturbance $C_{Ai}^{*}(s)$, when worked through the transfer function G(s), gives the particular nature of the output response $C_{Ao}^{*}(s)$.

Detail

In process control, we think of the gain K as a measure of how a permanent change in input C_{Ai}^{*} affects the output C_{Ao}^{*} in the long term. However, the *steady-state performance* of the reactor -- that is, how well it converts C_{Ais} to C_{Aos} -- is also indicated by the gain. From material balance (7.4.2), written at steady state, we find that

the gain depends on the reaction rate constant k and the reactor residence time τ_R.

$$\frac{C_{Aos}}{C_{Ais}} = K = \frac{F}{F+Vk} = \frac{1}{1+k\tau_R} \tag{7.4.7}$$

Low gain means good conversion of reactant A.

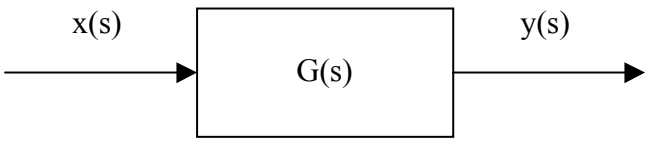
By placing the equations in standard form, we found that the time constant τ depends on the tank volume, the flow, and the reaction rate constant.

$$\tau = \frac{V}{F+Vk} = \frac{\tau_R}{1+k\tau_R} \tag{7.4.8}$$

These parameters are important individually, of course, but when we are concerned with the dynamic response, it is important to identify how they interact to affect the time constant. We see that the time constant is related to the reactor residence time τ_R, which we use in designing a stirred reactor to produce a desired outlet concentration of reactant. For understanding how that outlet concentration varies in time, however, the time constant is more significant. Notice that for no reaction, the time constant reduces to the residence time. As the reaction rate increases, the time constant decreases, indicating that the outlet concentration responds more quickly to disturbances

7.5 Block diagrams

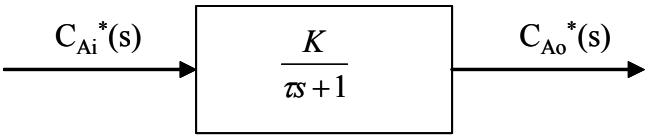
The *block diagram* is a graphical display of the process model in the Laplace domain. It comprises blocks and arrows, and thus resembles many other types of flow diagram. In our use with control systems, however, the arrows represent signals, variables that change in time, *which are not necessarily actual flow streams*. The block contains the transfer function, which may be as simple as a units conversion between x and y, or as complicated as a full chemical process. Remember that the transfer function incorporates all the dynamic information in the process model.



This diagram means

$$y(s) = G(s)x(s) \tag{7.5.1}$$

Returning to the mixed reactor in Section 7.4, we can represent the dynamic behavior of the reactor by a block diagram that is equivalent to Equation (7.4.5):

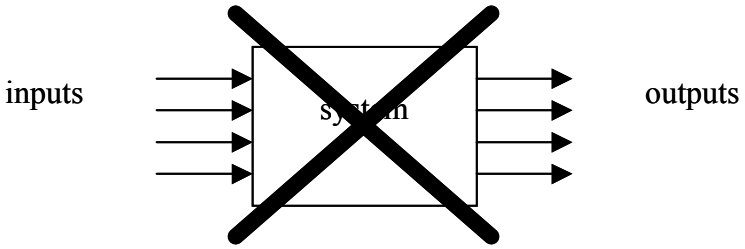


7.6 Block diagram structures

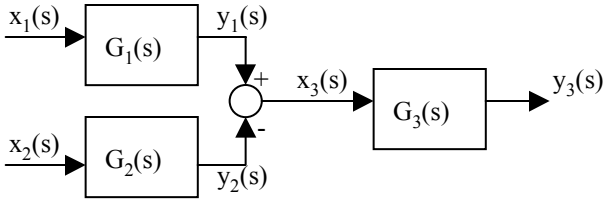
The real value of block diagrams is to represent the flow of signals among multiple blocks.

The Block Diagram Rules:

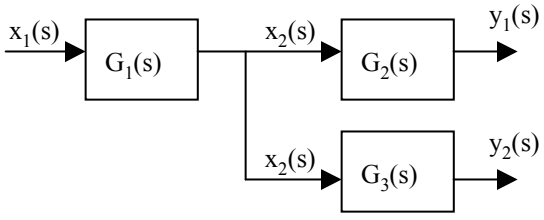
- only one input and output to a block. The figure in Section 7.1 was fine for its purpose, but does not qualify as a process control block diagram.



