THE MULTIPLE PLATE ANTENNA

by

ALLAN CARTER SCHELL

S.B., Massachusetts Institute of Technology (1956)

S.M., Massachusetts Institute of Technology (1956)

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Signature of Author  Allan Carter Schell
Department of Electrical Engineering, August 21, 1961

Certified by  __________
Thesis Supervisor

Accepted by  __________
Chairman, Departmental Committee on Graduate Students
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ABSTRACT

The multiple plate antenna is a very high resolution radio telescope that can be constructed at an economical cost per unit area. The aperture is divided into a number of reflectors, each individually positioned to redirect incoming radiation to a focus. This technique of zoning compels narrow band operation for fixed settings, but the adjustability of the plates permits operation over a wide frequency range and rapid and flexible beam steering.

The intended use of the antenna is to obtain surveys of the distribution of cosmic sources. The properties of the multiple plate antenna are determined with regard to its performance as a radio telescope. The analysis of the extent to which the antenna system reproduces the source variations leads to the use of spatial frequencies and the antenna transfer function, which measures the weighting that the antenna applies to the source angular spectrum.

The specification of the antenna requires a determination of the amplitude distribution across the aperture and the positions of the individual elements. In order to do this, the antenna and its intended use are characterized in the spatial frequency domain. A general error criterion is formulated, and the application of the calculus of variation and constraints intrinsic to antennas results in an integral equation that defines the condition for the optimum antenna. This equation can be used for radiative systems as the Wiener-Hopf equation is used for linear time domain systems. The antenna integral equation is of the form of a self-consistent equation and is amenable to iteration techniques. It is formulated for continuous apertures, where the answer is given as the amplitude distribution across the aperture, and for arrays, where the result is a density function that measures the number of elements per unit area across the aperture. This equation allows the determination of the arrangement of plates that best satisfies the performance objectives of the antenna system. A number of examples are given to demonstrate the technique. An analysis is presented of the effect of wavefront correlation as a limit to the size of the effective aperture.

Thesis Supervisor: Lan Jen Chu
Title: Professor of Electrical Engineering
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In the past two decades there has been a rapid growth of the science of radio astronomy. The techniques of radio astronomy have been greatly improved by new electronic instrumentation. Radio telescopes of very large size are in operation or under construction. Methods for the synthesis of apertures have been used with success.

Some of the important discoveries of radio astronomy in this period are the detection of the galactic hydrogen line, the identification of radiation from Jupiter, the radio observation of colliding galaxies, and the successful radar contacts with Venus and the sun.

One of the major requirements of radio astronomy is high resolution. There has been a continual growth in the size of radio astronomy telescopes and an increasing development of "skeleton" aerials such as the Mills cross and the interferometric systems.\(^1\)

Methods of interferometry and aperture synthesis have gained in popularity partially because of the enormous cost of the very large steerable paraboloids. It would appear unlikely that movable paraboloidal antennas larger than the 600-foot Navy structure presently under construction will be built.

One method of providing a more economical large aperture scanning antenna is to use a spherical reflector fixed on the earth with a movable line source feed mounted above it. A 1000-foot diameter antenna of this type is presently under construction at Arecibo, Puerto Rico.\(^2\) This technique, however, becomes very expensive for high frequencies and large diameters because of the difficulty of accurately positioning the feed structure above the reflector.
The National Science Foundation Advisory Panel on Radio Telescopes was convened to discuss methods of providing very high resolution for radio astronomy. One of the current topics of interest in radio astronomy is the determination of the structure of the galaxy. Of concern to cosmologists is the distribution of the large clouds of hydrogen that are present in the galaxy. These can be determined by measuring the emission line of neutral hydrogen at 1420.405 mcps. In addition, the observed shifts in the frequency provide a measure of the radial velocity of these clouds. A thorough survey of the hydrogen clouds in our galaxy has been made by the University of Leiden. However, the interpretation of the results of surveys of our galaxy requires a kinematic model in order to associate particular clouds and velocities with a position in the galaxy. Observation of another galaxy of the same type would permit an independent measurement of these properties and thus provide a much better picture of the galactic structure. A survey of the nearest galaxy, m31, to the same accuracy as the Leiden survey would require a radio telescope with a resolution of one minute of arc.

The topic of concern to the National Science Foundation Advisory Panel was to determine how an antenna of 1 min. beamwidth at 1420 mcps could be built. Several approaches were studied, such as aperture synthesis, a spherical reflector, and a cross consisting of a number of 200-foot reflectors. Another method was suggested by Professor Lan Jen Chu of M.I.T. His plan was to cover a large area with a number of reflectors, each individually positioned and oriented to redirect the incoming radiation from a particular direction with the correct phase to a focus located above the array of reflectors. A sketch of such an antenna is shown in Fig. 1. A central computing unit would be used to determine the adjustment of each plate in
Figure 1

A Sketch of a Multiple Plate Radio Astronomy Antenna
orientation and vertical position for a particular direction of radiation and wavelength. In the construction of such an antenna, the initial placement of the plates would be relatively unimportant provided general rules are observed, for each plate can be adjusted to have the proper path length to the focus.

In order to find the most economical cost per unit area, a compromise must be struck between the size of the individual plates and the number of plates used in an aperture of given size. The total cost of the antenna is a function of the number of plates used, and large plates require stronger and more accurate positioning units than small ones. For a plane circular aperture of radius \( a \), completely filled by square plates of side length \( d \), the number of plates needed is shown in Fig. 2.

Such an antenna would preserve the desired diffraction pattern over a narrow band of frequencies for a given setting. For other frequencies or angles of incidence, it would be necessary to readjust the plates in the aperture.

If the antenna were constructed over irregular terrain, its electromagnetic behavior would resemble that of a randomly zoned reflector, where the path lengths of adjacent sections of the antenna differed by unrelated numbers of wavelengths.

The use of a large number of plates, however, introduces additional freedom into the antenna design, for now the arrangement of the plates as well as the overall taper or illumination must be specified. In order to do this, one must characterize the antenna and its intended use. Then the antenna design may be optimized with regard to its specified performance goal.

This has been done by using the concept of spatial frequency, which is often applied to optics problems. The antenna is represented by its transfer
function, which is the Fourier transform of the antenna receiving cross section. The angular distribution of cosmic sources is represented by its spectrum of spatial frequencies. The use of the concept of spatial frequency and the analogy between communication theory and optics have been developed by Elias, O'Neill, and others, particularly with regard to the improvement of images in optical systems. The assessment of the quality of the lens or antenna has been carried out by Linfoot and Fellgett using as an error criterion the mean square difference between the source spectrum and the output of the device. This criterion was used by Drane and Parrent to find the best Tchebyscheff distribution for an eight element array scanning a source. In terms of radio astronomy measurements, Bracewell pointed out that the image at the output of the antenna could be considerably improved by a restoration process, and he determined the form of the restoration filter.

The central problem to be dealt with in this thesis is to find the antenna that best performs its intended use, as measured by an error criterion involving the system output. The source distribution is considered to be incoherent, so that the time average powers received from different directions add at the antenna output. The error criterion differs from that of Linfoot and Fellgett in that the actual output of the system, such as the output of the restoration process, is used in the comparison with the desired result. The analysis is applied to a general error criterion, although as an example the mean square difference between the source distribution and the restoration filter output is minimized.

In section II the analysis of the basic properties of the multiple plate antenna is presented to provide a familiarity with the antenna and
to introduce the aperture autocorrelation function. The power patterns of different arrangements are determined from the aperture correlation function, which is later modified to become the antenna transfer function.

The third section of the thesis develops the properties of the radiation patterns of incompletely filled apertures. The expected deviations of the field and power patterns of a particular array from the average are determined, and a probability of side lobe level is found. Also, the fraction of members of an ensemble of incompletely filled apertures with all lobes below a given level is determined.

The use of the multiple plate antenna as a radio astronomy instrument necessitates an analysis of the apparent temperature characteristics as well as the radiation pattern. In section IV the apparent antenna temperature is determined. This requires, in effect, an analysis of the efficiency of the antenna, for it is the area of ground uncovered in the spaces between plates that produces a large fraction of the undesired contributions to the apparent antenna temperature. Consideration of the geometry of the system leads to minimum temperature configurations.

In section V the equations for the antenna performance are developed. First, the antenna is characterized by its transfer function, and this is related to the physical spacings of the array. Then the noise and uncertainty associated with readings of the system output are discussed. The equations for the optimum restoration filter are derived for the case of an independent error in the output and an error associated with the pattern fluctuations. With this background, an error criterion represented in spatial frequencies is minimized subject to the constraint that a realizable antenna is obtained as an answer. It is this constraint that
provides the difficulty in the analysis, for without it, an answer can be obtained directly, often by inspection. However, such answers are often completely unrealizable as antennas. The result of the application of the calculus of variations and the use of constraints is an integral equation that defines the optimum antenna.

The solution of the integral equation for the antenna that minimizes the mean square difference between the source distribution and the output of the restoration filter is the central topic of section VI. The optimum field distribution across a continuous aperture is found by an iteration technique, and a number of examples are presented. It is shown that the aperture distribution can be found from the autocorrelation function to one arbitrary phase. A related problem is that of the optimum array of elements to use to obtain a sky map which, when restored in the presence of noise, will most faithfully reproduce the source distribution. Characterizing the antenna again by its spatial frequency transfer function, an integral equation is obtained that describes the density of antenna elements at different parts of the aperture. The optimum transfer function may be approximated by using a set of element spacings taken from a random sequence, the statistics of which are specified by the density function. In this manner a multiple plate antenna can be realized with a transfer function approximating the optimum function for the desired performance. A plot of the antenna element positions can resemble the locations of digits in pseudo-random codes, often used in computer sequences and radar detection techniques. Antennas of this sort may be considered as pseudo-random arrays and are, in a sense, a generalization of the Arsac, or constant spatial frequency response array.
A limiting factor to the resolution of a very large antenna is the degradation of the correlation between the incoming signal measured at two points in the aperture. In section VII the effect of correlation is discussed in terms of the mutual coherence function of Wolf. The propagation of the mutual coherence in two and three dimensions is derived, and the properties of the mutual coherence in the directions parallel and normal to the propagation are determined. A transform relation for the behavior of antennas in partially coherent fields is developed, and the normal gain of the coherence limited antenna is found. The normal gain in this sense corresponds to the maximum gain that can be obtained from an antenna that cannot change its characteristics as rapidly as the incoming wavefronts change. Examples illustrating the patterns of antennas in partially coherent fields are given.
II

BASIC CHARACTERISTICS OF MULTIPLE PLATE CONFIGURATIONS

The use of a large number of individually positioned plates to provide high resolution requires the specification of the arrangement of the plates as well as the overall taper or illumination. The basic characteristics of such an antenna can be studied before determining the rules for taper and placement. First, the path length for a particular plate from a reference wavefront to the focus is calculated, the relation between the aperture phase and plate size is determined, and the limitations of the antenna in terms of frequency bandwidth are found.

In section 2.4 the aperture autocorrelation is introduced, and the power pattern of the antenna is calculated from the autocorrelation function. The properties of different arrangements of incompletely filled apertures are determined to provide a familiarity with the aperture autocorrelation function.

2.1 Calculations of Path Length

The path length associated with a particular plate is measured along the normal of a reference plane wavefront to the center of the plate and from the plate center to the focal point. In Figure 3.a a view of the geometry in the plane including the vertical and the normal to the reference wavefront is given. A coordinate system is chosen with the origin at 0, an azimuthal angle $\varphi$, an elevation angle $\theta$, and the z axis along the vertical OF. The angle $\alpha$ is measured between the vertical and a ray from the plate center to the focal point F, which is located a distance OF = f above the origin. A wave is incident at an elevation angle $\theta_0$ to the
F is a focal point
A is a plate center
The location of plate centers is given by $z(\varphi, \alpha)$

**FIGURE 3a**
Path Length - No Plate Displacement

$\Delta' = \text{the displaced plate center position}$

$AA' = \varepsilon z$

**FIGURE 3b**
Effect of Plate Displacement on Path Length
vertical OF and at an azimuthal heading $\varphi_0$. The reference plane is chosen parallel to the wavefront and passing through the focal point F. The plate centers are located on a curve $Z = Z(\varphi, \alpha)$, $Z(0,0) = 0$, where $Z$ describes the height above 0 in the $z$ direction of the plate center with the coordinates $\varphi, \alpha$.

Then

$$\begin{align*}
AB &= C' A \sin \theta_0 \cos (\varphi - \varphi_0) \\
OF &= FA \cos \alpha \\
C'A &= FA \sin \alpha
\end{align*}$$

$$FA = \frac{f - Z}{\cos \alpha}$$

The path length is

$$CA + AF = (f - Z) \left[ \frac{1}{\cos \alpha} + \cos \theta_0 + \tan \alpha \sin \theta_0 \cos (\varphi - \varphi_0) \right]$$

A displacement in the vertical direction of the center of the plate produces a change in path length as shown in Fig. 3b. If the vertical displacement is denoted by $\delta z$, the change in path length is

$$-(\delta z \cos \theta_0 + \delta z \cos \alpha)$$

The total path length for the displaced plate center is

$$CA' + A'F = \Delta = (f - Z) \left[ \frac{1}{\cos \alpha} + \cos \theta_0 + \tan \alpha \sin \theta_0 \cos (\varphi - \varphi_0) \right] - \delta z (\cos \theta_0 + \cos \alpha)$$

The path length for the plate directly under the focus is $\Delta = f(1 + \cos \theta_0)$, and the difference in path length for the other parts of the reflector is

$$\Delta - \Delta_0 = \frac{f}{\cos \alpha} \left[ 1 - \cos \alpha + \sin \alpha \sin \theta_0 \cos (\varphi - \varphi_0) \right]$$

$$- \frac{Z}{\cos \alpha} \left[ 1 + \cos \alpha \cos \theta_0 + \sin \alpha \sin \theta_0 \cos (\varphi - \varphi_0) \right] - \delta z (\cos \theta_0 + \cos \alpha)$$
If the surface below the antenna is a horizontal plane, then \( Z = 0 \), and

\[
\Delta - \Delta_0 = \frac{f}{\cos \alpha} \left[ 1 - \cos \alpha + \sin \alpha \sin \theta_0 \cos (\varphi - \varphi_0) \right] - \delta z (\cos \theta_0 + \cos \alpha)
\]

If the surface below the antenna is a paraboloid of focal length \( f \), then

\[
OP = f, \quad FA = \frac{2f}{1 + \cos \alpha}, \quad O'A = FA \sin \alpha
\]

\[
AB = O' A \sin \theta_0 \cos (\varphi - \varphi_0), \quad O'K = O'F \cos \theta_0
\]

\[
CA + AF = \frac{2f}{1 + \cos \alpha} \left[ 1 + \cos \alpha \cos \theta_0 + \sin \alpha \sin \theta_0 \cos (\varphi - \varphi_0) \right]
\]

At \( \alpha = 0 \), \( \Delta_0 = f (1 + \cos \theta_0) \) and the effect of plate displacement is again \(-\delta z (\cos \theta_0 + \cos \alpha)\), giving

\[
\Delta - \Delta_0 = \frac{f}{\cos \alpha} \left[ \left( \frac{1 - \cos \alpha}{1 + \cos \alpha} \right) (1 - \cos \theta_0) + \frac{\sin \alpha}{1 + \cos \alpha} \sin \theta_0 \cos (\varphi - \varphi_0) \right] - \delta z (\cos \theta_0 + \cos \alpha)
\]

If the surface below the antenna is a section of a sphere with a Radius \( R = f \), then \( OP = FA = f \), \( O'K = f \cos \alpha \cos \theta_0 \)

\[
AB = f \sin \alpha \sin \theta_0 \cos (\varphi - \varphi_0)
\]

and

\[
\Delta = f \left[ 1 + \cos \alpha \cos \theta_0 + \sin \alpha \sin \theta_0 \cos (\varphi - \varphi_0) \right] - \delta z (\cos \theta_0 + \cos \alpha)
\]

At the origin, \( \Delta_0 = f (1 + \cos \theta_0) \) which gives as a result

\[
\Delta - \Delta_0 = f \left[ \cos \theta_0 (\cos \alpha - 1) + \sin \alpha \sin \theta_0 \cos (\varphi - \varphi_0) \right] - \delta z (\cos \theta_0 + \cos \alpha)
\]
2.2 Aperture Phase Error and Plate Size

A multiple plate antenna using curved plates of varying size would be much more costly than one that was constructed of identical flat plates. Therefore, the relations between phase error and plate size are determined in order to find the maximum size of flat plate that can be used without serious deterioration of antenna performance.

The error introduced by approximating a section of a paraboloid by a section of a tangent plane may be described by two effects. The first is the difference between a line and a parabola touching the line, and the second is the difference between a circle and a tangent line. These correspond to the errors along two normal cuts through the surface. The equation for a paraboloid is \( Z = \frac{1}{4f} r^2 \) where \( r \) is measured in the transverse plane and \( f \) is the focal length. Expanding this around a point,

\[
Z_o + \delta Z = \frac{1}{4f} r_o^2 + \frac{1}{2f} r_o \delta r + \frac{1}{4f} (\delta r)^2
\]

The first two terms are the equation of the tangent line, and the difference in \( z \) between the line and the parabola is \( \delta Z = \frac{1}{4f} (\delta r)^2 \). For on-axis signals \( (\theta_o = 0) \), the slope of the line is \( \frac{r_o}{2f} = -\tan \frac{\alpha}{2} \) and the change in path length in the \( z \) direction as a function of \( \delta l \), measured along the tangent line, is

\[
\delta Z = -\frac{1}{2r_o} \tan \frac{\alpha}{2} \cos \frac{\alpha}{2} (\delta r)^2 = -\frac{1}{2r_o} \tan \frac{\alpha}{2} \cos \frac{\alpha}{2} (\delta l_1)^2
\]

\[
\delta Z = -\frac{1}{4R} (\delta l_1)^2
\]

where \( R \) is the distance from the plate to the focus. The change in path length in the \( R \) direction as a function of \( \delta l_1 \), is

\[
\delta R = \frac{1}{2r_o} \tan \frac{\alpha}{2} \cos \frac{\alpha}{2} (\delta l_1)^2 = \frac{1}{4R} \sqrt{\frac{R+f}{2R}} (\delta l_1)^2
\]

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The total change in path length is

\[ \delta \Delta_1 = \frac{1}{4R} \left[ 1 - \sqrt{\frac{R^2 + f^2}{2R}} \right] (\delta l_1)^2 \]

This is the deviation in path length along a straight line approximation of a parabola. This represents the error incurred in one plane by the approximation of a part of a paraboloid by a plate. In a cut through the paraboloid normal to the axis a circle is generated, and the path length deviation caused by approximating part of the circle by a line is

\[ \delta \Delta_2 = \sqrt{R^2 + (\delta l_2)^2} - R = \frac{1}{2R} (\delta l_2)^2 \]

The errors are largest near the origin where \( R = f \), and cause path length changes of

\[ \delta \Delta = \frac{1}{2f} (\delta l_1)^2 + \frac{1}{2f} (\delta l_2)^2 \]

where \( \delta l_1 \) and \( \delta l_2 \) are distances measured in two normal directions from the point of tangency.

The mean square error produced by a square plate of side length \( d \) located near the origin is

\[ \delta \bar{\Delta}^2 = \frac{1}{d^2} \iint \delta \Delta^2 \, d l_1 \, d l_2 = \frac{7}{5} \left( \frac{d^2}{12f} \right)^2 \]

The plate may be displaced an amount \( \Delta Z \) to minimize the mean square error in path length. The optimum displacement near the origin is \( \Delta Z = \frac{1}{12} \frac{\delta^2}{f} \).

The mean square error then reduces to

\[ \delta \bar{\Delta}^2 = \frac{2}{5} \left( \frac{d^2}{12f} \right)^2 \]

for on-axis radiation on plates near the origin. For plates located away from the origin, the distance from the plate to the focus is larger, and the devia-
tion in path length is less. That is, plates located away from the origin are better approximations to pieces of a paraboloid than the plates at the origin.

If the antenna is being used to receive radiation at an angle $\theta_0$ from the zenith, the plates must approximate sections of paraboloids tipped to $\theta_0$. Again the maximum deviation in path length is at the origin, where the former equation for the deviation from a parabola can be used, provided a new focal length $f'$ given by $f' = \frac{2f}{1 + \cos \theta_0}$ is used. The parabolic error is then

$$\Delta_t = \frac{L}{4f} \left[ 1 + \sqrt{\frac{3 + \cos \theta_0}{4}} \right] (\delta l_t)^2$$

at the origin for rays at an angle $\theta_0$ from zenith. This expression differs imperceptibly from the on-axis case for values of $\theta_0$ less than 45°. The error in a normal plane is again the deviation of a tangent line from a circle, and is as before, $\Delta_2 = \frac{1}{2f} (\delta l_2)^2$

so that the total path error is

$$\Delta = \frac{L}{4f} \left[ 1 + \sqrt{\frac{3 + \cos \theta_0}{4}} \right] (\delta l_t)^2 + \frac{1}{2f} (\delta l_2)^2$$

2.3 Bandwidth Limitations and Control

The limitations of the antenna in terms of frequency bandwidth can be readily determined from the results of Sec. 2.1. Let the path length contain $Q$ wavelengths. Then, if the path length remains fixed, $d\Delta = Qd\lambda + \lambda dQ = 0$

$$\frac{dQ}{\lambda} = -Q \frac{d\lambda}{\lambda^2} = -\Delta \frac{d\lambda}{\lambda^2}$$

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In the path length expressions, $z$ is very small in comparison to $f$, and very little error is introduced in the general case by omitting the term multiplied by $z$ for bandwidth determination. The absolute phase is unimportant, so the path difference across the aperture is considered. The change in the number of wavelengths in the path is

$$dQ = -\left\{ \frac{f}{\cos \alpha} \left[ 1 - \cos \alpha \sin \theta \cos (\varphi - \varphi_0) \right] - \frac{z}{\cos \alpha} \left[ 1 + \cos \alpha \cos \theta_0 + \sin \alpha \sin \theta_0 \cos (\varphi - \varphi_0) \right] \right\} \frac{d\lambda}{\lambda^2}$$

If, on the other hand, $z$ is changed with frequency to keep the number of wavelengths in a given path constant, then

$$dQ = \frac{\lambda d\Delta^1}{\lambda^2} - \frac{d\lambda}{\lambda} \Delta^1 = 0$$

$$\frac{d\Delta^1}{d\lambda} \frac{dz}{d\lambda} = \Delta^1 \frac{d\lambda}{\lambda}$$

$$\frac{dz}{d\lambda} = -\frac{\Delta^1}{\lambda} \frac{1}{\cos \alpha + \cos \theta_0}$$

where $\Delta^1$ is $\Delta - \Delta_0$ as given in Sec. 2.2.

Similarly, the change in $z$ necessary to restore $Q$ to its original value after a shift in wavelength is found from

$$\frac{d\Delta^1}{dz} = \lambda \frac{dQ}{dz} \quad dz = \frac{\lambda}{d\Delta^1} \frac{dz}{dz}$$

$$\frac{dz}{\lambda} = \frac{1}{\cos \alpha + \cos \theta_0} dQ$$
The fractional change of wavelengths of \( z \) is thus seen to be nearly that of \( Q \). Roughly speaking, \( z \) would have to be controlled to about twice the desired aperture tolerance.

2.4 The Radiation Pattern

The use of many plates affords a greater area of compromise between the cost of the antenna and its performance. Instead of completely filling the aperture with plates, a distribution may be planned by which spaces between plates are allowed. This can be described in terms of plate density, or the ratio of the total plate area within a given region to the total area of the region. The use of fewer plates than necessary to completely fill the aperture results in a loss in gain and an increase in sidelobe level over the filled case. These deteriorations in performance may be relatively unimportant in comparison to the reduced cost of fewer plates. Antennas of this type have very high gain, and often the limitation of the device is not sensitivity but resolution. In this event it may be wise to sacrifice gain in order to increase resolution and the usefulness of the antenna as a radio astronomy instrument.

The calculation of the pattern of an antenna with an incompletely filled aperture can be carried out by conventional means when the plates are arranged in an orderly pattern. However, it may be found desirable to arrange the plates in a random manner across the aperture, so that the effect of regular omissions can be avoided. In this event, the calculation of the far field would be quite involved if the actual aperture function were used. The antenna can be satisfactorily described by statistical parameters, and general results
may be obtained. Using techniques of statistical analysis, it is possible to find the average behavior of a large number of antennas, each of which was formed under the same statistical conditions.

2.4.1 Pattern Calculations Using the Correlation Function

The field in the Fraunhofer or far-zone region is found using the assumptions that the deviations from constant phase in the aperture plane are small, and the field over the aperture is uniformly polarized in one direction, and neglecting the obliquity factor and high order terms in the aperture coordinates. The far field pattern is approximately

$$F(\theta, \phi) = \int \int g(x,y) e^{j k x \sin \theta \cos \phi} \cos \theta \sin \theta dy dx,$$

where $g(x,y)$ describes the field distribution across a projection of the antenna surface on a plane aperture having coordinates $x, y, z = 0$, $\theta$ is the polar angle measured from the vertical or the $z$ axis, and $\phi$ is the azimuthal or rotational angle in the $x$-$y$ plane.

The far field power pattern is

$$P(\psi_x, \psi_y) = F(\psi_x, \psi_y) F^*(\psi_x, \psi_y)$$

$$= \int \int g(x,y) e^{j x \psi_x + y \psi_y} dy dx \int \int g^*(x',y') e^{j x' \psi_x + y' \psi_y} dy' dx',$$

where $\psi_x = k \sin \theta \cos \phi$, and $\psi_y = k \sin \theta \sin \phi$.

The variables $x, y$, and $x', y'$ are independent, and a new set $x' = x + x', y' = y - y'$ may be introduced, yielding

$$P(\psi_x, \psi_y) = \int \int g(x,y) g^*(x + x', y + y') e^{j \psi_x x + \psi_y y} dy dx.$$
The aperture correlation function is now defined as
\[
C_A(\xi, \eta) = \frac{1}{A} \iiint g(x,y) g^*(x+\xi, y+\eta) \, dx \, dy
\]
where \(A\) is the area of the aperture.

The angular distribution of the far field power may be determined by taking the Fourier transform of the correlation function as
\[
P(\psi_x, \psi_y) = A \iint C_A(\xi, \eta) e^{j\psi_x \xi + j\psi_y \eta} \, d\xi \, d\eta
\]

The gain of the antenna is found from Parseval’s theorem, which may be written
\[
\frac{1}{4\pi^2} \iint P(\psi_x, \psi_y) \, d\psi_x \, d\psi_y = \iint g(x,y) g^*(x,y) \, dx \, dy = A C_A(0,0)
\]

It is necessary to examine the relationship between
\[
\iint P(\psi_x, \psi_y) \, d\psi_x \, d\psi_y
\]
and the total radiated power, which is
\[
\int_0^{2\pi} \int_0^\pi P(\theta, \phi) \sin \theta \, d\theta \, d\phi = P_T
\]
The element of area on the surface of the sphere of unity radius is \(\sin \theta \, d\theta \, d\phi\).

To find the element of area in the variables \(\psi_x, \psi_y\), the Jacobian \(\frac{\partial (\psi_x, \psi_y)}{\partial (\theta, \phi)}\) is determined, yielding the element of area
\[
dA = \frac{\partial (\psi_x, \psi_y)}{\partial (\theta, \phi)} \, d\theta \, d\phi
\]
The product \(d\psi_x \, d\psi_y\) may also be found by differentiating
\[
\psi_x = k \sin \theta \cos \phi \quad \text{and} \quad \psi_y = k \sin \theta \sin \phi
\]
yielding
\[
d\psi_x = k \cos \theta \cos \phi \, d\theta - k \sin \theta \sin \phi \, d\phi
\]
\[
d\psi_y = k \cos \theta \sin \phi \, d\theta + k \sin \theta \cos \phi \, d\phi
\]
In forming the product of these quantities it is necessary to remember that they are not numbers, but vectors, and the element of area is actually $dA = dl_1 \times dl_2$. Therefore, the product $d\delta \delta \theta = dq \delta \delta \varphi = 0$ and $d\theta \delta \delta \varphi = -dq \delta \delta \theta$. The result is

$$d\psi_x d\psi_y = k \sin \theta \cos \vartheta \, d\theta \, d\varphi$$

This expression differs from the element of area of a sphere in functional form by the factor $\cos \vartheta$. In certain cases this term may be a part of the antenna pattern not included in $P(\psi_x,\psi_y)$ and the result will be exact. In general, the radiated power must be concentrated in angles near $\theta = 0$ in order that this term have little effect. Alternately, the integral may be considered as the total power projected in the $z$ direction, or the integral of a projection of the power passing through the unit sphere on a gnomonic plane tangent at the zenith. If the power pattern is significantly different from zero only for small values of $\theta$, the limits of integration of $\psi_x$ and $\psi_y$ may be extended to $+\infty$ and $-\infty$ with the understanding that there is no contribution outside of the region around the zenith. Under these conditions

$$\int_{-\infty}^{\infty} P(\psi_x,\psi_y) \, d\psi_x \, d\psi_y = k^2 P_r = 4\pi^2 A C_\circ(0,0)$$

The power pattern is

$$P(\psi_x, \psi_y) = A \int C_\circ(x,\xi) \, e^{i\psi_x x + i\psi_y \xi} \, d\xi \, d\eta$$

The gain in the direction $\psi_{xo}$, $\psi_{yo}$ may be defined as

$$G(\psi_{xo}, \psi_{yo}) = \frac{P(\psi_{xo}, \psi_{yo})}{P_r} = \frac{k e^2 P(\psi_{xo}, \psi_{yo})}{\pi A C_\circ(0,0)}$$
and the forward gain as

\[ G_F = \frac{P(0,0)}{P_r/4\pi} = \frac{k^2 P(0,0)}{\pi A C_A(0,0)} \]

The use of the correlation function in the description of the pattern allows the computation of the effects of an aperture described by a regular function. As an example, consider a rectangular aperture of extent 2a in the x direction and 2b along the y axis, with a uniform in-phase field across the aperture. Then \( g(x,y) = 1 \quad |x| < a \), \( |y| < b \)

The correlation function may be found by averaging the product of \( g(x,y) \) and \( g(x + \xi, y + \eta) \). The result is

\[ C_R(\xi, \eta) = \left| \frac{a_x - \xi}{a_x} \right| \left| \frac{2b - \eta}{ab} \right| \quad |\xi| < 2a, |\eta| < 2b \]

which describes a rectangular base pyramid of unity height and of base area \( 4ab \). The Fourier transform of this yields

\[ P(\phi_x, \phi_y) = 16a^2 b^2 \frac{(\sin a \phi_x)^2 (\sin b \phi_y)^2}{(a \phi_x)^2 (b \phi_y)^2} \]

The power in the aperture is given by the value of \( C_A(0,0) \) multiplied by the area. In this case it is obviously \( 4ab \). The forward gain is

\[ G_F = \frac{k^2}{\pi(4ab)} P(0,0) = \frac{k^2}{\pi} 4ab = \frac{4\pi}{\lambda^2} (4ab) \]

2.4.2 Pattern Calculations of Incompletely Filled Apertures

By using the above described method, the pattern of an incompletely filled aperture may also be determined. However, a simpler and more general result can be obtained by finding the average power pattern of an ensemble of apertures,
each of which is formed by the same methods. Consider an ensemble of apertures, each composed of a large number of identical plates completely filling the aperture. Using a set of statistics such as a table of random binary numbers, the plate at the location $x_i, y_j$ of each aperture is either removed or not, so that after the removal process there are $M_{ij}$ plates remaining in the $x_i, y_j$ position of $N$ antennas. This procedure may be repeated for all $x$ and $y$. The average plate density at $x_i, y_j$ may be defined as $\frac{M_{ij}}{N}$, and the probability of finding a plate at the $x_i, y_j$ position of any aperture is $\frac{M_{ij}}{N}$. Of course, for a particular antenna, the plate is either there or it is not. However, this description allows an expected distribution to be calculated, and if the antenna contains sufficient plates both before and after the random plate removal, the actual pattern will approximate the statistical result.

If the probability of plate occurrence is the same for each location and the total number of remaining plates is an integer, $(M_{ij} = M)$, the number of members of the ensemble is $\frac{N!}{(N-M)!M!}$, which represents the number of ways that $M$ elements can be arranged in $N$ spaces. In this case there are exactly $M$ plates in each member antenna. If the probability of plate occurrence differs for each location, the number of members of the ensemble becomes arbitrarily large as the probability of plate occurrence at the position $x_i, y_j$, tends toward $\frac{M_{ij}}{N}$. The number of elements then in a particular array tends to but does not necessarily equal $\sum_i \sum_j \frac{M_{ij}}{N} = M$. If the array contains a large number of elements, however, the average number may be used as the number of elements in any array.

Having established a probable plate distribution, the autocorrelation function is computed from the sum of events and their probabilities of happening, as
\[ C_A(j,k) = \frac{1}{N} \sum_{m,n} g(x_m, y_n) \delta(x_{m+j}, y_{n+k}) p(m, n, m+j, n+k) \]

In this expression, \( g(x_m, y_n) \) represents the voltage or field strength at the plate located at \( x_m, y_n \), and the probability \( p(m, n, m+j, n+k) \) is that of finding a plate at \( x_m, y_n \) and a plate at \( x_{m+j}, y_{n+k} \). It is assumed that the probability of plate occurrence at one location is independent of any other, that the probability does not vary rapidly from one location to the next, and that there are a large number of locations. Under these conditions, a graph of the probability distribution across the aperture appears as a large number of blocks of equal or slowly varying height. The autocorrelation function may be computed from these values, but a considerably simpler technique results if the probability distribution is replaced by a continuous curve passing through the actual values at the plate centers. In this event the autocorrelation function becomes

\[ C_A = \frac{1}{A} \int \int g(x, y) \delta(x + \xi, y + \eta) p(x, y) p(x + \xi, y + \eta) \, dx \, dy \]

Once the autocorrelation function has been determined, the radiation pattern is found from its Fourier transform. The examples given below illustrate the technique.

(A) Uniform plate density rectangular aperture

If, in the ensemble of apertures, plates are randomly removed from the apertures under the same statistics for all positions, the resultant average plate density in the ensemble is the same for all positions. If there are \( M \) plates remaining in a row of \( N \) positions, the plate occurrence probability is \( P = \frac{M}{N} \). Let the aperture extent be \( 2a \) in the \( x \) direction and \( 2b \) in the \( y \) direction. If there were no adjacent plates the autocorrelation function would be

\[ \frac{M}{N} \left| \frac{d}{d} \right| \frac{d}{d} < d, \quad |\xi| < d, \quad |\eta| < d \]

"
For values of $\xi$ and $\eta$ greater than $d$, one may use the average plate density, for the removal of a particular plate is not a function of the removal of any other. Thus, there is a broad base to the autocorrelation function of the form

$$\frac{N(N-1)2a-\xi}{2a} \left| \frac{2b-\eta}{2b} \right| \quad \text{for} \quad |\xi| < 2a, \ |\eta| < 2b$$

The junction of the two functions represents the point where the averaging of a plate shifted with respect to itself by an amount less than $d$ steps, and the product of the two shifted average functions begins. The total autocorrelation function is

$$C_A = \frac{M(M-1)}{N(N-1)} \left| \frac{2a-\xi}{2a} \right| \left| \frac{2b-\eta}{2b} \right| \quad d < |\xi| < 2a, \ d < |\eta| < 2b$$

$$C_A = 0 \quad |\xi| > 2a, \ |\eta| > 2b$$

The transform of this function gives for the power pattern the sum of the average term $P_A$ and the spurious term $P_S$:

$$P(x, y) = 4ab \alpha^2 \frac{M}{N} \left( \frac{1-\frac{M-1}{N-1}}{\frac{\alpha}{\lambda}} \right)^2 \left( \frac{\sin \frac{\alpha x}{\lambda}}{\frac{\alpha x}{\lambda}} \right)^2 \left( \frac{\sin \frac{\alpha y}{\lambda}}{\frac{\alpha y}{\lambda}} \right)^2$$

$$+ \left( 4ab \right)^2 \frac{M(M-1)}{N(N-1)} \left( \frac{\sin \frac{\alpha x}{\lambda}}{\frac{\alpha x}{\lambda}} \right)^2 \left( \frac{\sin \frac{\alpha y}{\lambda}}{\frac{\alpha y}{\lambda}} \right)^2$$

and at the origin

$$P(0,0) = 4ab \left[ \frac{d^2 M}{N} \left( 1-\frac{M-1}{N-1} \right) + 4ab \frac{M(M-1)}{N(N-1)} \right]$$

The power in the aperture is $4ab \frac{M}{N}$ and the forward gain is

$$G_F = \frac{4\pi}{\lambda^2} \left[ \frac{M-1}{N-1} (4ab) + a^2 \left( 1-\frac{M-1}{N-1} \right) \right] = \frac{4\pi}{\lambda^2} (M \sigma^2)$$

Note that this assumes no loss of power between the plates.
(B) Circular aperture of uniform plate density

Similar calculations may be performed for a circular aperture of radius a from which plates have been removed at random, having a probability of plate occurrence of \( \frac{M}{N} \) for any position.

The autocorrelation function for a circular aperture composed of square plates of side length d is a surface of revolution of base radius 2a, surmounted by a pyramid of base length 2d (See Fig. 4a). This is given by

\[
C_a(\xi, \eta) = \frac{M}{N} \left( 1 - \frac{M-1}{N-1} \right) \left| \frac{d-\xi}{d} \right| \left| \frac{d-\eta}{d} \right| \\
+ \frac{M(M-1)}{N(N-1)} \frac{a}{\pi} \left[ \cos^{-1} \left( \frac{\sqrt{1-r^2}}{2a} \right) - \frac{\sqrt{1-r^2}}{2a} \sqrt{1 - \frac{r^2}{2a^2}} \right]
\]

The power pattern of this antenna is

\[
\mathcal{P}(\psi_x, \psi_y) = \frac{M}{N} \left( 1 - \frac{M-1}{N-1} \right) \pi a^2 \left( \frac{\sin \frac{d}{2} \psi_x}{\frac{d}{2} \psi_x} \right)^2 \left( \frac{\sin \frac{d}{2} \psi_y}{\frac{d}{2} \psi_y} \right)^2 \\
+ \frac{M(M-1)}{N(N-1)} \left( \frac{\pi a^2}{\lambda^2} \right)^2 \left( \frac{2 J_1(\lambda a \sqrt{\psi_x^2+\psi_y^2})}{a \sqrt{\psi_x^2+\psi_y^2}} \right)^2
\]

The power in the aperture is \( \frac{M}{N} \pi a^2 \) and the gain in the forward direction is

\[
G_F = \frac{4\pi}{\lambda^2} \left[ \frac{M-1}{N-1} \pi a^2 + \left( 1 - \frac{M-1}{N-1} \right) a^2 \right] = \frac{4\pi}{\lambda^2} \frac{M}{N} \pi a^2
\]

(C) Circular aperture of non-uniform plate density

If the probability of finding a plate at a given location is a function of the coordinates, the autocorrelation function shape is altered accordingly. The effect of plate removal is analogous to that of tapering the input power distribution. It is necessary to specify the ensemble average of the total number of plates and the form of the plate density. From this information and expressions for aperture tapers the resultant patterns can be obtained.
The Autocorrelation Function of a Circular Aperture Incompletely Filled by Square Plates

Figure 4a

The Average Far Field Power Pattern Components of an Incompletely Filled Aperture

Figure 4b
Let the feed illumination be such that each plate radiates the same amount of power, but plates are removed so that the plate density is

\[ p(r) = \frac{2M}{a^2} \left( 1 - \frac{r^2}{a^2} \right) \quad M \leq \frac{N}{2} \]

where \( M \) is the average number of plates in an aperture and \( N \) is the number of possible locations. The autocorrelation function is given by an involved relation, but its shape is that of a pyramid on a rounded surface of revolution, as shown in Fig. 4a. As there are results of tapered aperture calculations available, the only calculations needed are the height of the pyramid and the height of the average function at the origin. From elementary considerations, \( C_A(0,0) = \frac{M}{N} \) and the height of the average function at the origin is

\[
\frac{8\pi}{n a^3} \frac{M(M-1)}{N(N-\frac{3}{4})} \int_0^a (1 - \frac{r^2}{a^2})^2 r \, dr = \frac{4}{3} \frac{M(M-1)}{N(N-\frac{3}{4})}
\]

The forward gain factor is known to be \( \frac{3}{4} \) for this taper, and from this

\[
G_F = \frac{3}{4} \frac{4\pi a^2 A}{\lambda^2} = 4\pi \frac{P(0,0)}{P_F}
\]

\[
k^2 P_T = 4\pi a^2 C_A(0,0)
\]

\[
P_A(0,0) = 3\pi^2 A^2 \frac{\lambda^2}{k^2} C_A(0,0)
\]

\[
P_A(0,0) = \frac{M(M-1)}{N(N-\frac{3}{4})} (\pi a^2)^2
\]

The variation of the far field is that of a \( \Lambda_2 \) function, and it is found that

\[
P_A(\psi_x, \psi_y) = \frac{M(M-1)}{N(N-\frac{3}{4})} \Lambda_2^2 (a \sqrt{\psi_x^2 + \psi_y^2})
\]

To this function must be added the effect of the pyramidal top of the autocorrelation function, which contributes

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\[ P_s(\psi_x, \psi_y) = \frac{M}{N} \left( 1 - \frac{4}{3} \left( \frac{M-1}{N-3} \right) \pi a^2 \delta^2 \left( \frac{\sin \frac{\psi_x}{2}}{\frac{\psi_x}{2}} \right)^2 \left( \frac{\sin \frac{\psi_y}{2}}{\frac{\psi_y}{2}} \right)^2 \right) \]

giving for the total average power pattern

\[ P(\psi_x, \psi_y) = (\pi a^2) \left[ \frac{M}{N} \left( 1 - \frac{4}{3} \left( \frac{M-1}{N-3} \right) \pi a^2 \delta^2 \left( \frac{\sin \frac{\psi_x}{2}}{\frac{\psi_x}{2}} \right)^2 \left( \frac{\sin \frac{\psi_y}{2}}{\frac{\psi_y}{2}} \right)^2 \right) + \frac{M(M-1)}{N(N-1)} \pi a^2 \sum \left( a \sqrt{\psi_x^2 + \psi_y^2} \right) \right] \]

and for the gain

\[ G_F = \frac{4\pi}{\lambda^2} \left[ \frac{M-1}{N-3} \pi a^2 + \left( 1 - \frac{4}{3} \left( \frac{M-1}{N-3} \right) \delta^2 \right) \right] = \frac{4\pi}{\lambda^2} \left( \pi a^2 \right) \quad \frac{M}{N} \leq \frac{1}{2} \]

Note that the gain has not appreciably changed from that of the uniform plate density case, but the spurious sidelobe power is somewhat reduced. This antenna cannot offer gain higher than \( \frac{2\pi}{\lambda^2} \).

If, instead of uniform plate illumination and varying plate density, the case of uniform density and tapered illumination is considered, the following relations are obtained:

\[ p(r) = \frac{M}{N} \quad \text{taper} \quad g(r) = \frac{\pi}{a^2} \left( 1 - \frac{r^2}{a^2} \right) \]

\[ C_A(0,0) = \frac{M}{N} \]

\[ P_A(0,0) = \frac{3}{4} \frac{M(M-1)}{N(N-1)} \left( \pi a^2 \right)^2 \]

\[ P(\psi_x, \psi_y) = (\pi a^2) \left[ \frac{M}{N} \left( 1 - \frac{M-1}{N-3} \right) \pi a^2 \delta^2 \left( \frac{\sin \frac{\psi_x}{2}}{\frac{\psi_x}{2}} \right)^2 \left( \frac{\sin \frac{\psi_y}{2}}{\frac{\psi_y}{2}} \right)^2 \right] + \frac{3}{4} \frac{M(M-1)}{N(N-1)} \pi a^2 \sum \left( a \sqrt{\psi_x^2 + \psi_y^2} \right) \]

\[ G_F = \frac{4\pi}{\lambda^2} \left[ \frac{3}{4} \left( \frac{M-1}{N-3} \right) \pi a^2 \left( 1 - \frac{M-1}{N-3} \right) \delta^2 \right] \quad \frac{M}{N} \leq 1 \]
Note that for \( \frac{M}{N} \leq \frac{1}{2} \) the gain of this arrangement is somewhat lower than that of the former; however, the latter case holds for values of \( M \) up to and including \( N \).

The last arrangement is to have the power tapered as \( g^2(r) = \frac{5}{3}(1 - \frac{r^2}{a^2}) \) and the plate density tapered as \( \frac{3}{2} \frac{M}{N} \sqrt{1 - \frac{r^2}{a^2}} \) where \( \frac{M}{N} \leq \frac{2}{3} \).