- two signals may be summed at an explicit summing junction. The algebraic sign is indicated at the junction.



- a single signal may feed its value to multiple blocks. This does NOT indicate that the signal is split among the blocks.



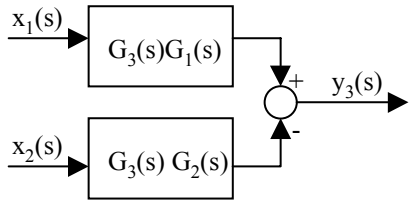
Block diagrams may be turned into equations by simple algebra. It is usually most convenient to start with an output and work backwards by substitution. In the summing diagram

$$\begin{aligned} y_3(s) &= G_3(s)x_3(s) \\ &= G_3(s)\big(y_1(s) - y_2(s)\big) \\ &= G_3(s)\big(G_1(s)x_1(s) - G_2(s)x_2(s)\big) \\ &= G_3(s)G_1(s)x_1(s) - G_3(s)G_2(s)x_2(s) \end{aligned} \qquad (7.6.1)$$

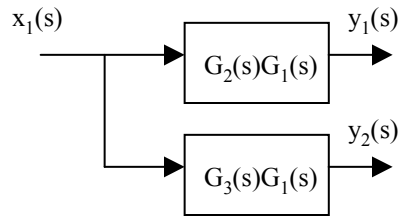
In the multiple assignment diagram

$$\begin{aligned} y_1(s) &= G_2(s)x_2(s) \\ &= G_2(s)G_1(s)x_1(s) \\ y_2(s) &= G_3(s)x_2(s) \\ &= G_3(s)G_1(s)x_1(s) \end{aligned} \qquad (7.6.2)$$

Similarly, equations may be turned into block diagrams. System (7.6.1) has two inputs and thus requires at least 2 blocks.



System (7.6.2) has two outputs for one input. Input x_1 is not split – its full value is sent to each of two blocks.



This pair of block diagrams is equivalent to the pair from which they were derived.

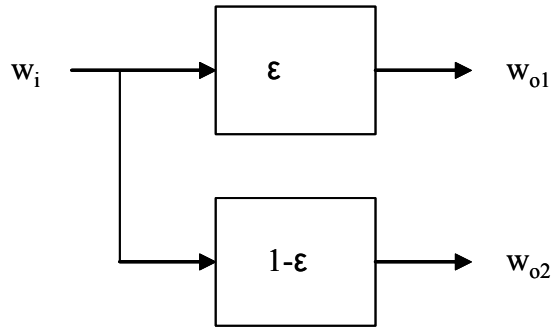
7.7 Splitting a signal

The block diagram rules contain a summing junction, but no explicit provision for splitting a signal. And yet, chemical processes frequently contain junctions at which a flowing stream is divided -- certainly the full value of the stream is not fed to each destination! To represent such splits on a block diagram, we feed the input signal into separate blocks. The transfer functions in these blocks express the manner in which the flow was split.

For example, suppose that an inlet flow always divides so that one branch receives fraction ε of the flow and a second receives the remainder. Then the material balance is

$$\begin{aligned} w_i &= w_{o1} + w_{o2} \\ &= \varepsilon w_i + (1-\varepsilon) w_i \end{aligned} \qquad (7.7.1)$$

and the block diagram is drawn



The gain of each transfer function is less than unity, showing that each output signal is diminished from the input value.

Detail

In fact, the transfer functions are nothing but gain, which implies that the response of the outlet to inlet disturbances is instantaneous. Of course, we assumed this in writing the material balance (7.7.1). For an incompressible flow, this is a good description.

The flow junction might behave so that the flow fraction ε depends on the magnitude of the inlet flow w_i. For a compressible flow, the dynamic response of the branches might differ, so that the transfer functions would depend on Laplace variable s. The complexity of the transfer functions depends on the detail of our modeling; however, the principle of splitting a signal on a block diagram is the same as in our simple incompressible case (7.7.1).

7.8 Describing systems

We began our study of dynamic systems by writing differential equations. Then we adopted the Laplace transformation of these equations as an equivalent description. Now we have introduced block diagrams as yet another description. That does it for descriptions -- we will now apply them to increasingly complicated systems. Whatever the means, our purpose is to calculate an output response for an input disturbance, as we have in many examples of first-order systems.