For this case, with \( M \) and \( N \) large,

\[
C_A(0,0) \approx \frac{M}{N} \left(1 - \frac{5}{4} \frac{M}{N}\right) + \frac{5}{4} \frac{M^2}{N^2}
\]

\[
\overline{P}_A(0,0) \approx \frac{3}{4} \times \frac{5}{4} \frac{M^2}{N^2} (\pi a^2)^2
\]

\[
\overline{P}(\psi_x, \psi_y) \approx (\pi a^2) \left[ \frac{M}{N} \left(1 - \frac{5}{4} \frac{M}{N}\right) \chi \left(\frac{\sin \frac{\psi_x}{2}}{\frac{\psi_x}{2}}\right) \left(\frac{\sin \frac{\psi_y}{2}}{\frac{\psi_y}{2}}\right) \right]^2 + \frac{15}{16} \frac{M^2}{N^2} \pi a^2 \Lambda_2 \left(a \sqrt{\psi^2 + \psi^2}\right)
\]

\[
G = \frac{4\pi}{\lambda^2} \left[ \frac{15}{16} \frac{M}{N} \pi a^2 + (1 - \frac{5}{4} \frac{M}{N}) a^2 \right] \quad \frac{M}{N} \leq \frac{2}{3}
\]

These values lie between those of the previous cases.

The equations of power and gain given above represent the average values of the quantities for an ensemble of antennas, each constructed from the same set of statistics. Each of the relations contains two terms, one of which is attributable to the average power distribution over the aperture and is denoted \( P_A \), and the other, called \( P_S \), to the average of the spurious radiation due to the omission of plates. The addition of the two effects to give the total power distribution may be justified by considering the two terms as independent of one another. The term \( P_A \) results from the regularity of the average, and
may be considered the desired pattern. The term $P_s$ reflects the imperfection of the device, and may be looked upon as a "noise", or the unwanted pattern. It is well to point out again that these terms are the result of averaging, and that the shape of a particular pattern would differ from these results. In the next section, measures of the deviations will be given. For the present, however, these relations give a good quantitative estimate of the ratios between the power radiated in the desired pattern and the spurious radiation. In Fig. 5 the average spurious level near the main beam is shown normalized to the main beam height, as a function of plate density and ratio of plate to aperture area, for the case of uniform illumination and uniform plate density. From this graph the spurious sidelobe levels for other tapers and plate densities may be deduced. These spurious levels are to be added to the desired or average sidelobes, much as noise power adds to a desired signal.

The graph of Fig. 6 shows the amounts of total power radiated on the average in the desired pattern and the spurious pattern, for the case of uniform plate density. In Fig. 7 the case of $1 - \frac{r^2}{a^2}$ plate density and uniform illumination is considered, and the graph of Fig. 8 illustrates the power distribution when a circular aperture has a plate density and a taper of the form $\sqrt{1 - \frac{r^2}{a^2}}$. 

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FIGURE 5

AVERAGE NEAR-IN SPURIOUS SIDELOBE LEVEL VS. PLATE DENSITY AND NUMBER OF PLATES
FIGURE 6

NORMALIZED RADIATED POWER VS. PLATE DENSITY

CASES 1 & 3 UNIFORM PLATE DENSITY
FIGURE 7

NORMALIZED RADIATED POWER VS. PLATE DENSITY

CASE 2: UNIFORM POWER ILLUMINATION
1-(△)² PLATE DENSITY, CIRCULAR APERTURE
NORMALIZED RADIATED POWER VS. NUMBER OF PLATES
CASE 4: $1-(\frac{r}{4})^2$ POWER TAPER CIRCULAR APERTURE
$\sqrt{1-(\frac{r}{4})^2}$ PLATE DENSITY
III

ARRAYS OF RANDOMLY POSITIONED PLATES

A complete radiating surface can be designed to provide a desirable radiation pattern, but the cost and difficulty of construction is great for large apertures. It is desired to know the effect of using partially filled apertures in order that performance similar to that of a large aperture antenna may be obtained from an array of smaller antennas.

Previously, the case of the random removal of plates from a large aperture has been considered. The aperture area is divided into a large number of plate locations or cells, and to each of these a probability of plate occurrence is assigned. The antenna performance is determined from the properties of the aperture autocorrelation function.

In this section, the plates are assumed to be positioned at random within the aperture. In an ensemble of arrays constructed in this manner, every location in the aperture is likely to have a plate centered there, while in the previous case the plate centers occurred only at a number of regular locations in the aperture.

A measured scattering pattern of a flat plate is used to provide a basis for the analysis of the properties of arrays of plates. The experimental curve is very close to the theoretical values and justifies their use in the analysis.

The properties of the field and power patterns for the case of an array of randomly located plates are found by using the analogy of this problem to that of the random walk. The probability of finding a certain level at one point in the sidelobe region of the pattern of a particular
array is determined. From this information the deviations from the average level of a pattern can be estimated. This prescribes the extent to which one antenna conforms to the average.

The ease of finding an array of randomly positioned plates with low sidelobes is estimated by finding the fraction of members of the ensemble of randomly positioned arrays that have all sidelobes below a specified level. This requires a sampling theorem for the pattern of a finite aperture to know the number of samples necessary to reconstruct the pattern. Also, the correlation between the pattern levels at different angles is found. A joint cumulative probability of sidelobe level is determined, and an upper bound to the pattern height between samples is set. The fraction of members of the ensemble of arrays with sidelobe levels below a specified level can then be stated.

3.1 Analysis of the Field and Power Patterns

Consider a square plate of side length d, oriented so that its sides are parallel to the coordinate axes x and y. The far field pattern of a single plate that has a uniform illumination across its surface of a field that is polarized in one direction should be

\[ \phi = \frac{d \sin \frac{\theta}{d} \sin \frac{\phi}{d} \sin \phi \sin \phi}{d \sin \frac{\theta}{d} \sin \frac{\phi}{d} \sin \phi \sin \phi}, \]

where \( \theta = k \sin \theta \cos \phi \) and \( \phi = k \sin \phi \). The results of an experimental measurement of the backscattering pattern of a 10\( \lambda \) square plate is shown in Fig. 9. Also plotted on the same graph is the theoretical pattern corresponding to a plate of that size. As may be seen, there is very little difference between the two patterns. It is assumed in the analysis that follows that the pattern of a single plate is given by the theoretical expression.

Let M square plates of side length d be placed at random in a rectangular aperture in the x-y plane, extending from -a to +a along the x axis
Backscattering Pattern of a 10 λ Square Plate

Figure 9
and from -b to +b along the y axis. The field pattern of this array is

\[ F(x, y) = \frac{\alpha^2 \sin^2 \psi_x \sin^2 \psi_y}{\psi_x^2 \psi_y^2} \sum_{m=1}^{M} f(x_m, y_m) e^{j(x_m \psi_x + y_m \psi_y)} \]

where \( f(x_m, y_m) \) is the normalized amplitude of the field at the plate located at \( x_m, y_m \). For simplification of the equations, the abbreviations

\[ S(a \psi_x, b \psi_y) = \frac{\sin^2 \psi_x \sin^2 \psi_y}{a \psi_x b \psi_y} \quad S_x = \alpha \frac{\sin^2 \psi_x}{\psi_x^2} \]

will be used throughout. Thus

\[ F(x, y) = S_x S_y \sum_{m=1}^{M} f(x_m, y_m) e^{j(x_m \psi_x + y_m \psi_y)} \]

If the probability of finding a plate is uniform across the aperture, the probability density may be written \( p(x, y) = \frac{1}{4a \psi_x b \psi_y} \), \( \iint_{-a}^{b} p(x, y) dx dy = 1 \)

The average voltage pattern of an ensemble of arrays is

\[ \langle F(x, y) \rangle = S_x S_y \sum_{m=1}^{M} f(x_m, y_m) e^{j(x_m \psi_x + y_m \psi_y)} \]

Using the probability density given above, the average field pattern becomes

\[ \langle F(x, y) \rangle = S_x S_y M \int_{-b}^{b} \int_{-a}^{a} f(x, y) e^{j(x \psi_x + y \psi_y)} \frac{1}{4a \psi_x b \psi_y} dx dy \]

However, the integral in the above expression is the normalized field pattern of the completely filled aperture, which may be designated

\[ F_0(\psi_x, \psi_y) = \frac{i}{4 a \psi_x b \psi_y} \int_{-a}^{a} \int_{-b}^{b} f(x, y) e^{j(x \psi_x + y \psi_y)} dx dy \]

yielding

\[ \langle F(\psi_x, \psi_y) \rangle = S_x S_y M F_0(\psi_x, \psi_y) \]

If the aperture illumination is uniform the average voltage pattern is

\[ \langle F(\psi_x, \psi_y) \rangle = S_x S_y M S(a \psi_x, b \psi_y) \]
The average power pattern for an array of randomly placed plates is

\[
\langle P(x, y) \rangle = \langle f(x, y) f^*(x, y) \rangle
\]

\[
= S_x S_y \left( \sum_m f^2(x_m, y_m) + \sum_{m \neq n} f(x_m, y_m) f(x_n, y_n) \right)^{1/2} (x_m - x_n) + j y_m (y_m - y_n)
\]

Considering the ensemble average, it can be seen that the first term within the brackets will range over all values of \( f(x, y) \). In the second term, the variables \( x_m, y_m \) and \( x_n, y_n \) are independent.

Thus \( \langle P(x, y) \rangle = S_x S_y \frac{M}{4ab} \left[ \int \int f^2(x, y) dx \, dy + 4ab M(M-1) F_o^2(x, y) \right] \)

For convenience the integral is denoted

\[
\frac{1}{4ab} \int_{x}^{y} f^2(x, y) dx \, dy = C_A(0)
\]

Then \( \langle P(x, y) \rangle = S_x^2 S_y^2 \left[ M C_A(0) + M(M-1) F_o^2 (x, y) \right] \)

and if the aperture illumination is uniform,

\[
\langle P(x, y) \rangle = S_x^2 S_y^2 \left[ M + M(M-1) S^2(a_x, b_y) \right]
\]

The variance of the field pattern is

\[
\sigma_F^2 = \langle PP^* \rangle - \langle P \rangle^2 = \langle P \rangle - \langle P \rangle^2
\]

Therefore

\[
\sigma_F^2 = S_x^2 S_y^2 \left[ M C_A(0) + M(M-1)F_o^2 (x, y) - S_x^2 S_y^2 M^2 F_o^2 (x, y) \right]
\]

\[
= M S_x^2 S_y^2 \left[ C_A(0) - F_o^2 (x, y) \right]
\]

If the aperture illumination is uniform, the variance becomes

\[
\sigma_F^2 = M S_x^2 S_y^2 \left[ 1 - S^2(a_x, b_y) \right]
\]
The variance of the field pattern gives an estimate of the average amount of power radiated in a given direction that exceeds that of the squared average field pattern.

The variance of the power pattern may be computed the mean square deviation from the average power pattern. This is

\[ \sigma_P^2 = \langle |P(x, y) - \langle P(x, y) \rangle|^2 \rangle \quad \text{but } P \text{ is real, so} \]

\[ \sigma_P^2 = \langle P^2(x, y) \rangle - \langle P(x, y) \rangle^2 = \langle \text{FF*FF*} \rangle - \langle P \rangle^2 \]

The first term is

\[ \text{FF*FF*} = S_x S_y \sum_{m} \sum_{n} \sum_{j} \sum_{k} e^{j(x_m x_j + x_n y_j + y_n y_k)} \]

This product has \(M^4\) terms. While the variables \(x_m, y_m, x_n, y_n, \ldots\) are random and thus range from \(-a\) to \(+a\) for \(x\) and \(-b\) to \(+b\) for \(y\), the occurrence of combinations such as \(x_m = x_n\) produces terms which must be examined closely, for the average depends upon the particular combination of \(m, n, j, \) and \(k\). These combinations are listed below. The \(y\) variation has been omitted for clarity.

**Table I**

<table>
<thead>
<tr>
<th>Combination</th>
<th>No. of Terms</th>
<th>Product</th>
<th>Average of Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m = n = j = k)</td>
<td>(M)</td>
<td>(\sum_{m} f(x_m, y_m))</td>
<td>(M \langle \Delta \rangle (0))</td>
</tr>
<tr>
<td>(m = n, j = k \neq n)</td>
<td>(M(M-1))</td>
<td>(\sum_{m} f^2(x_m) \sum_{j} f^2(x_j))</td>
<td>(M(M-1) \langle \Delta \rangle (0))</td>
</tr>
<tr>
<td>(m = k, j = n \neq k)</td>
<td>(M(M-1))</td>
<td>(\sum_{m} f^2(x_m) \sum_{j} f^2(x_j))</td>
<td>(M(M-1) \langle \Delta \rangle (0))</td>
</tr>
<tr>
<td>(m = n \neq j, k \neq j)</td>
<td>(M(M-1))</td>
<td>(\sum_{m} f^3 e^{j\psi x_m} \sum_{k} f e^{-j\psi y_k})</td>
<td>(M(M-1) F_0(\psi) F_0(\psi))</td>
</tr>
</tbody>
</table>
### Table I (cont'd)

<table>
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<th>Combination</th>
<th>No. of Terms</th>
<th>Product</th>
<th>Average of Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>m=n=k, j≠k</td>
<td>$M(M-1)$</td>
<td>$\sum_m f^3 e^{-i\psi_m} \sum_j f e^{i\psi_j}$</td>
<td>$M(M-1) F_{30}(\psi) F_0(\psi)$</td>
</tr>
<tr>
<td>m=j=k, n≠j</td>
<td>$M(M-1)$</td>
<td>$\sum_m f e^{-i\psi_m} \sum_n f^4 e^{i\psi_n}$</td>
<td>$M(M-1) F_{30}(\psi) F_0(\psi)$</td>
</tr>
<tr>
<td>n=j=k, k≠n</td>
<td>$M(M-1)$</td>
<td>$\sum_m f e^{i\psi_m} \sum_n f^3 e^{i\psi_n}$</td>
<td>$M(M-1) F_{30}(\psi) F_0(\psi)$</td>
</tr>
<tr>
<td>m=n, j≠k or n</td>
<td>$M(M-1)(M-2)$</td>
<td>$\sum_m f^2 \sum_n f e^{i\psi_x} \sum_{\kappa} f e^{i\psi_y}$</td>
<td>$M(M-1)(M-2) C_A(0) F_2^2(\psi)$</td>
</tr>
<tr>
<td>m=k, n≠k or j</td>
<td>$M(M-1)(M-2)$</td>
<td>$\sum_m f^2 \sum_n f e^{-i\psi_m} \sum_{\kappa} f e^{i\psi_y}$</td>
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<td>$\sum_m f e^{i\psi_m} \sum_n f^2 e^{-2i\psi_m} \sum_{\kappa} f e^{i\psi_y}$</td>
<td>$M(M-1)(M-2) F_{20}(2\psi) F_0^2(\psi)$</td>
</tr>
<tr>
<td>m=j, n=k≠j</td>
<td>$M(M-1)$</td>
<td>$\sum_m f^2 e^{i2\psi_m} \sum_n f e^{-2i\psi_{km}}$</td>
<td>$M(M-1) F_{30}^2(2\psi)$</td>
</tr>
<tr>
<td>m≠j, j≠k</td>
<td>$M(M-1)(M-2)(M-3)$</td>
<td>$\sum_m f \sum_n f e^{i\psi_m} \sum_{\kappa} f e^{i\psi_y} \sum_{\kappa} f e^{i\psi_{km}}$</td>
<td>$M(M-1)(M-2)(M-3) F_0^4(\psi)$</td>
</tr>
<tr>
<td>m≠k, n≠j</td>
<td>$M(M-1)(M-2)$</td>
<td>$\sum_m f e^{i\psi_m} \sum_n f e^{-i\psi_m} \sum_{\kappa} f e^{i\psi_y}$</td>
<td>$M(M-1)(M-2) F_{30}^2(2\psi)$</td>
</tr>
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<td>$M(M-1)(M-2)$</td>
<td>$\sum_m f e^{i\psi_m} \sum_n f e^{-i\psi_m} \sum_{\kappa} f e^{i\psi_y}$</td>
<td>$M(M-1)(M-2)(M-3) F_0^4(\psi)$</td>
</tr>
</tbody>
</table>

Combining similar terms,

$$\langle P^2(\psi) \rangle = M C_{2\eta}(0) + 2M(M-1) C_A^2(0) + 4M(M-1) F_{30}(\psi) F_0(\psi) + 4M(M-1)(M-2) C_A(0) F_0^2(\psi)$$

$$+ 2M(M-1)(M-2) F_{20}(2\psi) F_0^2(\psi) + M(M-1) F_{30}^2(2\psi) + M(M-1)(M-2)(M-3) F_0^4(\psi)$$
where 

\[ C_A(b) = \frac{1}{4ab} \int_{-b}^{b} \int_{-b}^{b} f(x,y) \, dx \, dy \] 

\[ C_B(b) = \frac{1}{4ab} \int_{-b}^{b} \int_{-b}^{b} f^2(x,y) \, dx \, dy \] 

\[ F_{m_0}(x,y) = \frac{1}{4ab} \int_{-b}^{b} \int_{-b}^{b} f^{m_0}(x,y) e^{i\pi x y} \, dx \, dy \] 

\[ F_{e_0}(x,y) = \frac{1}{4ab} \int_{-b}^{b} \int_{-b}^{b} f^{e_0}(x,y) e^{i\pi x y} \, dx \, dy \]

The square of the average power is

\[ \langle P \rangle^2 = S_x S_y \left[ M^2 C_A^2 + 2M^2(M-2)C_A C_B F_{m_0} + M^2(M-2)^2 F_{e_0} \right] \]

and the variance is

\[ \sigma_p^2 = S_x S_y \left[ M C_A^2 + M(M-2)C_A^2 + 4M(M-2)F_{m_0} + 2M(M-1)(M-2)C_A F_{m_0} + 2M(M-1)(M-2)F_{e_0} \right] \]

For the case of uniform illumination, \( F_{m_0} = F_o = S \) , \( C_A = C_B = 1 \) and

\[ \langle P^2(x,y) \rangle = S_x S_y \left[ M(M-1) + 4M(M-2)S^2 + 2M(M-1)(M-2)S^2 \right] \]

The variance for the uniform illumination case is

\[ \sigma_p^2 = S_x S_y M(M-1) \left[ 1 + 2(M-2)S^2 - 2(M-3)S^2 \right] \]

when \( x = y = 0 \), \( \sigma_p^2(0) = M(M-1) \left[ 1 + 2M - 4M + 6 + 2M - 4 + 1 \right] = 0 \)

and for large \( M \)

\[ \sigma_p^2 \approx 2M^3 S^2 \left[ 1 - 2S^2 + S \right] \]
3.2 The Behavior of the Pattern Function

The field pattern function \( F(\psi_x, \psi_y) = S_x S_y \sum_m^M f(x_m, y_m) e^{i \psi_x x_m + i \psi_y y_m} \)
may be considered as the sum of \( M \) independent steps of arbitrary direction and length proportional an even function \( f \) of the variables \( x_m \) and \( y_m \). The variables \( x_m \) and \( y_m \) are randomly selected from values between \(-a\) and \(+a\) for \( x_m \) and \(-b\) to \(+b\) for \( y_m \). The pattern function is rewritten

\[
F(\psi_x, \psi_y) = S_x S_y \left[ \sum f(x_m, y_m) \cos(\psi_x x_m + \psi_y y_m) + j f(x_m, y_m) \sin(\psi_x x_m + \psi_y y_m) \right]
\]

The power pattern is

\[
P(\psi_x, \psi_y) = S_x S_y \left[ (\sum X_m)^2 + (\sum Y_m)^2 \right]
\]

and the average power pattern is

\[
\langle P(\psi_x, \psi_y) \rangle = S_x^2 S_y^2 \left[ \langle (\sum X_m)^2 \rangle + \langle (\sum Y_m)^2 \rangle \right]
\]

If the two terms of \( P \) are normally distributed, independent, and have equal variance and zero mean, the distribution is called a Rayleigh distribution. If all conditions but the last are met, the distribution becomes a modified Rayleigh distribution. The power may then be written, with no loss of generality, as

\[
P(\psi_x, \psi_y) = \left[ (a + \sum X_m)^2 + (\sum Y_m)^2 \right]
\]

If \( M \) is a large number, the distributions of \( X \) and \( Y \) will tend to be normal. The variance of \( X \) is

\[
\sigma_X^2 = \langle (\sum X_m)^2 \rangle - \langle (\sum X_m) \rangle^2
\]
\[
\begin{align*}
\langle \sum_{m} f^2(x_m, y_m) \cos^2(\psi_x x_m + \psi_y y_m) \rangle - M F_0^1(\psi_x, \psi_y) \\
= M \left[ \frac{1}{2} C_n(0) + \frac{1}{2} F_{20}(2a\psi_x, 2\psi_y) - F_0^2(\psi_x, \psi_y) \right]
\end{align*}
\]

where \( C_n(0) = \frac{1}{4ab} \int_{-b}^{b} \int_{-a}^{a} f(x, y) \, dx \, dy \), \( F_{20}(2a\psi_x, 2\psi_y) = \frac{1}{4ab} \int_{-b}^{b} \int_{-a}^{a} f(x, y) \cos 2\psi_x x \cos 2\psi_y y \, dx \, dy \)

and

\( F_0(\psi_x, \psi_y) = \frac{1}{4ab} \int_{-b}^{b} \int_{-a}^{a} f(x, y) \cos \psi_x x \cos \psi_y y \, dx \, dy \), \( f \) is even.

In a similar manner the variance of \( Y \) is found to be

\[ \sigma_Y^2 = M \left[ \frac{1}{2} C_n(0) - \frac{1}{2} F_{20}(2a\psi_x, 2\psi_y) \right] \]

The condition \( \sigma_x^2 = \sigma_y^2 \) is met at a number of points where

\[ F_{20}(2a\psi_x, 2\psi_y) = F_0^2(\psi_x, \psi_y) \]

These do not, in general, correspond to the nulls of either pattern. Thus, if the illumination of the aperture is tapered, the distribution is not Rayleigh at the nulls.

For the case of uniform illumination the variances become

\[ \sigma_x^2 = M \left[ \frac{1}{2} + \frac{1}{2} S(2a\psi_x, 2b\psi_y) - S^2(a\psi_x, b\psi_y) \right] \]

and

\[ \sigma_y^2 = M \left[ \frac{1}{2} - \frac{1}{2} S(2a\psi_x, 2b\psi_y) \right] \]

At the nulls of \( S(a\psi_x) \) or \( S(b\psi_y) \) the variances become \( \sigma_x^2 = \sigma_y^2 = \frac{1}{2} M \). Since the number \( M \) of elements in the array is large, the distributions of \( X \) and \( Y \) may be assumed to be normal.
The conditions for the independence of $X$ and $Y$ are that their joint probability density is Gaussian and $\langle XY \rangle = 0$.

But $XY = \sum f(x_m, y_m) \cos(\psi x_m + \psi y_m) \sum f(x_n, y_n) \sin(\psi x_n + \psi y_n)$

and $\langle XY \rangle = \frac{M}{4ab} \int_{-b}^{b} \int_{-a}^{a} f(x, y) \cos \psi x \sin \psi y \sin \psi x \cos \psi y \, dx \, dy$

$= \frac{M}{4ab} \int_{-b}^{b} \int_{-a}^{a} f(x, y) \sin 2\psi x \sin 2\psi y \, dx \, dy$

$= 0$, as $f(x, y)$ is even in $x$ and $y$.

Thus, two of the three necessary conditions for a modified Rayleigh condition have been met. The distributions of $X$ and $Y$ have been shown to be uncorrelated, and are assumed to be normal. In the works of Ruze and Rondinelli, where the radiating element positions are fixed and error currents are added to the no-error distribution, the variances $\sigma_X^2$ and $\sigma_Y^2$ are equal, and the field strength obeys a modified Rayleigh distribution. However, in the case considered here where the plates are randomly placed and the field strengths at the plates are specified by their location, the variances are not equal at all points.

The determination of the probability density function $W(r)$ for the magnitude of two independent normal distributions $X$ and $Y$ with unequal variances and a non-zero mean is given below.

The distribution of $Y$ is

$$W(Y) = \frac{1}{\sqrt{2\pi} \sigma_Y} e^{-\frac{Y^2}{2\sigma_Y^2}}$$

and the distribution of $X$, which has a mean of $\mu$ is

$$W(X) = \frac{1}{\sqrt{2\pi} \sigma_X} e^{-\frac{(X-\mu)^2}{2\sigma_X^2}}$$

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The distribution of \( r = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2} \) is \( \mathcal{W}(r) \), where

\[
\mathcal{W}(r) = \int_{-\pi}^{\pi} \mathcal{W}(x) \mathcal{W}(y) \, r \, d\theta, \quad x = r \cos \theta, \quad y = r \sin \theta
\]

Substituting,

\[
\mathcal{W}(r) = \frac{r}{2\pi \sigma_x \sigma_y} e^{\frac{-1}{2} \left( \frac{r^2}{\sigma_x^2} + \frac{r^2}{\sigma_y^2} \right)} \int_{-\pi}^{\pi} e^{\frac{-1}{2} \left[ r^2 \cos^2 \theta \left( \frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2} \right) - \frac{2ar}{\sigma_y^2} \cos \theta \right]} \, d\theta
\]

The integral is of the form

\[
\int_{-\pi}^{\pi} e^{b \cos \theta + c \cos \theta} \, d\theta = e^{\frac{b}{2}} \int_{-\pi}^{\pi} e^{b \cos \theta + \frac{b}{2} \cos 2\theta} \, d\theta
\]

To evaluate this integral, the integrand is expanded as a Fourier series,

\[
e^{iz(r-\frac{1}{2})} = \sum_{n=-\infty}^{\infty} J_n(z) r^n
\]

where \( r = e^{j\omega t} \), \( z = -j\omega \)

Then

\[
e^{\omega \cos \theta} = J_0(j\omega) + 2 \sum_{n=1}^{\infty} e^{-jn\frac{\pi}{2}} J_n(j\omega) \cos n\theta
\]

or

\[
e^{\omega \cos \theta} = I_0(\omega) + 2 \sum_{n=1}^{\infty} I_n(\omega) \cos n\theta
\]

and the integral is

\[
e^{\frac{b}{2}} \int_{-\pi}^{\pi} \left[ I_0(c) + 2 \sum_{n=1}^{\infty} I_n(c) \cos n\theta \right] \left[ I_0\left( \frac{b}{2} \right) + 2 \sum_{m=1}^{\infty} I_m\left( \frac{b}{2} \right) \cos 2m\theta \right] \, d\theta
\]

This integral is zero for any term for which \( n \neq 2m \). The result of integration is

\[
2\pi \, e^{\frac{b}{2}} \left[ I_0(c) I_0\left( \frac{b}{2} \right) + 2 \sum_{m=1}^{\infty} I_m\left( \frac{b}{2} \right) I_{2m}\left( c \right) \right]
\]

where \( b = \frac{r^2}{2}\left( \frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2} \right) \) and \( c = \frac{ar}{\sigma_y^2} \)

The probability distribution is
Several results can be deduced as limiting cases. First, if the variances are equal, a modified Rayleigh distribution is obtained, as

$$W(r) = \frac{2r}{\sigma_x \sigma_y} e^{-\frac{r^2}{2} (\frac{1}{\sigma_x^2} \frac{1}{\sigma_y^2})} \int_0^\infty \left( \frac{1}{\sigma_x^2} \frac{1}{\sigma_y^2} \right) \mathcal{I}_0 \left( \frac{2\sigma_x}{\sigma_y} \right)$$

Second, if the mean value $a$ is zero, but the variances remain equal, a somewhat similar distribution results, as

$$W(r) = \frac{r}{\sigma_x \sigma_y} e^{-\frac{1}{2} r^2 \left( \frac{1}{\sigma_x^2} + \frac{1}{\sigma_y^2} \right)} \int_0^\infty \left( \frac{1}{\sigma_x^2} \frac{1}{\sigma_y^2} \right) \mathcal{I}_0 \left( \frac{r^2}{2} \left( \frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2} \right) \right)$$

The probability of occurrence at an angle $\psi_x, \psi_y$ of a particular value $F' = \frac{E}{\sigma_x \sigma_y}$ of the voltage pattern of the array of randomly distributed plates may be written as

$$W(F) = \frac{F}{\sigma_x \sigma_y} e^{-\frac{1}{2} \left( \frac{E^2}{\sigma_x^2} + \frac{E^2}{\sigma_y^2} \right)} \int_0^\infty \left( \frac{1}{\sigma_x^2} \frac{1}{\sigma_y^2} \right) \mathcal{I}_0 \left( \frac{E^2}{\sigma_x^2} \right) \mathcal{I}_0 \left( \frac{E^2}{\sigma_y^2} \right) + 2 \sum_{n=1}^\infty \sum_{m=1}^\infty \mathcal{I}_0 \left( \frac{E^2}{\sigma_x^2} \right) \mathcal{I}_0 \left( \frac{E^2}{\sigma_y^2} \right)$$

The cumulative probability, or the probability that the magnitude of the field will exceed a specified value, is equal to

$$W(F > F_x) = \int_{F_x}^{\infty} W(F) dF = 1 - \int_0^{F_x} W(F) dF$$

In dealing with the above integral for a particular $\psi_x$ and $\psi_y$ it is important to note the variance at these coordinates. For $\psi_x$ and $\psi_y$ near zero, the variance is small. This reflects the physical situation.
of all the elements adding in the main beam regardless of their location. At large values of \( a_{X} \psi \) or \( b_{Y} \psi \), however, the two variances tend towards the value of \( \frac{1}{2} C_{A} (0) \). Excluding the region around the main beam from attention, the terms \( \frac{1}{\sigma_{x}^2} \) and \( \frac{1}{\sigma_{y}^2} \) are small and nearly equal, and the term \( \left( \frac{1}{\sigma_{x}^2} - \frac{1}{\sigma_{y}^2} \right) \) can be neglected. Similarly, the average pattern is very small compared to the variance for large \( a_{X} \psi \) and \( b_{Y} \psi \). From consideration of the probability function in the regions of the far out side lobes, one is led to expect only minute deviations from a Rayleigh distribution of

\[
\mathcal{W} (F') = \frac{2F'}{\mu} e^{-\frac{F'^2}{\mu}}
\]

where \( \sigma^2 = \sigma_{x}^2 + \sigma_{y}^2 \), \( F_0^2 << \sigma^2 \), \( a_{\psi_x}, b_{\psi_y} >> 1 \) and a cumulative probability

\[
\mathcal{W} (F' > F_{z} ') = e^{-\frac{F_{z}^2}{\mu}}
\]

This result comes about from the convenient fact that the variances for all values of \( \psi_{X} \) and \( \psi_{Y} \) except the main beam region are very large in comparison to the average pattern or the expected side lobes. This is not the case in an analysis of current errors in an array, where the error currents ( and hence the variance ) may be quite small in comparison to the desired values. In the above analysis the random variables are not errors but are the positions of the radiating plates, and as these can range from one end of the aperture to the other, the variances become very large for large apertures. As a result, the expected pattern magnitude nearly obeys a Rayleigh distribution at appreciable angles.

Previously, the average power pattern has been found to be

\[
\langle P \rangle = S_{x}^{2} S_{y}^{2} \left[ M C_{A} (0) + M (M-1) F_{o}^{2} (\psi_{X}, \psi_{Y}) \right]
\]
At angles where the second term is small,

\[ \langle F^2 \rangle \approx \frac{S_x^2}{S_y^2} M C_A(0) \]

The probability that the magnitude of the field pattern exceeds \( \varepsilon F_{rms} \) is

\[ \mathcal{W}(F > \varepsilon F_{rms}) = \mathcal{W}(F > \varepsilon \sqrt{\langle F^2 \rangle}) = e^{-\varepsilon^2} \]

and thus is obtained the result that the probability of sidelobes that are greater than \( \varepsilon \) times the r.m.s. field is \( e^{-\varepsilon^2} \). This is the cumulative sidelobe probability at an angle \( \varphi_x, \varphi_y \). To find the number of sidelobes in a pattern, it is necessary to determine the correlation between the pattern at \( \varphi_x^1 \) and \( \varphi_x^2 \). This is

\[ \rho_{12} = \langle [F(\varphi_x^1, \varphi_y^1) - \langle F(\varphi_x^1, \varphi_y^1) \rangle][F(\varphi_x^2, \varphi_y^2) - \langle F(\varphi_x^2, \varphi_y^2) \rangle] \rangle \\
= \langle F(\varphi_x^1, \varphi_y^1)F(\varphi_x^2, \varphi_y^2) \rangle - \langle F(\varphi_x^1, \varphi_y^1) \rangle \langle F(\varphi_x^2, \varphi_y^2) \rangle \]

But \( \langle F \rangle = F_0 \) and \( F_0(\varphi_x, \varphi_y) = \int f(x, y) e^{j \varphi_x x + j \varphi_y y} dx dy \)

which yields

\[ \rho_{12} = M \left[ F(\varphi_x^1 - \varphi_x^2, \varphi_y^1 - \varphi_y^2) - F_0(\varphi_x^1, \varphi_y^1) F_0^*(\varphi_x^2, \varphi_y^2) \right] \]

A set of points are uncorrelated or linearly independent if the correlation \( \rho \) formed between any pair of points is zero. As an example, consider the case of uniform illumination, or \( f(x, y) = \frac{1}{4ab} \)

Then

\[ \rho_{12} = M \left[ S a(\varphi_x^1 - \varphi_x^2), b(\varphi_y^1 - \varphi_y^2) - S(a\varphi_x^1, b\varphi_y^1) S(a\varphi_x^2, b\varphi_y^2) \right] \]

If \( \varphi_x^1 = \frac{n_x}{\lambda} \) and \( \varphi_x^2 = \frac{n_x}{\lambda} \), then for any \( n \neq m \), \( \rho_{nm} = 0 \).

Therefore, all points in \( \varphi_x \) and \( \varphi_y \) that occur at the nulls of the uniform illumination pattern are linearly independent. For a rectangular aperture of area \( 4ab \) there are \( \left( \frac{4a}{\lambda^2} + 1 \right) \left( \frac{4b}{\lambda^2} + 1 \right) \) of these points within the range of \(-k \leq \varphi_x, \varphi_y \leq +k\).
The pattern of an antenna of finite size is determined by a set of samples in an analogous manner to that used for time network analysis. Let

\[ F(\psi_x, \psi_y) = \int_{-\alpha}^{\alpha} \int_{-\beta}^{\beta} f(x, y) e^{j \psi_x x + j \psi_y y} \, dx \, dy \]

and consider \( f \) to be the product of a periodic function

\[ f_p(x, y) = \sum_{m, n} f(x - 2na, y - 2nb) \]

and a window function \( \text{rect}(\frac{x}{2a}, \frac{y}{2b}) \)

Then

\[ f_p(x, y) = \sum D_{nm} e^{-i \frac{2\pi mn}{2a} x - i \frac{2\pi nm}{2b} y} \]

and

\[ D_{nm} = \frac{1}{4ab} \int_{-\alpha}^{\alpha} \int_{-\beta}^{\beta} f(x, y) e^{i \frac{2\pi mn}{2a} x + i \frac{2\pi nm}{2b} y} \, dx \, dy \]

When \( \tau_x = \frac{2\pi n}{2a} \), \( \tau_y = \frac{2\pi m}{2b} \), \( F(\psi_x, \psi_y) = D_{nm} \), and so

\[ f_p(x, y) = \frac{1}{4ab} \sum F \left( \frac{n\tau}{2a}, \frac{m\tau}{2b} \right) e^{-i \frac{2\pi mn}{2a} x - i \frac{2\pi nm}{2b} y} \]

Since \( f(x, y) = f_p(x, y) \text{rect}(\frac{x}{2a}, \frac{y}{2b}) \), \( F(\psi_x, \psi_y) = \int_{-\alpha}^{\alpha} \int_{-\beta}^{\beta} f_p(x, y) e^{i \psi_x x + i \psi_y y} \, dx \, dy \)

\[ F(\psi_x, \psi_y) = \frac{1}{4ab} \sum F \left( \frac{m\tau}{2b}, \frac{n\tau}{2a} \right) \frac{\sin a(\psi_x - \frac{2\pi n}{2a})}{a(\psi_x - \frac{2\pi n}{2a})} \frac{\sin b(\psi_y - \frac{2\pi m}{2b})}{b(\psi_y - \frac{2\pi m}{2b})} \]

Let the far field pattern have samples that are negligible outside \( |\psi_x| \leq k \)

The samples occur for every \( \tau_x = \frac{n\pi}{2a} \), \( \tau_y = \frac{m\pi}{2b} \), and over a range of \(-k \leq \psi_x, \psi_y \leq +k\) the total number of samples is

\[ \left( \frac{4a}{\lambda} + 1 \right) \left( \frac{4b}{\lambda} + 1 \right) \]

Thus the number of samples necessary to specify the pattern equals the total number of uncorrelated points of the pattern of the uniformly illuminated random array. The probability of the magnitude of the
pattern at any of the sample points exceeding a specified value $F_1$ is

$$W_{all \; below \; F_1} = \left[ 1 - e^{-\frac{F_1}{a}} \right]^{(\frac{a}{\lambda} + 1)(\frac{b}{\lambda} + 1)} \approx \left[ 1 - e^{-\frac{F_1}{a}} \right]^{15a b \lambda}$$

and the fraction of arrays in the ensemble that have all sample points below $\varepsilon F_{rms}$ is approximately

$$\frac{1}{15a \lambda}$$

The percentage of members of an ensemble of arrays constructed by locating elements within an area $4ab$ that have low sidelobes is very small unless the array contains a very large number of elements. As an example, an array containing $5 \times 10^3$ elements out of a possible $1 \times 10^4$ has an average side lobe level of -37 decibels. The fraction of arrays in the ensemble that have all sidelobes more than 27 db down is .55.

The pattern of the array can be reconstructed by using the values of the pattern at the sample points. Consider one of the arrays with all values at the sample points less than or equal to $\varepsilon F_{rms}$. The pattern is

$$F(\phi_x, \phi_y) = \sum_{n=-N_x}^{N_x} \sum_{m=-N_y}^{N_y} F \left( \frac{n a \phi_x}{2a}, \frac{m b \phi_y}{2b} \right) \sin \alpha(\phi_x - \frac{\alpha n}{a}) \sin \beta(\phi_y - \frac{\beta m}{b}) \frac{a(\phi_x - \frac{\alpha n}{a})}{b(\phi_y - \frac{\beta m}{b})}$$

The maximum value of $F$ is

$$F_{max} = \sum_{n, m} \varepsilon F_{rms} \sin \alpha(\phi_x - \frac{\alpha n}{a}) \sin \beta(\phi_y - \frac{\beta m}{b}) \frac{a(\phi_x - \frac{\alpha n}{a})}{b(\phi_y - \frac{\beta m}{b})}$$

which cannot exceed

$$\sum_{n, m} \frac{\varepsilon F_{rms}}{a b (\phi_x - \frac{\alpha n}{a})(\phi_y - \frac{\beta m}{b})}$$

This sum is less than

$$\sum_{n} \sum_{m} \frac{16 \varepsilon F_{rms}}{n^2 (2 \pi n)(2 \pi m)}$$

Approximating the above expression by an integral yields

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\[ F_{\text{max}} < \frac{16 \, E}{\pi^2} \, F_{\text{rms}} \, \ln (2N_x + 1) \, \ln (2N_y + 1) \]

or

\[ F_{\text{max}} < \frac{16 \, E}{\pi^2} \, F_{\text{rms}} \, \ln \left( \frac{4 \, x}{\lambda} + 1 \right) \, \ln \left( \frac{4 \, y}{\lambda} + 1 \right) \]

This is an upper bound to the sidelobes of the pattern function of that fraction of members of the ensemble that have all sample point values less than \( \varepsilon F_{\text{rms}} \). For the most part, the pattern lobes will be considerably less than this upper bound, and the majority will be near the average sidelobe level. However, the requirement that all lobes be low is a stringent one, and a very large number of elements and a well filled array are necessary to satisfy it.
IV

THE APPARENT TEMPERATURE OF THE ANTENNA

An antenna consisting of a large number of plates can be designed to provide a desirable radiation pattern from the standpoint of resolution and side-lobes. However, if the antenna is to be used as a radio astronomy instrument, it is necessary that the apparent temperature of the antenna exhibit desirable characteristics. In general, an attempt is made to keep the contributions to the apparent antenna temperature from sources other than the signal as low as possible. The methods of radiometry do not necessarily require that the antenna temperature be low in order to have high sensitivity; however, fluctuations of the undesired part of the antenna temperature cause a limit to the performance.

In this section the apparent temperature of a multiple plate array is determined. The effect of the ground beneath the antenna is discussed, and calculations of the reflected and absorbed energy are performed, based on experimental data of ground characteristics. These results are used to determine the apparent temperature of a multiple plate array in which a fraction of the area of the ground beneath the aperture is illuminated. Another arrangement of plates is considered where the ground is completely in the geometric shadow of the feed for the case of a vertically directed radiation pattern. The change in the area of the ground uncovered due to the motion of plates in directing the beam is calculated.

4.1 General

The apparent temperature of an object viewed by an antenna is defined as the temperature of a black body subtending the same solid angle from the an-
tenna as the object and producing the same flux density at the antenna. The relation between the emission of a black body and its temperature is given by Planck's law,

\[
E = \frac{2\pi c}{\lambda^3} \frac{1}{e^{\frac{hc}{k\lambda T}} - 1}
\]

where \( E \) is the energy emitted per steradian per second per unit frequency band per unit area, \( h \) is Planck's constant, \( k \) is Boltzmann's constant, \( c \) is the velocity of light, and \( T \) is the temperature of the body. At radio frequencies, where \( hc \ll k \lambda T \), this may be approximated by \( E = \frac{2\pi T}{\lambda^2} \), the classical Rayleigh-Jeans statement of black body radiation.

Consider an aerial in thermodynamic equilibrium with a body at a temperature \( T \). Let the antenna be matched by a load also at a temperature \( T \). The load will deliver an amount of thermal power \( kT \Delta f \) in a frequency band \( \Delta f \) to the antenna and a fraction of this, \( \alpha kT \Delta f \), will be absorbed by the body. Since the system is in equilibrium, an equal amount of power will be delivered by the body to the aerial. If the system is not in thermal equilibrium, this power will be available so long as the body temperature remains unchanged. If there are a number of bodies at different temperatures the power available to the antenna is

\[
P = \sum \alpha_n kT_n \Delta f = k \Delta f T_a
\]

where \( T_a \) is the effective antenna temperature. Thus \( T_a = \sum \alpha_n T_n \) with \( \alpha_n \) being the fraction of the power radiated from the antenna that is absorbed in the \( n \)th body. This may be written in an integral form using the antenna gain as

\[
T_a = \frac{\int T_b(\theta, \phi) G(\theta, \phi) d\Omega}{\int G(\theta, \phi) d\Omega} = \frac{1}{4\pi} \int T_b(\theta, \phi) G(\theta, \phi) d\Omega
\]
where \( T_b \) is the brightness temperature in the direction \( \theta, \varphi \).

If there is a power loss ratio of \( \varepsilon \) in the antenna by components at temperature \( T_o \), the final antenna temperature is

\[
T_{fa} = (1 - \varepsilon) T_a + \varepsilon T_o
\]

where \( T_a \) is the effective temperature of a lossless antenna of identical physical dimensions. Discussions of these relations and the principles behind them may be found in Pawsey\(^6\) and in several recent articles.\(^7,10\)

### 4.2 The Reflection Characteristics of the Ground

The ground beneath the antenna has a profound effect upon the antenna temperature. If the antenna were transmitting, a fraction of the power would pass between the plates and strike the ground. A part of this power would be absorbed and the remainder would be reflected. If the antenna were used for reception, energy from unwanted directions would be reflected by the ground into the feed along with thermal energy radiated by the ground.

In order to determine the relative amounts of reflected and absorbed power, it is necessary to know the bistatic reflection properties of the ground at the frequencies of interest. Results obtained from an L band measurement program\(^11\) indicate that surfaces such as asphalt roadways or short grass act essentially as coherent smooth scatterers. The relative strength of the reflected specular wave is given by the Fresnel coefficients

\[
R_h = \frac{\cos \alpha - (\varepsilon - \sin^2 \alpha)^{\frac{1}{2}}}{\cos \alpha + (\varepsilon - \sin^2 \alpha)^{\frac{1}{2}}}
\]

\[
R_v = \frac{\varepsilon \cos \alpha - (\varepsilon - \sin^2 \alpha)^{\frac{1}{2}}}{\varepsilon \cos \alpha + (\varepsilon - \sin^2 \alpha)^{\frac{1}{2}}}
\]
where $R_h$ is the reflection coefficient for a wave with the incident E field parallel to the surface, $R_v$ is the reflection coefficient for the orthogonal polarization, $\alpha$ is the angle of incidence measured from the vertical and $\varepsilon$ is the apparent relative dielectric constant of the medium. For surfaces such as short grass or asphalt the measured relative dielectric constant is about 5. With this information, the effect of the ground on the apparent antenna temperature may be estimated for the case of frequencies in the vicinity of 1000 Mcps and for smooth terrain.

4.3 The Effect of Ground Reflection and Absorption on the Antenna Temperature

In order to obtain a quantitative description of the antenna temperature it is necessary to assume a model for the antenna. The arrangement to be considered is a large number of square plates positioned to provide a directional radiation pattern. The ground beneath the plates is considered planar insofar as the relative height at any point is a negligible fraction of the diameter of the antenna. However, in the analysis of the ground reflection, the height is assumed to vary irregularly but slowly over large intervals, resulting in a diffusion of the specular reflection.

Consider an incompletely filled aperture of $M$ plates of side length $d$, arranged within a circular aperture of radius $a$ as described in Sec. 2.4.2. The antenna will be analyzed in transmission of $\pi a^2$ watts of power. The feed pattern is such that each plate radiates the same amount of power, and the ground at the locations where there are no plates receives the same power density as do the plates. The power reflected by the plates is $\frac{M}{N} \pi a^2$
and the power striking the ground is \((1 - \frac{N}{N}) \pi a^2\) where the effects of feed illumination of areas outside the antenna and diffraction over the plate edges is neglected. The principal component of the field that strikes the area near the center of the aperture is polarized horizontally regardless of the feed polarization. For the calculations that follow, the fields are considered to be horizontally polarized, although similar expressions may be evaluated for the orthogonal feed polarization.

If the surface of the ground at the location of a missing plate is sufficiently smooth to reflect the incident field coherently, the angular distribution of the reflected field from one location is

\[
E(\theta, \phi) \sim E_0 \cos \theta e^{i k r_n(a_n)} \frac{\sin k \frac{d}{2} \sin \theta \cos \phi - \sin \alpha_n \cos \phi_n}{\sin \theta \cos \phi - \sin \alpha_n \cos \phi_n} \frac{\sin k \frac{d}{2} \sin \theta \sin \phi - \sin \alpha_n \sin \phi_n}{\sin \theta \sin \phi - \sin \alpha_n \sin \phi_n}
\]

where \(a_n\) is the angle between the direction of the feed and the vertical, and \(\phi_n\) is the azimuthal coordinate of the location. The resultant field pattern from the contributions of all spaces in the array is

\[
E_{1}(\theta, \phi) \sim \sum_{i=1}^{N-N} E_0 \cos \theta e^{i k r_n(a_n)} \frac{\sin k \frac{d}{2} (\psi_x - k \sin \alpha_n \cos \phi_n)}{(\psi_x - k \sin \alpha_n \cos \phi_n)} \frac{\sin k \frac{d}{2} (\psi_y - k \sin \alpha_n \sin \phi_n)}{(\psi_y - k \sin \alpha_n \sin \phi_n)}
\]

where \(r_n\) is the sum of the distance from the feed to the nth plate center and the distance from the plate center to a reference plane in the direction \(\theta, \phi\). It is safe to assume that the quantity \(k r_n\) will vary in an irregular manner as a function of \(n\), and all possible values of \(e^{ikr_n}\) will be equally likely to occur. Physically, this corresponds to a surface smoothness such that each space acts as a coherent scatterer with the phase centers of the spaces not arranged in a regular pattern but at different and irregular
heights. Under these conditions the power pattern is

\[ P_r(\theta, \phi) \sim \sum_{n=1}^{N-M} E_0^2 \cos^2 \theta R_n^2(\alpha_n) \frac{\sin^2 \left( \frac{\psi_x - k \sin \alpha \cos \phi}{\psi_x - k \sin \alpha \cos \phi_n} \right)}{\psi_x - k \sin \alpha \cos \phi_n} \frac{\sin^2 \left( \frac{\psi_y - k \sin \alpha \sin \phi}{\psi_y - k \sin \alpha \sin \phi_n} \right)}{\psi_y - k \sin \alpha \sin \phi_n} \]

If the aperture contains a very large number of plates and spaces, the character of the reflected power due to the ground at the spaces can be found from a different approach. Consider an element of surface area located at an angle \( \alpha \) from the feed and subtending a solid angle \( \Delta \Omega_1 \), at the feed. The surface will intercept an amount of energy from the feed, and a fraction \( R_h^2 \) will be reflected in the solid angle \( \Delta \Omega_2 \) centered about the specular reflection angle. If the surface is planar and the reflected wave is coherent, the solid angle \( \Delta \Omega_1 = \Delta \Omega_2 \) and the incident power is reflected into the far field with a change in intensity due to the reflection coefficient and a change in direction corresponding to the laws of reflection. If there are a very large number of spaces in the aperture, radiation will be supplied to nearly all directions and the effect of the surface in terms of the reflected power is as if an image feed were positioned beneath the aperture plane at a distance equal to the feed height above the plane. The power pattern of the image feed is that of the real feed multiplied by the plate density and by the square of the reflection coefficient. For the case of uniform plate density (see Sec. 2.4.2)

\[ P_{\text{image}}(\theta, \phi) = \left(1 - \frac{M}{N}\right) R_h^2(\alpha) P_{\text{feed}}(\alpha, \phi) \]

In general the feed pattern is symmetric in \( \phi \), and ideally would extend only from \( \alpha = 0 \) to \( \alpha = \alpha_0 \), the angle at which the outer extremity of the aperture is reached. The reflected power will be a broad pattern without rapid variations.
The mean square value of the reflection coefficient can be found by integrating the average contributions of each part of the aperture area. A circular area of ground with the reflection characteristics described in Sec. 4.2 is assumed. To evaluate the integral, the relation between $\alpha$ and $r$ must be determined. If the feed is at a height $f$ above the aperture plane, the radial distance in the aperture is $r = f \tan \alpha$, and the mean square reflection coefficient over the aperture area is

$$\overline{R_h^2} = 2\pi \int_0^{\alpha_o} R_h^2 (\alpha) f^2 \tan \alpha \sec^2 \alpha \, d\alpha$$

for uniform aperture illumination. Substituting for $R_h$ yields

$$\overline{R_h^2} = 2\pi \int_0^{\alpha_o} \left[ \frac{\cos \alpha - (\varepsilon - \sin^2 \alpha) \frac{f}{\sqrt{\varepsilon}}}{\cos \alpha + (\varepsilon - \sin^2 \alpha) \frac{f}{\sqrt{\varepsilon}}} \right] f^2 \tan \alpha \sec^2 \alpha \, d\alpha$$

This integral can be closely approximated by the simpler expression

$$\overline{R_h^2} \approx 2\pi f^2 \int_1^{\cos \alpha_o} \left( \frac{x - \frac{f}{\sqrt{\varepsilon}}}{x + \frac{f}{\sqrt{\varepsilon}}} \right) \frac{1}{x^3} \, dx$$

$$\overline{R_h^2} \approx \pi f^2 \left[ \tan^{-1} \omega_o + \frac{8 (2 + \sqrt{\varepsilon})}{\sqrt{\varepsilon} (1 + \sqrt{\varepsilon}) \cos \alpha_o + \sqrt{\varepsilon}} - \frac{8 (2 \cos \alpha_o + \sqrt{\varepsilon})}{\sqrt{\varepsilon} \cos \alpha_o (\cos \alpha_o + \sqrt{\varepsilon})} + \frac{16}{\varepsilon} \ln \frac{\cos \alpha_o + \sqrt{\varepsilon}}{\sqrt{\varepsilon} (1 + \sqrt{\varepsilon}) \cos \alpha_o} \right]$$

Inserting the expression for the aperture area $A = \pi a^2 = \pi f^2 \tan^2 \alpha$, $A \approx 1 + \frac{1}{\tan \omega_o} \left[ \frac{8 (2 + \sqrt{\varepsilon})}{\sqrt{\varepsilon} (1 + \sqrt{\varepsilon}) \cos \alpha_o + \sqrt{\varepsilon}} - \frac{8 (2 \cos \alpha_o + \sqrt{\varepsilon})}{\sqrt{\varepsilon} \cos \alpha_o (\cos \alpha_o + \sqrt{\varepsilon})} + \frac{16}{\varepsilon} \ln \frac{\cos \alpha_o + \sqrt{\varepsilon}}{\sqrt{\varepsilon} (1 + \sqrt{\varepsilon}) \cos \alpha_o} \right]$.

A few values of $\overline{R_h^2}$ versus $\alpha_o$ are given below for a relative dielectric constant $\varepsilon = 5$.

<table>
<thead>
<tr>
<th>$\alpha_o$ (degrees)</th>
<th>$\overline{R_h^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>.24</td>
</tr>
<tr>
<td>60</td>
<td>.30</td>
</tr>
<tr>
<td>75</td>
<td>.49</td>
</tr>
</tbody>
</table>
The power that strikes the ground and is not reflected is absorbed. If the antenna is used for transmitting, this will reduce the efficiency of the antenna. If the antenna is used for reception, noise power generated by the ground will be radiated to the feed. The amount of power lost in the ground is

$$\int_{A} P_{\text{feed}} \times (1 - R_{h}^{2}) \times (1 - \text{plate density}) dA = \int P_{\text{abs}} dA$$

which for a circular aperture with uniform plate density and uniform illumination is $$\pi a^{2}(1 - \frac{M}{N})(1 - R_{h}^{2})$$.

The apparent temperature of the antenna in reception may now be found using the statements of thermodynamic balance of Sec. 4.1. The antenna is considered to have no loss other than the ground absorption. The apparent temperature consists of three parts: the first is due to the radiation pattern of the plates, the second, to the reflection from the ground, and the third, to the absorption by the ground. The contribution to the antenna temperature by the plates is found from the relation derived previously

$$T_{\text{plates}} = \frac{1}{4\pi} \int T(\theta, \varphi) G(\theta, \varphi) d\Omega$$

and the contribution of the spaces is

$$T_{\text{spaces}} = \frac{\int T(\theta, \varphi) Pr(\theta, \varphi) d\Omega}{\int P_{\text{feed}} dA} + \frac{\int P_{\text{feed}} (1 - R_{h}^{2})(1 - \text{plate density}) dA}{\int P_{\text{feed}} dA}$$

As an illustration of the effect of the various terms, consider a uniformly illuminated circular aperture of radius a that contains M of a possible N plates of area $d^{2}$. The locations of the plates are chosen as described.
in Sec. 2.4.2. The feed is located at a height $f$ above the aperture so that the maximum angle of feed radiation is $\alpha_o = \tan^{-1} \frac{a}{f}$. The apparent antenna temperature is

$$T_d = \frac{1}{4\pi} \int T(\theta,\phi) G(\theta,\phi) d\Omega + \frac{\int T_{\text{mb}}(\theta,\phi) d\Omega}{\pi a^2} + (1 - \frac{M}{N})(1 - \frac{R^2}{R_n^2}) T_o$$

If the sky distribution consists of a constant ambient temperature $T_{\text{amb}}$ and a source of temperature $T_s$ that subtends a solid angle $\Omega_s$ that is narrower than the main beam, the apparent antenna temperature is

$$T_a = T_s \Omega_s \left[ \frac{M}{N} \left( \frac{M a}{\lambda^2} \right)^2 \right] + T_{\text{amb}} \left( 1 - \frac{M}{N} \right) \frac{R^2}{R_n^2} + T_o \left( 1 - \frac{M}{N} \right) \left( 1 - \frac{R^2}{R_n^2} \right)$$

The sensitivity of the antenna temperature to that of the source is

$$\frac{\partial T_{\text{ant}}}{\partial (T_s \Omega_s)} = \frac{G}{4\pi} = \frac{M}{N} \left( \frac{M a}{\lambda^2} \right)^2$$

which clearly indicates the direct relation between sensitivity and area when the antenna is receiving power from a source narrower than the antenna beam.

If the source size is considerably greater than the main beam of the antenna but small in comparison to the beamwidth of the pattern of a single plate, the apparent antenna temperature is

$$T_a \approx T_s \left( \frac{M}{N} \right) \left( \frac{M a}{\lambda^2} \right)^2 + T_{\text{amb}} \left( 1 - \frac{M}{N} \right) \frac{R^2}{R_n^2} + T_o \left( 1 - \frac{M}{N} \right) \left( 1 - \frac{R^2}{R_n^2} \right)$$

From this expression the dependence of the antenna sensitivity on the ratio $M/N$ may be inferred. If $M/N$ is very small, changes in $T_s$ will be masked by the last term if $T_s$ is comparable with or less than $T_o$, which is 300°K. Fluctuations in the value of the last term can cause false
readings of signals. Similarly, the effect of changes in the ambient background temperature is greater as M/N decreases. Of course, if the contributions of the last two terms remain constant over an observation period, the source temperature may be determined; however, T_amb and T_o are random signals and thus by their very nature are subject to fluctuations. Therefore it is desirable to keep the M/N ratio as close to unity as is commensurate with the cost and resolution objectives of the antenna.

4.4 The Effect of Plate Arrangement on the Apparent Antenna Temperature

It is not necessary that all parts of the aperture be filled in order to maintain a low apparent antenna temperature. In the preceding examples the spaces between the plates received the same power per unit area as the plates. This would correspond to a feed height that is large in comparison to the radius of the antenna. Consider the case of a feed height f above a circular aperture consisting of many plates arranged on an approximately planar surface. The plates will be tilted at angles to reflect incoming radiation into the feed and will thereby shadow the ground from the feed. If the plate spacings are properly adjusted, at one beam position the plates can prevent any geometric rays from the feed from striking the ground within the aperture area. Thus, to a first approximation, there will be no contribution to the apparent antenna temperature from the surface of the ground beneath the antenna for one beam position.

Let the plates be arranged in concentric rings with the plates in each ring sufficiently close to one another so that very little power passes between adjacent plates. The spacing between corresponding plates
in adjacent rings is to be such that, for power reflected vertically, 
(i.e., the beam position at $\theta_0 = 0$), no rays pass between the plates.
If the plate width is $d$, this condition is met for a plate separation of

$$ S = d\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} \tan \alpha\right) $$

where $\alpha$ is the angle between the ray to the feed and the vertical. If
an antenna with this plate arrangement is steered so that incident radia-
tion is received from a direction $\theta_0$, $\phi_0$, the distance that is uncovered
between two corresponding plates on adjacent rings is

$$ x_1 = d\left[\cos\frac{\alpha}{2} - \cos\left(\frac{\alpha}{2} - \frac{\theta_0 \cos(\phi_0 - \phi)}{2}\right) + \sin\frac{\alpha}{2} \tan -\sin\left(\frac{\alpha}{2} - \frac{\theta_0 \cos(\phi_0 - \phi)}{2}\right) \tan \alpha\right] $$

Similarly, the distance uncovered between adjacent plates in the same
ring is $x_2 = d\left[1 - \cos\left(\frac{\theta_0}{2} \sin(\phi - \phi_0)\right)\right]$.

The amount of uncovered area will be approximately

$$ A_u = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 - \cos\left(\frac{\theta_0}{2} \cos(\phi - \phi_0)\right) - \sin\left(\frac{\theta_0}{2} \cos(\phi - \phi_0)\right) \frac{\sin \frac{\alpha}{2} \cos \tan \alpha}{\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} \tan \alpha} r dr d\phi $$

$$ + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 - \cos\left(\frac{\theta_0}{2} \sin(\phi - \phi_0)\right)}{\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} \tan \alpha} r dr d\phi $$

For small values of the beam angle $\theta_0$, the contributions from the terms
other than those involving $\sin\left(\frac{\theta_0}{2} \cos(\phi - \phi_0)\right)$ may be neglected, yielding

$$ A_u \approx \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\left(\frac{\theta_0}{2} \cos(\phi - \phi_0)\right) \tan \frac{\alpha}{2} r dr d\phi \approx \theta_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan \frac{\alpha}{2} r dr $$

The relation between $r$ and $\alpha$ is $r = f \tan \alpha$. - 70 -
Thus \[ A_u = \theta_o \int_0^{\alpha_o} \tan \frac{\alpha}{2} f \left( \frac{2 \sin \alpha}{\cos^3 \alpha_o} \right) d\alpha \approx 2 f^2 \theta_o \int_0^{\alpha_o} \frac{\sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{\cos \frac{\alpha}{2} \cos^3 \alpha} d\alpha \]

\[ A_u \approx \frac{f^2 \theta_o}{2} \left[ \frac{\sin \frac{\alpha_o}{2}}{\cos^2 \alpha_o} - \frac{\sin \frac{\alpha_o}{2}}{2 \cos \alpha_o} + \frac{1}{4 \sqrt{2}} \ln \frac{\sqrt{2} \sin \frac{\alpha_o}{2}}{2 \sin \frac{\alpha_o}{2} + 1} \right] \]

The fractional uncovered area for a beam angle of \( \theta_o \) is

\[ \frac{A_u}{A} = F \theta_o \approx \frac{\theta_o}{2\pi} \left( \frac{1}{\tan \alpha_o} \left[ \frac{\sin \frac{\alpha_o}{2}}{\cos^2 \alpha_o} - \frac{\sin \frac{\alpha_o}{2}}{2 \cos \alpha_o} \right] + \frac{1}{4 \sqrt{2}} \ln \frac{2 \sin \frac{\alpha_o}{2} - 1}{2 \sin \frac{\alpha_o}{2} + 1} \right) \]

The fractional uncovered area per unit beam angle is given for values of maximum feed angles.

<table>
<thead>
<tr>
<th>( \alpha_o )</th>
<th>45</th>
<th>60</th>
<th>75</th>
<th>degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F / 2\pi )</td>
<td>.30</td>
<td>.41</td>
<td>.54</td>
<td></td>
</tr>
</tbody>
</table>

This illustrates the extent to which the ground may be kept in the geometric shadow of the feed. A rough estimate of the resultant antenna temperature may be obtained by assuming that the ground reflection coefficient is equal to its root mean square value for all angles of incidence. The apparent antenna temperature is approximately

\[ T_a \approx \frac{1}{4\pi} \int T(\theta, \phi) \delta(\theta, \phi) + F \theta_o R_n \frac{\overline{r}}{Pr \alpha} + T_o (1-R_n) F \theta_o \]

The reduction of the effect of the last two terms by the factor \( F \theta_o \) is clear. For the value \( \theta_o = 0 \) the power lost is not zero as this expression indicates. It is necessary to use an improved expression for accurate evaluation of the contributions at very small beam angles.
The total plate area in the aperture for this arrangement of plates is

\[ A_p = \int_0^{2\pi} \int_0^\alpha \frac{d\phi}{S} \, r \, dr \, d\phi = 2\pi \int_0^{\alpha_0} \frac{1}{\cos^2 \frac{\alpha}{2} + \sin^2 \frac{\alpha}{2} \tan \alpha} \int^\frac{\tan \alpha \sec^2 \alpha \, d\alpha}{\cos^2 \frac{\alpha}{2} + \sin^2 \frac{\alpha}{2} \tan \alpha} \]

\[ = 8\pi f^2 \int_0^{\alpha_0} \frac{\sin^2 \frac{\alpha}{2} \, d\alpha}{(2\cos^2 \alpha - 1)^2} = -8\pi f^2 \int_1^{\alpha_0} \frac{dx}{(2x^2 - 1)^2} \]

\[ A_p = 8\pi f^2 \left[ \frac{\cos \alpha_0}{\cos \alpha_0} - 1 + \ln \frac{2\cos \alpha_0}{\cos \alpha_0 + 1} \right] \]

The fraction of the total aperture area occupied by the plates is

\[ \frac{A_p}{A} = \frac{\alpha}{\tan^2 \alpha} \left[ \frac{\cos \frac{\alpha}{2}}{\cos \alpha_0} - 1 + \frac{\ln \frac{2\cos \frac{\alpha}{2}}{\cos \alpha_0 + 1}}{\frac{\alpha}{2}} \right] \]

which is given below for a few values of \( \alpha_0 \).

<table>
<thead>
<tr>
<th>( \alpha_0 )</th>
<th>45</th>
<th>60</th>
<th>75</th>
<th>degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{A}{A} )</td>
<td>.89</td>
<td>.72</td>
<td>.47</td>
<td></td>
</tr>
</tbody>
</table>

Thus it is seen that a judicious plate arrangement can result in an appreciable reduction in the number of plates without large undesired contributions to the apparent antenna temperature.

4.5 Interpretation of Results

The performance of a radio telescope is evaluated by the characteristics of the apparent antenna temperature. As the purpose of the instrument is to determine the pattern of the sky brightness temperature,
it is desirable to have high resolution, rejection of signals from other than the principal direction, and a low residual temperature. The resolution and sidelobes of the multiple plate array have been discussed previously. The concern of the above analysis has been the effect of the spaces between plates of the array and how the surface of the ground at these spaces contributes unwanted noise to the antenna output. The ground reflects sky noise to the feed in proportion to the square of the Fresnel reflection coefficients if the surface is smooth. Experimental investigations have shown that surfaces such as asphalt roadways and short grass reflect energy as smooth surfaces with a relative dielectric constant \( \varepsilon_r \approx 5 \). The noise available at the feed from the ground may be determined from this knowledge of the reflection properties of the surface. These calculations lead to an apparent antenna temperature composed of three parts; a term due to the plates of the antenna, a term representing the noise power reflected by the spaces between plates, and a term due to the noise power generated by the ground. Examination of this expression shows the desirability of filling a major fraction of the aperture area with plates. Judicious arrangement of the plates and feed may lead to partial shadowing of the ground by the plates, and calculations for one arrangement are presented, showing a considerable reduction in the unwanted terms of the antenna temperature for a given ratio of space to plate area.
CHARACTERIZATION AND OPTIMIZATION OF THE ANTENNA SYSTEM

A multiple plate antenna can be designed with either large or small spaces between plates and with any amplitude taper across the array of plates. In sections 2.4 and 3.2 the radiation patterns of different arrangements of plates have been analyzed. If large numbers of plates are used, the radiation patterns are narrow beams with relatively low side-lobes, and the differences among arrangements of plates are not clear. An assessment of the merits of different arrangements is necessary in order to choose among them. This requires an evaluation of the antenna with regard to its intended use. For some purposes, desirable goals may be maximum gain and low sidelobes. In general, however, these goals are but rules of thumb and provide a quantitative measure of the difference between the performance of one antenna system and another for only a limited number of uses. To weigh the merits of the antenna, an error criterion must be developed that accurately describes the extent to which the antenna satisfies the system requirement. This necessitates a knowledge of the intended use and a description of the system.

In a radio astronomy system used to survey a distribution of incoherent sources, the measured noise temperature at the receiver output is not a true description of the source distribution. The antenna has weighted the input data by its transfer function. If this coloration is to be removed, the data at the receiver output must be processed. This operation is known as restoration. In an antenna system that is free from noise and
uncertainty, all harmonics of the input signal that pass through the antenna can be restored to their original value. If there are errors in measurement, an error is present in the restored output, and the choice of the antenna used to collect the data will affect the amount of error in the restored output.

In section 5.1 the antenna transfer function for incoherent distributions is determined, and in section 5.2 the effect of noise and spurious responses on the antenna output is discussed. In section 5.3 the optimum restoration process for a given antenna is found. An arbitrary error criterion is considered in section 5.4, and the calculus of variation is applied to yield the defining equations for the antenna that minimizes the error. Intrinsic to the derivation of these equations is that a realizable antenna is obtained as an answer.

5.1 The Antenna Transfer Function for Incoherent Sources

When the antenna is used to survey a distribution of sources, the output is a function of the sources, the antenna properties, and the source correlation. If the correlation between the fields received at the antenna from any two sources is negligibly small after an observation period, the source distribution may be termed incoherent. In this case,

$$\overline{P_0} \propto (E_1 + E_2)^2 = \overline{P_1 + P_2}$$

where \(\overline{P_0}\) is the time averaged output power from the antenna for two incoherent inputs \(E_1\) and \(E_2\). The antenna is, therefore, linear in power for incoherent sources. A superposition relation can be written

$$\overline{P_0} (\psi_x, \psi_y) = \iint \overline{S} (\psi'_x, \psi'_y) A_r (\psi'_x - \psi_x, \psi'_y - \psi_y) \ d\psi'_x \ d\psi'_y$$
where $\overline{P}_0(\psi_x, \psi_y)$ is the time averaged output power at the antenna terminals for a given direction $\psi_x, \psi_y$, $\overline{S}(\psi_x, \psi_y)$ is the intensity of the source distribution, and $A_r(\psi_x, \psi_y)$ is the receiving cross section of the antenna. These quantities are assumed not to vary over the frequency band of interest. Such a situation might occur in a radio astronomy survey, where a distribution of discrete sources is being mapped by receiving the radiation over a band of frequencies that is small in comparison to the center frequency. While the field strengths from all sources add with their particular phase at any instant of time, the detector time constant is sufficiently large so that the output power of the antenna appears as a linear sum of the source powers, weighted by the antenna power pattern.

The power received by the antenna from a single point source of intensity $S$ at $\psi_x, \psi_y$ is

$$P_r(\psi_x, \psi_y) = A_r(\psi_x, \psi_y) S(\psi_x, \psi_y)$$

$$= G(\psi_x, \psi_y) A_r S(\psi_x, \psi_y)$$

$$= \frac{\lambda^2}{4\pi} G(\psi_x, \psi_y) S(\psi_x, \psi_y)$$

where $G(\psi_x, \psi_y)$ is the antenna gain, previously derived as

$$G(\psi_x, \psi_y) = \frac{P(\psi_x, \psi_y)}{P_r/4\pi}$$

with

$$P(\psi_x, \psi_y) = A \int C_A(\xi, \eta) e^{i \psi_x \xi + i \psi_y \eta} d\xi d\eta$$

and

$$k^2 P_r = 4 \pi^2 A C_A(0,0)$$

as shown in section 2.4.
This yields
\[ G(\psi_x, \psi_y) = \frac{4\pi}{\lambda^2} \int \frac{C_A(\xi, \eta)}{C_A(0,0)} e^{j \psi_x \xi + j \psi_y \eta} d\xi d\eta \]

This is a statement that the gain pattern of the antenna is proportional to the Fourier transform of the antenna aperture autocorrelation function.

Taking Fourier transforms of both sides of the superposition relation yields
\[ p_A(u,v) = \gamma(u,v) s(u,v) \]

The transformed equation shows the antenna output to be the product of the input spectrum and the antenna transfer function. The transform variables \( u \) and \( v \) can be considered as spatial frequencies, representing the rapidity of fluctuations of signals with respect to their angular coordinates \( \psi_x \) and \( \psi_y \). This is analogous to the use of the frequency domain in linear circuit theory, where multiplicative relations hold in the frequency domain and convolutions in the time domain. The development of these concepts by Elias and their subsequent use by O'Neill and others has resulted in a close association between the areas of communication theory and optics.

The antenna transfer function, \( \gamma(u,v) \), is
\[ \gamma(u,v) = \left| \frac{C_A(\xi, \eta)}{C_A(0,0)} \right|_{\xi = u, \eta = v} \]
and represents the weighting of the spectral distribution of the input by the antenna. The spatial frequencies \( u \) and \( v \) are equal to the spacings \( \xi \) and \( \eta \). The antenna sensitivity to a particular spatial frequency is determined by the value of the aperture autocorrelation at the spacing equal to the spatial frequency. The aperture autocorrelation function is a measure of the occurrence of different spacings in
the antenna, and the relative numbers of these spacings give the relative antenna sensitivity to the spatial frequency spectrum.

5.2 Noise and Uncertainty in the Antenna System

With any measurement of a desired signal there is associated an uncertainty in the reading of the desired signal. Typical causes of unwanted signals or noise present in the measurement of incoherent source distributions by an antenna are changes in the gain of the detection circuitry, errors in the output and recording apparatus, variations in the antenna pattern due to wind or thermal effects, antenna positioning error, inaccuracies in timing and calibration, mismatch of the r-f circuits, noise due to feed spillover and transmission line loss, and the fluctuation of the detector output due to the inherent limitations of the receiver. The last of these effects was analyzed by Dicke in his description of the radiometer. It has been shown that the root mean square fluctuation in apparent antenna temperature for a double sideband receiver with an equivalent antenna noise temperature $T_D$, a noise bandwidth $B$ and a detection time constant $\tau_D$ is

$$\langle \Delta T \rangle_{\text{r.m.s.}} = \frac{\pi F T}{8 (2B \tau_D)^{1/2}}$$

If a fixed temperature, appreciably higher than the source temperature, is used for a comparison temperature for the radiometer, the r.m.s. output fluctuations will be nearly proportional to this comparison temperature and may be considered as additive noise. Similarly, if the contributions to the apparent temperature of the antenna from feed spillover, spaces between sections of the antenna, and transmission line loss are large in comparison to the input
temperature from the source distribution, the r.m.s. output fluctuations will be almost independent of the source temperature.

If the comparison temperature is equal to or less than the source temperature and other contributions are small, the r.m.s. level of the output error fluctuations depends on the source temperature. However, for low temperature surveys with modern receivers the output fluctuations due to the radiometer are small if the comparison temperature is very low, and they may be masked by other effects. Thus the radiometer incremental sensitivity is determined by its output fluctuations, and these are independent of the input temperature if the comparison temperature or other loss contributions are relatively high, and they can be very small if the source and the comparison temperature are low.

Many of the other causes of error, such as recorder inaccuracies and calibration errors, can also be considered to be independent of the source distribution. For the purpose of the following analysis an additive independent noise, representing the above described effects, is considered to be present at the output of the antenna. This noise causes an uncertainty in the observation of the output antenna temperature at each and every angle of observation, and the correlation between the noise at one angle of observation and the noise at a different angle is assumed to be negligible. Thus

$$P_R(\psi_1, \psi_2) = P_A(\psi_1, \psi_2) + N(\psi_1, \psi_2)$$

and forming the autocorrelation functions yields

$$R_R(\Delta \psi_x, \Delta \psi_y) = \int P_R(\psi_x, \psi_y) P_R(\psi_x + \Delta \psi_x, \psi_y + \Delta \psi_y) d\psi_x d\psi_y$$

$$R_N(\Delta \psi_x, \Delta \psi_y) = \int N(\psi_x, \psi_y) N(\psi_x + \Delta \psi_x, \psi_y + \Delta \psi_y) d\psi_x d\psi_y$$
Taking Fourier transforms of both sides yields the power spectra in spatial frequency of the angular distribution of sources and noise. While it is true that $P_A$ and $N$ are measures of noise power, the antenna system is linear in time average power for incoherent sources, and the Fourier transforms of the autocorrelation functions formed above represent the average spectral content of the source and noise distribution. Thus they are "power" spectra of power, but it is important to realize this is the "power" spectrum of the angular variation of the time averaged input power distribution.

The autocorrelation functions can be used to describe the average source characteristics for an ensemble of measurements of a source distribution or for an ensemble of source distributions. However, the resemblance of the ensemble average autocorrelation function to that of a single member of the ensemble will depend on the statistics involved in the source generation, and each case must be considered in its own light. For example, an ensemble of source distributions could be formed by locating a number of incoherent point sources at random in $\psi_x$ and $\psi_y$. The autocorrelation function of a member of the ensemble would be a narrow spike at the origin with, for most cases, a few lesser spikes located elsewhere in $\Delta \psi_x$ and $\Delta \psi_y$. The spatial frequency spectrum of this source distribution would be a broad flat curve for any member of the ensemble, and the ensemble average spatial frequency spectrum would be a good representation of the frequency content of any member of the ensemble. While it is clear that the

$$R_R(\Delta \psi_x, \Delta \psi_y) = R_A(\Delta \psi_x, \Delta \psi_y) + R_N(\Delta \psi_x, \Delta \psi_y)$$
ensemble average autocorrelation function and the autocorrelation function of any member of the ensemble are usually not identical, it is necessary to find a representative description of the source statistics of a particular set of experiments, and the ensemble average autocorrelation function is a logical choice.

The Fourier transform of the noise autocorrelation function is the spatial frequency power spectrum of the time averaged noise power. This may have any shape so long as it is everywhere positive, but one might expect the noise spectrum to be flat over the range of spatial frequencies considered for the antenna problem. This would correspond to the error in reading the antenna temperature at one angle being independent of the error at a different angle of observation.

Another source of error that deserves consideration is the fluctuation in the output due to changes in the antenna pattern. This effect might be caused by wind temperature, weather conditions, or different weight distributions as the antenna is moved. Consider an antenna power pattern that consists of $P_A$, the expected or average pattern, and $P_S$, the spurious pattern that represents the variations from the average pattern caused by the above mentioned effects. For any member of an ensemble of experiments, the total power pattern $P = P_A + P_S$ has a Fourier transform $\tau$ that is the antenna transfer function for that experiment. The transfer function also consists of two parts $\tau_A$ and $\tau_S$, and the ensemble average of $\tau$ is

$$<\tau> = \tau_A$$

while the ensemble mean square of $\tau$ is

$$<\tau^2> = \tau_A^* \tau_A + \tau_A^* <\tau_S> + \tau_A <\tau_S^* > + <\tau_S \tau_S^* > = \tau_A \tau_A^* + <\tau_S \tau_S^* >$$
5.3 The Restoration Process

The output of the antenna is the convolution of the source distribution with the antenna point source power pattern; in terms of the spatial frequency spectrum, this represents a coloration of the input spectrum by the antenna transfer function. The output of the antenna is, therefore, not a true representation of the source distribution. In order to obtain a more accurate estimate of the source, a process of restoration is necessary. This is somewhat analogous to the concepts of "whitening" and the matched filter in time domain circuitry. The definitive method of restoration of antenna signals in the presence of noise is that of Bracewell, and a modification of his technique follows.

Consider the output of the antenna to consist of three parts: the source distribution convolved with the average antenna pattern, the convolution of the source with the spurious antenna pattern, and an additive error term due to the receiver and other inaccuracies. In the spatial frequency domain, these terms are

\[ p_0(u,v) = r_A s + r_S s + n \]

Let the above output be fed into a restoration filter that has a transfer function \( r_R \), with the restriction that \( r_R = 0 \) when \( r_A = 0 \) so that no attempt is made to restore a signal that did not pass through the antenna on the average. A measure of the performance error of the total antenna and restoration system is the squared difference between the output of the restoration filter and the source distribution, averaged over all spatial frequencies for which \( r_A \neq 0 \) and over the ensemble of experiments. By Parseval's
This corresponds to the ensemble average of the squared difference of these components of the angular source variation and the output of the restoration filter, averaged over all angles. This is

\[ \langle \bar{e}^2 \rangle = \int \left\langle \left| \mathbf{S} - \mathbf{R} \left( \mathbf{r}_A \mathbf{s} + \mathbf{r}_S \mathbf{n} \right) \right|^2 \right\rangle \ dudv \]

where \( A \) is the region of non-zero \( r_A \). Assuming independence between \( s \) and \( n \) and a noise of zero mean,

\[ \langle \bar{e}^2 \rangle = \int_A \left\{ \left\langle s^2 \right\rangle \left( 1 - r_A r_A^* + r_A r_A^* (r_S r_S^* + \left\langle r_S r_S^* \right\rangle) + r_A r_A^* \left\langle n^2 \right\rangle \right) \right\} dudv \]

Minimizing this error for each value of \( u \) and \( v \) gives

\[ r_R = \frac{r_A^*}{(r_A r_A^* + \left\langle r_S r_S^* \right\rangle + \left\langle n^2 \right\rangle \left\langle s^2 \right\rangle)} \]

and

\[ \langle \bar{e}^2 \rangle = \int \left[ \left( \frac{\left\langle n^2 \right\rangle}{\left\langle s^2 \right\rangle} + \frac{\left\langle r_S r_S^* \right\rangle}{\left\langle s^2 \right\rangle} + (r_A r_A^* \left\langle r_S r_S^* \right\rangle) \right) \right] \langle s^2 \rangle \ dudv \]

If there were no noise or spurious antenna response,

\[ r_R = \frac{1}{r_A} \quad \text{and} \quad \langle \bar{e}^2 \rangle = 0 \]

and the restoration process would produce an exact likeness of all harmonics of the signal that pass through the antenna, and no other harmonics. This is called by Bracewell the principal solution of the source distribution.

5.4 The Optimization of the Antenna System

The extent to which an antenna performs its intended function can be described by formulating an error criterion involving the difference between the system output and a desired output. In general, the absolute
value of the difference or the value raised to an integral power is averaged over all angles to provide a meaningful measure of the antenna system error. Whatever error criterion is used, however, must involve the actual output of the system, not necessarily the output terminals of the antenna, for the desired optimum antenna is the one that provides the least average error at the output of the system, and that need not be the one that provides the maximum signal power at the antenna terminals.

Let the error criterion for a particular antenna system, expressed in the spatial frequency domain, be denoted by

$$ I = \int_A E(\zeta, \eta) \, d\zeta \, d\eta $$

where A is the region of non-zero \( \zeta \), the antenna transfer function. The function E might typically be the square of the difference between the product of \( \zeta \) and the source spectrum and some other operation on the source spectrum. Consider a variation of I such that

$$ \delta I = \int_A \delta E \, d\zeta \, d\eta = \int_A \frac{\partial E}{\partial \zeta} \delta \zeta \, d\zeta \, d\eta $$

The variation in \( \zeta \) must be consistent with the constraint that is the autocorrelation function

$$ \tau(\zeta, \eta) = \int g(x, y) \, g^*(x+\zeta, y+\eta) \, dx \, dy $$

where \( g(x, y) \) is normalized so that \( \tau(0,0) = 1 \). The antenna transfer function is assumed to exist from \(-2a\) to \(+2a\) in the \( u \) direction and from \(-2b\) to \(+2b\) in the \( v \) direction. A variation in \( g \) causes
\[ \delta \tau = \int g(x,y) \delta g^*(x+u,y+v) \, dx \, dy + \int \delta g(x,y) \, g^*(x+u,y+v) \, dx \, dy \]
and
\[ \delta I = \iint \frac{\partial E}{\partial \tau} \left[ g(x,y) \, \delta g^*(x+u,y+v) + \delta g(x,y) \, g^*(x+u,y+v) \right] \, dx \, dy \, du \, dv \]

which, with an appropriate change in limits, may be written
\[ \iint \frac{\partial E}{\partial \tau} \left[ \delta g^*(x,y) \, g(x-u,y-v) + g^*(x+u,y+v) \, \delta g(x,y) \right] \, dx \, dy \, du \, dv \]

Reversing the order of integration and using \( T^*(-u) = 2 \),
\[ \iint \int \int \left( \frac{\partial E}{\partial g} \, \delta g^*(x,y) \, g(x+u,y+v) + \frac{\partial E}{\partial g} \, g^*(x+u,y+v) \, \delta g(x,y) \right) \, du \, dv \, dx \, dy \]

where the limits have been written to show the range of \( x \) and \( y \).

The fundamental constraint of any antenna is that its cross section, averaged over all angles, is \( \frac{\lambda^2}{4\pi} \). In terms of the transfer function
\[ \tau(0,0) = 1 = \int gg^* \, dx \, dy = \int (g+\delta g)(g^*+\delta g^*) \, dx \, dy \]

As this holds for any \( \delta g \), no matter how small,
\[ \int g(x,y) \, \delta g^*(x,y) \, dx \, dy + \int g^*(x,y) \, \delta g(x,y) \, dx \, dy = 0 \]

This is true for any variation in \( g \), whether real or imaginary, as
\[ \delta g = \varepsilon \eta_r(x,y) + i \varepsilon \eta_i(x,y) \]

and thus
\[ \int_b^a \int_{-a}^b g_r \, \eta_r \, dx \, dy = 0 \]

To find a minimum of the integral, it is necessary that
\[ \frac{d}{d\varepsilon} \delta I \bigg|_{\varepsilon=0} = 0 \]
For this to be true, either the integrand is zero, or

\[ g_r(x, y) = -i \frac{A_r}{\lambda} \int_{-b}^{b} \int_{-a}^{a} \left[ \frac{\partial E}{\partial z} \right] g(x+u, y+v) \, du \, dv \]

where \( g = g_r + jg_i \), and the constants \( A_r \) and \( A_i \) are determined such that

\[ \int_{-b}^{b} \int_{-a}^{a} g_g \, dx \, dy = 1 \]

If the aperture illumination \( g(x,y) \) is real, the defining equations for the optimum antenna of extent \( 2a \) in the \( x \) direction and \( 2b \) in the \( y \) direction are

\[ g(x,y) = A \int_{-b}^{b} \int_{-a}^{a} \frac{\partial E}{\partial z} g(x+u, y+v) \, du \, dv \]

The first of the above pair of equations constitutes the necessary condition for determining the optimum antenna, and the second provides the correct normalization. In general, if \( E \) is a complicated function of \( \tau \), the equations are not easily solvable.

The range of values of \( x \) and \( y \) do not allow \( g(x+u, y+v) \) to be shifted completely across \( \frac{\partial E}{\partial z} \). Therefore, the above integral equations cannot be solved simply by taking their Fourier transforms, as this leads to

\[ G(\psi_x, \psi_y) = A \int_{-b}^{b} \int_{-a}^{a} G(\psi_x', \psi_y') \left[ \frac{\partial E}{\partial z} \right]_{\text{transformed}} \frac{\sin a(\psi_x' - \psi_x)}{a(\psi_x' - \psi_x)} \frac{\sin b(\psi_y' - \psi_y)}{b(\psi_y' - \psi_y)} \, d\psi_x' \, d\psi_y' \]

where \( g(x, y) \) is assumed to be real.
In a sense, the above equations are analogous to the Wiener-Hopf equation. There also it is the restricted range of the shift variable that prevents the simple solution by transforms. These equations differ from the Wiener Hopf equation because the constraint from which these equations evolved is that the average antenna cross section is a constant. The Wiener-Hopf equation stems from the fact that the output of a passive linear system is zero until there is an input. The above equations are formulated for any error criterion, and need not apply only to the mean square error.
REALIZATION OF THE OPTIMUM ANTENNA

The defining equations have been determined in section 5.4 for the antenna that satisfies the condition for maximum performance based on a chosen error criterion. An important freedom in the formulation of these equations is that the error criterion can be applied to the actual output of the system, which is not necessarily the antenna output terminals. For many problems, it is desired to obtain the minimum error form of the signal after data processing or other operations on the antenna output. In these cases, an error criterion based on the antenna output usually does not yield the best system output. An example of this is an antenna with a restoration filter, used to survey a distribution of incoherent sources. The antenna that provides at the restoration filter output the closest resemblance to the source distribution is defined not necessarily by its gain or sidelobe level, but by a relation involving the ratio of signal to noise and uncertainty of readings.

There are two areas of interest for the solution of the optimum antenna equations. The first of these is to determine the best aperture distribution for a continuous aperture of fixed size used for a given purpose. In section 6.1 this problem is handled directly by the relations found previously. The error criterion involving the source and the restored output provides an integral equation that is not simply solved by analytic means. However, the equations lend themselves to self-consistent solutions, and can be evaluated by graphical means or by machine computation. A number of examples are calculated by an iterative
method to show the forms of the solution for different conditions.

The defining integral equations involve both the transfer function $\tau$ and the aperture distribution $g$. The manner in which a known finite range autocorrelation specifies a finite range generating function is discussed in section 6.2. It is shown that the finite range autocorrelation imposes enough constraints to define a unique function within one arbitrary constant. Therefore, the mixing of $\tau$ and $g$ in the integral equations does not imply a wide class of solutions but rather that the finite range autocorrelation and its finite range generating function are closely bound together.

The second topic of concern is the determination of the optimum array of a fixed number of elements for a specific use. The case of very large numbers of elements is considered, for which a description of the plate distribution is required that ignores the minute detail and presents the gross, pertinent effects. To achieve this, the density function, as defined in section 2.4.2, is used, and the defining integral equation involving this function is found in section 6.3. Another problem of interest is to determine the optimum array if the source contains much more detail than can be resolved by the given number of elements when closely spaced. The necessary density of elements is dictated by the signal to noise ratio, and the element locations may be chosen by using a random code.

6.1 The Optimum Continuous Aperture of Fixed Dimensions

A common problem of antenna design is to determine the radiation pattern of the primary feed for a reflector antenna. For many purposes, the feed pattern is chosen to provide a secondary pattern with low sidelobes. For astronomy, it is not always necessary that the first few sidelobes be very low, especially if a restoration process is to be applied to the antenna output. From the standpoint of the spatial frequency spectrum, it is
important that the best weighting of the source harmonics is obtained.

A simple special case of optimization is the maximization of the output power of the antenna. The error criterion may be written as

\[ E = s(1 - \tau) \]

where the source power spectrum is \( s(u,v) \) and the antenna output is \( \tau(u,v)s(u,v) \). For a point source \( s(u,v) = s_0 \), and \( \frac{\partial E}{\partial \tau} = -s_0 \).

The integral equations are

\[ g(x,y) = -A s_0 \int \int g(x+u,y+v) \, du \, dv, \quad 1 = -A s_0 \int \int \tau \, du \, dv \]

and are solved by

\[ g(x,y) = \frac{1}{4ab}, \]

showing the well known result that a uniform distribution receives the maximum output power from a point source.

For a set of equations that are more applicable to radio astronomy, consider the error criterion evolved in section 5.3 involving the mean square difference of the source distribution and the restored output of the antenna. Assuming that \( \tau \) is real and considering only the variation in \( x \) and \( u \),

\[ E = \frac{\langle n^1 \rangle + \langle \tau^1 \rangle}{\langle n^1 \rangle + \langle \tau^2 \rangle + \langle \tau^1 \rangle} \]

A reasonable assumption for the spectra of \( \langle n^2 \rangle \), \( \langle s^2 \rangle \), and \( \langle \tau^1 \rangle \) is that they are flat within the range of values of non-zero \( \tau \). This implies that the sources are narrower than the antenna beamwidth, the errors are independent of angle, and that narrow spurious lobes occur at random in the pattern. Setting \( \langle s^2 \rangle = 1 \), \( \langle n^2 \rangle = c \), \( \langle \tau^1 \rangle = d \), and \( c + d = \sigma^2 \).
\[ E = \frac{\sigma^2}{\sigma^2 + \gamma^2} \]

and \[ \frac{\partial E}{\partial \gamma} = -\frac{\sigma^2 \gamma}{(\sigma^2 + \gamma^2)^{\frac{3}{2}}} \]

The integral equation becomes

\[ g(x) = -A \int_{-a}^{a} \frac{\sigma^2 \gamma}{(\sigma^2 + \gamma^2)^{\frac{3}{2}}} g(x+u) \, du \]

where the constant \( A \) is determined by

\[ 1 = -A \int_{-2a}^{2a} \frac{\sigma^2 \gamma^2}{(\sigma^2 + \gamma^2)^{\frac{3}{2}}} \, du \]

An iterative procedure for solving this equation can be stated as follows. First, assume as a solution \( g_0(x) \) some reasonable aperture distribution, such as a constant, and compute its autocorrelation function. Second, carry out the indicated integration graphically or otherwise, obtaining the first solution, \( g_1(x) \), as an answer. Third, use \( g_1(x) \) in the same manner as \( g_0(x) \) to obtain \( g_2(x) \), a better approximation to the true solution of the equation. Examples of this technique are illustrated. In Figure 10, for the case of \( \sigma^2 = 1 \), the trial distribution, \( g_0 \), is a constant. The first iteration yields \( g_1(x) \) as shown. The same procedure is illustrated for \( \sigma^2 = .25 \) in Figure 11, and for \( \sigma^2 = 0.1 \) in Figure 12. In every case the iterative method has produced a function that has quickly converged to a smooth aperture distribution that further iterations have not substantially changed.

The trial functions used above are even functions of \( x \), and the iterative method preserves this nature. An odd function of \( x \) is used as \( g_0(x) \) for \( \sigma^2 = .25 \) in Figure 13. The resultant \( \tau \) is tending toward
Figure 12

(a) \( g_0(x) \)
(b) \( \frac{\partial^2 E}{\partial x^2} \)
(c) \( g_1(x) \)
(d) \( \gamma_{12} \)
(e) \( g_x(x) \)
(f) \( \sigma \)
the transfer function shown in Figure 13. This transfer function, when squared, is nearly equal to the square of the ideal transfer function found in Figure 11. However, impulse functions are necessary in the odd aperture distribution to produce a \( r \) that yields as low an error as that derived from an even function. Therefore, for continuous aperture distributions without impulses, an even aperture distribution is preferred. This selection of even over odd aperture functions is illustrated in Figure 14, where, as a trial solution, a \( g_o(x) \) containing both even and odd terms is used. After two iterations the aperture distribution is nearly even, and further iteration yields the even result. This example also shows the stability of the iteration method for such cases, in that two substantially different trial functions yield, in a few iterations, nearly the same result.

This procedure can be applied to the solution of the integral equations for other spectral distributions or for other error criteria.

6.2 The Specification of a Function by its Autocorrelation

A function that exists over a finite range of its variable produces an autocorrelation function that also exists over a finite range. If the autocorrelation function is given, the function generating the autocorrelation can be derived by using the properties of its Fourier transform.

Consider a function \( g(x) \) that exists over a region \( |x| < a \) and that has the property

\[
\int_{-\infty}^{\infty} g^* g \, dx < \infty
\]
Let this function have a Fourier transform

\[ F(\psi) = \int_{-a}^{a} g(x) e^{i\psi x} \, dx \]

and an autocorrelation function

\[ \tau(u) = \int_{-a}^{a-u} g(x) g^*(x+u) \, dx \quad u > 0 \]

\[ = \int_{-a-u}^{a} g(x) g^*(x+u) \, dx \quad u < 0 \]

The autocorrelation function is zero for all values of \(|u| \geq 2a\). The Fourier transform of the autocorrelation function is

\[ P(\psi) = \int_{-2a}^{2a} \tau(u) e^{i\psi u} \, du \]

which is related to the Fourier transform of the function \( g \) by

\[ P(\psi) = F(\psi) F^*(\psi) = |F(\psi)|^2 \]

The magnitude of \( F(\psi) \) is determined by \( P(\psi) \), and it would appear that the phase of \( F(\psi) \) is lost by forming the autocorrelation function. However, the phase of \( F \) is contained in \( P \). A statement of the sampling theorem is that the Fourier transform \( F(\psi) \) of a function \( g(x) \) that is limited to \( |x| < a \) and zero elsewhere can be constructed from a set of samples spaced at intervals of \( \frac{2\pi}{2a} \) in \( \psi \). This is derived in section 3.2 and can be written

\[ F(\psi) = \sum_{m} F(m \frac{2\pi}{2a}) \frac{\sin(\psi m)}{(\psi - m \frac{2\pi}{2a})} \]

where

\[ g(x) = \frac{1}{2a} \sum_{m} F(m \frac{2\pi}{2a}) e^{-i m \frac{2\pi}{2a} x} \quad \text{for } |x| < a \]

Similarly, the autocorrelation function exists over a finite range
and its transform can be represented as

\[ P(\psi) = \sum_k P(\kappa \frac{2\pi}{4a}) \frac{\sin 2\alpha (\psi - \kappa \frac{2\pi}{4a})}{2\alpha (\psi - \kappa \frac{2\pi}{4a})} \]

At each of the samples of \( P \) corresponding to \( k \) even, the magnitude of one coefficient of the Fourier harmonics of \( g \) is determined. The even samples of \( P \), therefore, give the magnitude of the Fourier harmonics of \( F \). Consider a number \( 4N+1 \) of samples of \( P \) such that the remaining samples make a negligible contribution to \( g(x) \) and to the distribution of \( P(\psi) \) for \(|\psi| < \frac{N\pi}{\alpha} \). For any physical problem a sufficiently large \( N \) can be chosen to satisfy this condition. These samples constitute \( 4N+1 \) constraints on the harmonics of \( g(x) \). The samples at even \( k \) determine the magnitude of the coefficients of \( 2N+1 \) harmonics. This leaves \( 2N \) constraints to be imposed on the coefficients of the harmonics of \( g(x) \). Each of the odd samples of \( P \) is independent of all other samples and imposes an independent condition on the Fourier transform of \( g(x) \). These conditions can be written

\[ P(\kappa \frac{2\pi}{4a}) = \sum_n \sum_m F(m \frac{2\pi}{2a}) \hat{F}^*(n \frac{2\pi}{2a}) \frac{\sin \pi \left( \frac{\psi}{2} - m \right)}{\pi \left( \frac{\psi}{2} - m \right)} \frac{\sin \pi \left( \frac{\psi}{2} - n \right)}{\pi \left( \frac{\psi}{2} - n \right)} \]

or, setting \( k = 2l+1 \)

\[ P[ (2l+1) \frac{2\pi}{4a} ] = \sum_n \sum_m \hat{F}(m \frac{2\pi}{2a}) \overline{\hat{F}}(n \frac{2\pi}{2a}) \frac{(-i)^{n+m}}{\pi^2 (l-m+\frac{1}{2})(l-n+\frac{1}{2})} \]

There are \( 2N \) of these equations, each corresponding to an odd value of \( k \). Each equation has a real part that must be satisfied. Writing \( F(m \frac{2\pi}{2a}) = a_m + jb_m \) gives

\[ P[ (2l+1) \frac{2\pi}{4a} ] = \sum_{n,m} \left[ a_m a_n + b_m b_n \right] \frac{(-i)^{n+m}}{\pi^2 (l-m+\frac{1}{2})(l-n+\frac{1}{2})} \]

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The imaginary part of the equation is
\[ \sum_{n,m} \left[ a_m b_n - a_n b_m \right] \frac{(-i)^{m+n}}{\pi^2 (l-m+\frac{1}{2})(l-n+\frac{1}{2})} \]
which is zero for a symmetric sum.

At the even k samples of P the constraints are
\[ P(2l \frac{\pi k}{2a}) = a_k^2 + b_k^2 \]

These constraints provide a matrix of equations that can be solved for the real and the imaginary parts of the coefficients of the Fourier harmonic expansion of \( g(x) \). In a region of the spectrum from \(-\frac{N\pi}{a}\) to \(+\frac{N\pi}{a}\), the Fourier transform of the autocorrelation function supplies \( 4N+1 \) constraints for the determination of \( g(x) \), and this is enough to specify \( 2N+1 \) of the Fourier harmonics of \( g(x) \) to one arbitrary phase. The number \( N \) can be chosen to be arbitrarily large, and by Parseval's theorem
\[ \int P \, dx = \int g \, g^* \, dx < \infty \]

\( N \) can always be chosen sufficiently large so that an arbitrarily small amount of the transform of the autocorrelation function remains outside the range of consideration.

Therefore, an autocorrelation function that exists over a finite range and is zero elsewhere specifies a function \( e^{j\phi} g \), where \( \phi \) is an arbitrary constant. Both the autocorrelation function and \( g \) are assumed to have a Fourier transform. The function \( g \) is non-zero over half the range of the autocorrelation function and is completely specified by the autocorrelation function. If the independent variable of \( g \) were time,
a finite time autocorrelation would completely determine the finite
time generating function to an arbitrary shift in time. A reflection
of the function, \( g(-x) \), also is an acceptable solution.

6.3 The Optimum Array

The defining equations for the optimum array are found by using
a modification of the procedure described in section 5.4. The important
distinction between the problem of finding the optimum array of elements
for an intended use and that of finding the optimum continuous distribution
is that a position as well as an amplitude must be specified for every
element. If there are few elements in the array, this can be conveniently
done; however, if there are very many elements, an approach involving the
gross nature of the array is desirable, rather than a technique for the
proper assignation of the amplitude and position of each and every element.
It would be expected from physical considerations that the position of a
few elements in a large array is relatively unimportant. There are a
number of arrangements that provide nearly the same average error for a
given set of conditions. Therefore, a method of dictating a class of
arrays to satisfy the requirements is found. These arrays are described
by a density function, which relates the number of elements in a given
area to the number when the area is filled, as discussed in section 2.4.2.
In this manner the locations of elements are not given, but the essential
characteristic of the antenna, the transfer function, can be specified.
For arrays consisting of very large numbers of elements, the transfer
function of a single array can approach very nearly the transfer function
formed with the density function. For the optimization procedure, the
constraint equation becomes

$$\int g(x,y)g(x,y)p(x,y)\,dx\,dy = \text{constant}$$

where \(g(x,y)\) is the envelope of the amplitude taper across the array and is assumed to be real, and \(p(x,y)\) is the density function. The aperture is assumed to be rectangular and to extend from \(-a\) to \(+a\) in the \(x\) direction and from \(-b\) to \(+b\) in the \(y\) direction. It follows that

$$\int_{-b}^{b} \int_{-a}^{a} p(x,y)\,dx\,dy = \frac{M}{N} \cdot 4ab$$

where \(M\) is the number of elements in the array and \(N\) is the number of elements necessary to fill the entire aperture.

The defining equations for the array that satisfies the desired error criterion are

$$g(x,y) = B \int_{-b}^{b} \int_{-a}^{a} \frac{\partial E}{\partial t} \, g(x+u,y+v)\,p(x+u,y+v)\,du\,dv$$

$$1 = B \int_{-2b}^{2b} \int_{-2a}^{2a} \text{ } \frac{\partial E}{\partial t} \, \tau \, du\,dv$$

where the second equation determines the normalization of \(g(x,y)\).

Consider the case of a one dimensional distribution of elements with a uniform amplitude taper. The equations become

$$1 = B \int_{-a}^{a} \frac{\partial E}{\partial t} \, p(x+u)\,du \quad \text{if } |x| < a, \quad p \leq 1$$

$$1 = B \int_{-2a}^{2a} \frac{\partial E}{\partial t} \, \tau \, du$$

A solution to these equations may be obtained by iterating the following equations:
Choosing a trial $p_0(x)$ and carrying out the integration will yield a $g_{p_0}(x)$ that is, in general, not a constant. For the second round, $p_0(x)$ is weighted by $g_{p_0}(x)$ to supply a $p_1(x) = g_{p_0}(x) p_0(x)$. This yields a $g_{p_1}(x)$, which in turn can be used for the weighting of $p_2(x) = g_{p_1}(x) p_1(x)$. This procedure may be continued until a reasonably flat $g_p(x)$ is obtained. This technique is illustrated in Figure 15, where the error function

$$E = \frac{\sigma^2}{\sigma^2 + \tau^2}$$

is used for an array with $\frac{M}{N} = 0.5$. After two rounds a satisfactory $p(x)$ is obtained.

An important restriction to these equations is that the density function cannot exceed unity. In determining the optimum density function, the limitations imposed upon the allowed solutions may prevent significant changes in the antenna transfer function. For example, if the amplitude taper were specified, and the ratio of the total number of elements present to the total possible number was $\frac{M}{N} = 0.9$, the amount of variation allowed for the density function would be small indeed. Under these conditions, it may not be possible to satisfy the integral equations, and the array that produces the minimum system error can occur at one of the limits of the allowed variation. Varying the amplitude taper as well as the density function
Figure 15
provides enough freedom to solve this problem. The cases of interest are those where \( \frac{A_1}{\bar{N}} \) is appreciably less than one. This is apparent from physical reasoning, for it is clear that the arrangement of elements in a nearly full aperture cannot be significantly changed, while a sparsely filled array has the freedom necessary to permit appreciable alteration of its weighting of the input signal.

The above discussion pertains to the determination of the density function for an array of a fixed number of elements within an aperture of fixed size. Implicit in the analysis is that the spatial frequencies that can be received by the antenna are contained within \( 2\alpha \), and that the frequency content of the source outside this range is not considered. However, it may well be that the antenna response range covers but a small portion of the total source spectrum. An example of this is the survey of discrete sources in the UHF range, where antennas that could receive the major part of the spatial frequency spectrum would require lengths of several miles. For cases such as these, a more meaningful measure of the antenna performance is the extent to which the antenna system reproduces the source distribution over the range of spatial frequencies of the source. This requires the determination of the optimum density function for an array of a fixed number of elements without a predetermined limit to the dimensions of the aperture. Problems of this sort can be handled by considering the aperture extent to be equal to the range of the source spectrum, which is assumed to be very large in comparison to the space occupied by the given number of elements when tightly packed. The density function obtained by solving the iterated integral
equations can then be approximated by setting the relatively small parts of the curve equal to zero. Another method of obtaining an approximation to the optimum array is to minimize the error integral without regard to the constraint that \( \tau \) is an autocorrelation function, but requiring that the average antenna cross section be a constant.

Consider an incoherent source distribution that has a flat spatial frequency spectrum from \( u=-2L \) to \( u=+2L \). The variation in \( v \) will be disregarded. This distribution is scanned by a linear array of \( M \) elements of length \( d \) arranged along the \( x \) axis over an extent \( 2a<2L \). All elements are uniformly illuminated with a constant phase in the aperture plane. The error criterion used is the mean square difference between the source distribution and the restored output, as discussed in section 5.3, averaged over the source spatial frequencies. This is

\[
\bar{E}^2 = 1 - \frac{\int_{-2L}^{2L} S^2 du}{\int_{-2L}^{2L} \sigma^2 du} + \frac{\int_{-2L}^{2L} \Sigma^2 du}{\int_{-2L}^{2L} \sigma^2 du} = 1 - \frac{1}{4L} \int_{-2a}^{2a} \frac{\tau^2}{\sigma^2 + \tau^2} du
\]

where the noise spectrum present in the system is a constant, denoted \( \sigma^2 \).

The constraint that the elements are equally illuminated leads to

\[
\int_{-2a}^{2a} \tau du = Md
\]

and a new error integral may be written

\[
I' = \int (E + \lambda \tau) du ,
\]

for which a condition for the minimum error is

\[
\frac{\partial E}{\partial \tau} + \lambda = 0 , \quad \lambda = -\frac{\partial E}{\partial \tau}
\]
This yields
\[ \frac{\partial \xi}{\partial \varepsilon} = -\frac{2 \sigma \varepsilon}{(\sigma^2 + \varepsilon^2)^2} = -\lambda, \quad \xi = \text{const.} = \frac{\lambda \alpha}{4a}, \quad |u| < 2a. \]

Clearly, a function that is a constant over an interval and zero elsewhere is not an autocorrelation function. However, an array of a very large number of elements can have a correlation function that drops from unity at \( u=0 \) to a prescribed value at \( u=d \), remains nearly constant over a large range of values of \( u \), and drops rapidly to zero at \( u=2a \). For the purposes of analysis, the actual autocorrelation function can be approximated by a constant over the aperture range.

This leads to
\[ \overline{\xi^2} = 1 - \frac{1}{4L} \left( \frac{\lambda \alpha}{4a} \right)^2 \int_{-2a}^{2a} \frac{(\frac{\lambda \alpha}{4a})^2}{\sigma^2 + (\frac{\lambda \alpha}{4a})^2} \, du = 1 - \frac{\lambda \alpha}{L (4a \sigma)^2 + (\lambda \alpha)^2} \]

The error is a minimum when
\[ \frac{\partial \overline{\xi^2}}{\partial \alpha} = 0 \quad \Rightarrow \quad \frac{(\lambda \alpha)^2}{(4a \sigma)^2 + (\lambda \alpha)^2} + \frac{8a^2 \lambda^2 \alpha^2 \sigma}{[(4a \sigma)^2 + (\lambda \alpha)^2]} \]

or
\[ 2a = \frac{\lambda \alpha}{2 \sigma}. \]

Defining \( 2a = N \sigma \), where \( N \) is the number of elements required to completely fill the aperture, a noise to signal level of \( \sigma \) leads to an incompletely filled aperture of average density
\[ \frac{M}{N} = 2a \]

and a resultant error of
\[ \overline{\xi^2} = 1 - \frac{\sigma}{2L} = 1 - \frac{\lambda \alpha}{8 \sigma L}, \quad \sigma \leq \frac{1}{2}, \quad L \geq a. \]

Thus the necessary density of elements for optimum performance is dictated by the signal to noise ratio. Having determined the aperture extent for a given problem, the value of \( a \) may be used in the limits of the integral equations previously derived for an optimum array to yield
the density function for the antenna. In this manner a physically realizable antenna is assured.

Antennas formed by this process represent the optimum antenna for the intended use in that they supply to the restoration filter the greatest range of the source spectrum at the highest level commensurate with the noise and constraints on the antenna system. An antenna of this type provides equal sensitivity to a wide range of spatial frequencies. An arrangement of four elements that yields such a transfer function is described by Arsac, who termed it the 0-1-4-6 antenna and discussed its response as a radio astronomy antenna. The Arsac array is a linear array characterized by having each spacing other than those at small regions at \( u=0 \) and the edge of the transfer function occur only once in the array. In general, however, the relative height of the flat portion of the transfer function is dictated by the noise and uncertainty present in the measurements if the output is to be the best representation of the input.

The density function provides a measure of the average relative number of elements along the aperture. Any arrangement that provides the desired antenna transfer function is a solution to the problem. In general, a particular arrangement will yield an approximation to the transfer function. A method of element placement for an array of a very large number of elements is to subdivide the aperture into cells of width \( d \) and to use a table of random digits as the generator of a random code with an average equal to the density function. For example, a density of 0.7 could be formed over a region of the aperture by taking
a reading from a table of random digits ranging from 0 to 9 for every element cell. If the digit read were 6 or less, an element would be located in the cell, while if the digit were 7,8, or 9, no element would be located there. In this manner an arrangement of elements would evolve that resembled a random code. In general, the density function varies across the aperture, and it is necessary to change the statistics of element placement. This is done by changing the range of the digits that permit an element in a cell.

For the case of small $\frac{M}{N}$ ratios but large $N$, the random code is capable of providing a fairly smooth autocorrelation function without extended regions of zero value, while most regular arrangements of widely spaced elements yield a transfer function with large spikes and regions of no response.

This technique is particularly amenable to very large arrays, for the density function is usually a broad, slowly varying function, and the use of a random code to provide element locations results in a better approximation as the total number of elements increases. The antenna is characterized by the autocorrelation function and this in turn by the density function. These describe the gross properties of the antenna and are relatively unaffected by the location of a few elements of the array. Thus there is a class of antennas, each with a different arrangement of elements but all generated by the same density function that will provide an approximation to the transfer function that minimizes the system output error.
In section 6.3 it has been found that an array of elements can be scattered over a very large area to provide high resolution if the spurious responses and errors in measurement are small. The restoration process can provide the necessary reconstruction of the source spectral components to yield the correct form of the source distribution, so long as the signal to noise ratio is consistent with the chosen density of antenna elements. A limiting factor to the resolution of a very large antenna is the degradation of the correlation between the incoming signal measured at two points in the aperture. The lack of correlation of the signal across a wave front may be due to effects such as a turbulent atmosphere or ionospheric refraction. An analysis based on the properties of the mutual coherence function provides a technique for evaluating antenna properties in the presence of partially coherent fields.

The basis for the theory of partially coherent waves, as formulated by Wolf, is presented in section 7.1. The propagation of mutual coherence in two dimensions is given in section 7.2, followed by the mutual coherence of the field produced by a plane source. Also in section 7.3 the difference between the mutual coherence measured parallel and normal to the direction of propagation is derived. This is applied in section 7.4 to the analysis of antenna properties. A relation is found for the maximum gain that can be obtained from a fixed antenna in the presence of partially coherent fields. The figure for normal gain represents a limit just as the normal gain derived by Chu determines a limit for low Q antennas and monochromatic
coherent waves. A transform technique for the analysis of the diffraction patterns of antennas in partially coherent fields is presented in section 7.5, and a number of examples are given.

7.1 The Theory of Partially Coherent Waves

In the past, the phenomena of interference and diffraction of electromagnetic waves have been described largely with the assumptions of perfect coherence for small sources of narrow spectral width or total incoherence for extended sources emitting a broad spectrum.

The formulation of a theory of partial coherence in 1955 by E. Wolf opened the way to consistent methods of dealing with problems involving signals of finite spectral width and sources of finite physical extent. Earlier treatments of the subject are discussed by Parrent. The purpose of this section is to develop techniques for the application of this theory to the problems of antenna analysis and design, and to show the upper limit imposed upon antenna performance by partially coherent waves.

Basic to the theory of partial coherence is the measure of the correlation between the vibrations at two points \( P_1 \) and \( P_2 \) in the wave field of a finite polychromatic source. A single component of the vector field may be expressed as

\[
V^\tau(t) = \int_0^\infty a(v) \cos[\varphi(v) - 2\pi vt] \, dv
\]

The Hilbert transform of this real term is

\[
V^i(t) = \int_0^\infty a(v) \sin[\varphi(v) - 2\pi vt] \, dv
\]

and the complex quantity

\[
V(t) = V^\tau(t) + i \, V^i(t)
\]
is the analytic signal of $V^r$ as determined by Gabor.\textsuperscript{23} The mutual coherence function of the wave field is defined as

$$
\Gamma_{12}(\tau) = \langle V_1(t+\tau)V_2^*(t) \rangle
$$

where $V_1(t+\tau)$ is the analytic signal associated with the real disturbance at point $P_1$ and time $t+\tau$, and $V_2^*$ is the conjugate of the analytic signal associated with the real disturbance $V_2^r$ at point $P_2$ and time $t$. When the points $P_1$ and $P_2$ coincide and $\tau = 0$, the mutual coherence function measures the time average intensity at $P_1$. The sharp brackets represent a time average defined as

$$
\langle V_1(t+\tau)V_2^*(t) \rangle \equiv \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} V_1(t, t+\tau)V_2^*(t, t) \, dt
$$

where $V(t, t) = V^r(t, t) - V^i(t, t)$

and

$$
V^r(\tau, t) = \begin{cases} V^r(t) & |t| < \tau \\ 0 & |t| > \tau \end{cases}
$$

and

$$
V^i(\tau, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{V^r(\tau, t')}{t'-t} \, dt'
$$

The mutual coherence function is normalized to

$$
\gamma_{12}(\tau) = \frac{\Gamma_{12}(\tau)}{\sqrt{\Gamma_{11}(0) \Gamma_{22}(0)}}
$$

where $\Gamma_{11}(0)$ and $\Gamma_{22}(0)$ are the intensities at the points $P_1$ and $P_2$. The function $\gamma_{12}(\tau)$ is called the complex degree of coherence, and from the Schwartz inequality

$$
0 \leq |\gamma_{12}(\tau)| \leq 1
$$

In vacuum the mutual coherence function obeys the two wave equations

$$
\nabla_s^2 \Gamma^s(P_i, P_2, \tau) = \frac{1}{c^2} \frac{\partial^2 \Gamma^s(P_i, P_2, \tau)}{\partial \tau^2}
$$

$s = 1, 2$
7.2 The Propagation of Mutual Coherence in Two Dimensions

The manner in which the mutual coherence propagates in two dimensions is found from the solution to the wave equations by the application of Green's theorem.

Consider two functions $U_1$ and $U_2$ that are continuous and have continuous partial derivatives in a plane area $S$ and along the line $l$ bounding this area. The divergence theorem may be written

$$ \int_S \nabla \cdot U \, dS = \int_L U \cdot n \, dl $$

and Green's second theorem is

$$ \int_S [u_1 \nabla^2 u_2 - u_2 \nabla^2 u_1] \, dS = \oint_L n \cdot (u_1 \nabla u_2 - u_2 \nabla u_1) \, dl $$

The product $n \cdot \nabla U$ is called the normal derivative of $U$ and is written $\frac{\partial U}{\partial n}$. Let the function $U_1$ obey the homogeneous wave equation within $S$, and let $U_2$ obey the homogeneous wave equation at all points within $S$ except $P$, where $U_2$ is singular. Applying Green's theorem to all of $S$ except a small area $S'$ centered on $P$, the left side of the equation becomes zero, and

$$ \oint_L (u_1 \frac{\partial U_1}{\partial n} - u_2 \frac{\partial U_2}{\partial n}) \, dl = -\oint_{l'} (u_1 \frac{\partial U_1}{\partial n} - u_2 \frac{\partial U_2}{\partial n}) \, dl' $$

The behavior of $U$ near $P$ is specified to be

$$ U_2 \to \log \rho \quad , \quad \frac{\partial U_2}{\partial r} \to \frac{l}{\rho} $$

where $\rho$ is measured from the point $P$. The path $l'$ is a small circle of radius $\rho$ centered about $P$. Then

$$ \oint_L (u_1 \frac{\partial U_1}{\partial n} - u_2 \frac{\partial U_2}{\partial n}) \, dl = -\int_0^{2\pi} \left( u_1 \frac{l}{\rho} - \frac{\partial U_1}{\partial \rho} \ln \rho \right) \rho d\rho $$

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and choosing $\rho$ arbitrarily small yields the result

$$U_2(P) = -\frac{1}{2\pi} \oint_L \left( U_1 \frac{\partial U_2}{\partial n} - U_2 \frac{\partial U_1}{\partial n} \right) dl$$

The path of integration, $\ell$, includes a line source $L$ and a large semi-circular arc which closes the path of integration in the plane.

The function $U_2$ is chosen to satisfy the wave equation and the singularity at $P$. At distances far from the source the function should have an $r$ variation of $\frac{e^{ikr}}{r}$, or that of a diverging cylindrical wave. In addition, the boundary condition $U_2=0$ along the line containing $L$ is imposed. These conditions are satisfied by

$$U_2 = \frac{\pi}{2} \left[ H_0^{(i)}(kr) - H_0^{(i)}(kr') \right]$$

where $H_0^{(i)}(kr)$ is the Hankel function of the first kind, $r$ is measured from $P$ to a point on the line source, and $r'$ is the distance from the line source to $P'$, which is the image of $P$ reflected about the line source. Substituting into the integral relation and neglecting any contributions from the circular arc,

$$U(P) = -\frac{k}{2} \int_L U_1 \frac{\partial U_2}{\partial n} dl$$

The normal derivative of $H_0(kr)$ is $kH_1(kr)\cos \theta$, where $\cos \theta$ is the angle between the direction of $r$ and the normal to the line. The relation becomes

$$U(P) = -\frac{k}{2} \int_L U_1 \frac{H_1^{(i)}}{L} \cos \theta \, dl$$

Application of this result to the solution of the wave equation for the propagation of one spectral component of the coherence,

$$\left[ \nabla^2_s + \kappa^2(s) \right] G_{12}(s) = 0$$

$s = 1, 2$
where \( S_1 \) and \( S_2 \) are two independent source points and \( P \) is the point of observation. Repeating this procedure for a second point of observation, \( P_2 \), gives

\[
G(P, P_2, \nu) = \frac{k^2}{4} \int \int G(S_1, S_2, \nu) H_{1}(kr_1) H_{1}(kr_2) \cos \theta_1 \cos \theta_2 d\ell_1 d\ell_2
\]

as the solution to the coherence of the two dimensional field produced by one spectral component of a line source. The complete solution is the integral over all frequencies,

\[
\Gamma(P, P_2, \tau) = \frac{k^2}{4} \int \int \int \int G(S_1, S_2, \nu) H_{1}(kr_1) H_{1}(kr_2) \cos \theta_1 \cos \theta_2 e^{-2\pi i \nu \tau} d\ell_1 d\ell_2
\]

which for the quasi-monochromatic case is

\[
\Gamma(P, P_2, \tau) = \frac{k^2}{4} e^{-2\pi i \nu \tau} \int \int \Gamma(S_1, S_2, 0) H_{1}(kr_1) H_{1}(kr_2) \cos \theta_1 \cos \theta_2 d\ell_1 d\ell_2
\]

For large values of \( kr \), the quasi-monochromatic case becomes

\[
\Gamma(P, P_2, \tau) = \frac{k^2}{2\pi} e^{-2\pi i \nu \tau} \int \int \Gamma(S_1, S_2, 0) e^{\frac{ik(\gamma-\nu)}{r^2}} \cos \theta_1 \cos \theta_2 d\ell_1 d\ell_2 \quad kr_{1,2} \gg 1
\]

These relations describe the mutual coherence between two points of the wave field of a partially coherent source in terms of the mutual coherence function associated with the source.

7.3 The Mutual Coherence of the Field Produced by a Plane Source

The solution to the pair of wave equations in three space dimensions,

\[
\left[ \frac{\partial^2}{\partial s^2} + k^2(\nu) \right] G_{12}(\nu) = 0 \quad s = 1, 2
\]
where
\[ G_{12}(\nu) = \int_{-\infty}^{\infty} G_{12}(t) e^{i\nu t} dt, \quad \nu > 0; \quad G_{12}(\nu) = 0, \quad \nu < 0 \]
is found by the repeated application of Green's theorem.

Let the function \( U \) satisfy the homogeneous wave equation everywhere within a volume bounded by the surface \( S \). A second function \( U_2 \) is chosen to satisfy the homogeneous wave equation everywhere within \( S \) except at the point \( P_1 \). Green's theorem for these functions may be written
\[
\int_{V} (\nabla \cdot U_2 - \nabla \cdot U_1) dV = \int_{S} (\nabla \cdot U_2 - \nabla \cdot U_1) \cdot n dS
\]
The surface may be altered to include a small spherical cap \( S' \) around the point \( P_1 \). The left side of the above equation is then zero, and if the behavior of \( U_2 \) in the vicinity of \( P_1 \) tends to \( r^{-1} \) measured from \( P_1 \), the above expression becomes
\[
U_2(\mathbf{r}) = -\frac{i}{4\pi} \int_{S} (\nabla \cdot U_2 - \nabla \cdot U_1) \cdot n dS
\]
The surface \( S \) includes the plane source \( \sigma \) and a hemispherical cap over the source plane. The contributions to the above integral from the cap may be neglected, and the function \( U_2 \) is chosen to satisfy the radiation condition at large distances from the source and to be zero over \( \sigma \). A function that satisfies these conditions and that of the singularity at \( P_1 \) is
\[
U_2 = \frac{e^{ikr}}{r} - \frac{e^{ikr'}}{r'},
\]
where \( r \) is measured from \( P_1 \) to a point on \( \sigma \) and \( r' \) is measured from the image of \( P_1 \) to \( \sigma \), in the manner described by Sommerfeld. Then
\[ U_i(P_i) = -\frac{i}{4\pi} \int U_i \nabla U_i \cdot n \, d\sigma \]

and \( \nabla U \cdot n = \frac{\partial U}{\partial r} \cos \theta \) where \( \cos \theta \) is the angle between the normal to \( U \) and \( r \). The integral becomes

\[ U_i(P_i) = -\frac{i}{4\pi} \int U_i \frac{2(ikr-1)}{r^2} \cos \theta \, d\sigma \]

Application of the above result to the solution of the wave equations for the coherence,

\[ \left\{ \nabla^2_s + k^2_s(v) \right\} G_{s2}(v) = 0 \quad s = 1, 2 \quad k = \frac{2\pi v}{c} \]

leads to the result

\[ G(P_1, P_2, v) = \frac{i}{4\pi} \int \int G(S_2, S_2, v) \left( \frac{1 - ikr}{r^2} \right) \left( \frac{1 + ikr}{r'^2} \right) \cos \theta_2 e^{ikr_1} d\sigma_2 d\sigma'_2 \]

To obtain the complete solution for all frequency components of \( G \) this equation is transformed to yield

\[ \Gamma(P_1, P_2, r) = \frac{i}{4\pi} \int \int \cos \theta_1 \cos \theta_2 \left[ 1 + \frac{r_1 - r_2}{c} \frac{\partial}{\partial t} - \frac{r_1 r_2}{c^2} \frac{\partial^2}{\partial t^2} \right] \Gamma(S_1, S_2, t - \frac{r_1 - r_2}{c}) d\sigma_1 d\sigma'_2 \]

In the quasi-monochromatic case, the far field approximation to the above relation is

\[ \Gamma(P_1, P_2, r) = \frac{k^2}{4\pi^2} e^{-\frac{2\pi r}{r_2}} \int \int \cos \theta_1 \cos \theta_2 \Gamma(S_1, S_2, \omega) e^{ikr_1} d\sigma_1 d\sigma'_2 \quad |r| \ll \frac{1}{\Delta \omega} \]

The points \( P_1 \) and \( P_2 \) are two independent points of observation, and \( S_1 \) and \( S_2 \) are two independent points on the plane source.

If the point \( P_1 \) is located a small distance \( h \) measured along the normal
to $\sigma$ from $S_1$, the equation for one spectral component of the mutual coherence function is

$$G(P, P, \nu) \sim \frac{1}{4\pi^2} \int_{\sigma} \int_{\sigma} G(S_1, S_1, \nu)(1 + i\kappa r_2) \cos \theta_{12} e^{i \kappa r_2} \frac{d \sigma_1 d \sigma_2}{\sigma^2 r_2^2} e^{- \frac{i \kappa r_2}{\sigma^2 r_2^2}}$$

Let the element of area be $dS_i = 2\pi \rho d\rho$, where $h^2 + \rho^2 = r_i^2$

The principal contribution to the integral is in the vicinity of $\rho = 0$ and the upper integration limit is written as $\Lambda$, where $\Lambda$ is chosen large enough to include all appreciable contributions to the integral. Then

$$G(P, P, \nu) = \frac{1}{2\pi} \int_{0}^{\Lambda} G(S_1, S_2, \nu)(1 + i\kappa r_2) \cos \theta_{12} e^{i \kappa r_2} \frac{h^2 \rho^2 e^{i \kappa h^2 \rho^2}}{(h^2 + \rho^2)^{3/2}} d\rho$$

If $h$ is sufficiently small, $G(S_1, S_2, \nu)$ will not vary significantly over the $\rho$ integration and may be removed from the integral, yielding

$$G(P, P, \nu) = \frac{1}{2\pi} \int_{0}^{\theta} G(S_1, S_2, \nu)(1 + i\kappa r_2) \cos \theta_{12} e^{i \kappa r_2} \frac{h^2 \rho^2 e^{i \kappa h^2 \rho^2}}{(h^2 + \rho^2)^{3/2}} d\rho$$

Introducing a new variable $u = \frac{\rho}{h}$

$$G(P, P, \nu) = \frac{1}{2\pi} \int_{0}^{\theta} G(S_1, S_2, \nu)(1 + i\kappa r_2) \cos \theta_{12} e^{i \kappa r_2} \frac{h^2 \rho^2 e^{i \kappa h^2 \rho^2}}{(1 + u^2)^{3/2}} du$$

As the limit $h = 0$ is approached, the integral over $U$ tends to unity and the expression is

$$G(S_1, S_2, \nu) = \frac{1}{2\pi} \int_{0}^{\theta} G(S_1, S_2, \nu)(1 + i\kappa r_2) \cos \theta_{12} e^{i \kappa r_2} d\sigma_2$$
In the quasi-monochromatic case with \( kr_2 \gg 1 \), the coherence becomes

\[
\Gamma(s, p_1, \tau) = \frac{ik}{2\pi} e^{-im_1\tau} \int \Gamma(s, s_1, 0) e^{\frac{-ikr_2}{r_1^2}} d\sigma \quad |\tau| \ll \frac{1}{\omega}\]

Let the angle between \( S_1P_2 \) and the normal to \( \sigma \) be \( \frac{\pi}{2} - \phi \), and introduce an element of area \( d\sigma = 2\pi \rho_2 d\rho_2 \) where \( \rho_2 \) is measured from \( S_1 \) which is the origin of the plane normal passing through \( P_2 \).

Then

\[
\Gamma(s, p_1, \tau) = \frac{ik}{2\pi} e^{-im_1\tau} \int \Gamma(s, s_1, 0) \frac{\alpha_b}{r_1^2} \frac{1}{k} \sqrt{\rho_2^2 + h^2} d\rho_2
\]

For \( kr_2 \gg 1 \), the exponent is assumed to be slowly varying only in the region around \( \rho_2 = 0 \), where it is approximated by the first and third terms of its Taylor expansion as

\[
\Gamma(s, p_1, \tau) \approx \frac{ik}{2\pi} e^{-im_1\tau} \int \Gamma(s, s_1, 0) e^{\frac{-i\rho_2}{r_1^2}} \frac{\alpha_b}{\rho_2} d\rho_2
\]

If \( \Gamma(s, s_1, 0) \) is a slowly varying function in comparison to \( \cos kr_2 \), the expression becomes

\[
\Gamma(s, p_1, \tau) \approx \frac{ik}{2\pi} e^{-im_1\tau} e^{\frac{-i\rho_2}{r_1^2}} \int_0^{\infty} e^{-\frac{i\rho_2}{r_1^2}} \frac{\alpha_b}{\rho_2} d\rho_2 \quad |h| \ll \frac{1}{\omega}
\]

where it is necessary to restrict \( h \) by a condition analogous to that on \( \phi \).

Neglecting all contributions to the integral from zones other than the first, the coherence becomes

\[
\Gamma(s, p_1, \tau) \approx e^{-im_1\tau} e^{\frac{-i\rho_2}{r_1^2}} \Gamma(s, s_1, 0)
\]

In terms of the distances \( S_1P_2 \) and \( S_1S' \),

\[
S_1S' = S_1P_2 \cos \varphi
\]

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and if the coherence on $\sigma$ is a function of the separation between the points $S_1$ and $S_2$, the result is

$$\Gamma[(s_1, -P_1), \sigma] = e^{i(2\pi/k_s)(S_1 - S_2) \sin \phi} \Gamma[(s_2, -P_2) \cos \phi, \sigma]$$

This result is of importance in understanding the reception properties of antennas in the presence of partially coherent waves, and it affords a definition of the direction of propagation of a wave disturbance. If the mutual coherence is measured for different orientations in space in a region of temporal and spatial stationarity, the direction of propagation of a partially coherent field may be defined as the normal to the plane in which the mutual coherence is real when $\tau = 0$. The coherence measured between two points in a line at an angle to the direction of propagation of partially coherent wavefronts is, apart from a phase factor, the same as the coherence between the projections of these points in the plane that is perpendicular to the direction of propagation.

The range of validity of this expression is obvious from a physical interpretation of the limits imposed on $\tau$ and $\h$. The requirement of quasi-monochromatism means that the time shift $\tau$ between the signals at two points should not be so great that different wave trains are being compared. The limit on the size of $\h$ similarly requires that different parts of the same wave train be compared. The condition that the mutual coherence be a slowly varying function compared to $\cos k\tau$ implies that the line $S_1 S'_1$ lie in the plane normal to the direction of propagation. If this is not so, a rapidly varying component of $\Gamma_{12}$ exists and the earlier expression for the propagation of mutual coherence involving the double integral over the source must be evaluated.
7.4 The Application of Coherence Theory to Antenna Reception or Transmission of Partially Coherent Waves

The mutual coherence function can be used for the description of antenna performance when partially coherent waves are incident upon or emitted by the antenna aperture. The intensity at the focus as a function of the direction of the incoming radiation may be viewed as the antenna pattern on reception of waves of the particular degree of coherence. Similarly, the average intensity per unit solid angle radiated by a partially coherent source may be considered to be the antenna pattern on transmission of partially coherent waves.

The form of the correlation between the signal at different points of the wavefront has been the subject of a number of investigations, and four different functions appear in the literature. The simplest of these is the exponential correlation, which describes the correlation between the signals at two points separated by a distance \( d \) as \( e^{-\frac{d}{\lambda}} \), where \( \lambda \) is a constant for a given situation and is defined as the correlation length.

The reception pattern of a uniform line source illuminated by quasi-monochromatic partially coherent waves with an exponential correlation gives a simple example of the application of coherence theory. Consider a uniform linear antenna of length \( 2a \), ranging from \(-a\) to \(+a\) along the x axis. Incident upon this antenna are wavefronts for which the mutual coherence measured at two points \( S_1 \) and \( S_2 \) in the line normal to the direction of propagation is

\[
\Gamma_{12}(\tau) = \mathcal{I}_0 e^{-\frac{|S_1 - S_2|}{\lambda}} e^{-2\pi \nu \tau}
\]

The mutual intensity along the x axis is then

\[
\Gamma_{12}(\phi) = \mathcal{I}_0 e^{-\frac{|x_1 - x_2| \cos \theta}{\lambda}} e^{-2k(x_1 - x_2) \sin \theta}
\]
The power collected by the antenna is the sum of the contributions from all sections of the antenna and is written as

\[ I = \int_{-a}^{a} \int_{-a}^{a} I_0 e^{-i(x-x_1)\cos\theta} e^{-i\psi(x-x_1)} dx_1 dx_2 \]

where \( \psi = k \sin\theta \), and a multiplicative factor \( \cos\theta \) represents the directivity of each infinitesimal receiving element. This expression may be integrated to yield

\[ I = I_0 \cos\theta \left\{ \frac{4a\cos^3 \theta}{\ell} - \frac{2(\cos^2 \theta - \psi^2) - 2a\cos\theta}{(\cos^2 \theta + \psi^2)} \right\} \left[ 4(\cos^2 \theta - \psi^2) - 2a\cos\theta \sin^2 \phi a \right] \]

This equation may be written

\[ I = I_0 \cos\theta \left\{ \frac{4a\cos^3 \theta}{\ell} - \frac{2(\cos^2 \theta - \psi^2)(1 - e^{-2a\cos\theta})}{(\cos^2 \theta + \psi^2)} - 4e^{-2a\cos\theta} \left( \frac{\cos\theta - \psi^2}{\ell} \right)^2 \sin^2 \phi a \right\} \]

which is of the form

\[ I = A + B \sin^2 \frac{\psi a}{(\phi a)^2} - C \sin \frac{\psi a}{2\phi a} \]

If the correlation length 1 is large in comparison to the antenna length 2a, the second term predominates and the pattern tends toward

\[ I_{\phi} \cos^2 \theta (2a)^{-1} \sin^2 \frac{\psi a}{(\phi a)^2} \]

for the limiting case of monochromatic coherent waves.

If the antenna length is much larger than the correlation length the intensity at the focus is

\[ I = \frac{4I_0 a \ell \cos\theta}{(1 + k\ell^2 \sin^2 \theta)} \]

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which places in evidence the linear relation between aperture length and intensity and shows a forward intensity of $4I_0\alpha l$ and a half power beamwidth of $2\tan^{-1}\left(\frac{1}{\kappa l}\right)$, corresponding to an aperture of a size approximately equal to twice the correlation length. This expression is not valid for $kl \sim 1$, where the mutual coherence is not a slowly varying function compared to the phase terms in the integral over the aperture. However, the case of $kl \sim 1$ would imply virtually no correlation between two points separated by a half wavelength, which is a very unlikely physical phenomenon.

For $a \gg l$ and $\pi l \gg 1$, the forward gain is

$$G = \frac{4I_0\alpha l}{\pi^2/4\pi} = \frac{2}{\pi} kl = \frac{4\alpha l}{\lambda}$$

which corresponds to the normal gain of an omnidirectional antenna of length $2l$. This figure represents a limit to the gain obtainable in the presence of partially coherent waves, just as the normal gain for monochromatic coherent waves determines a limit for low Q antennas.

A similar case has been treated by Parrent and Skinner, who derived the field produced by a line source emitting partially coherent waves. In their problem the correlation across the aperture is specified to be the same regardless of source orientation. This corresponds to a transmitting situation, where partially coherent waves are diverging from the source and are being observed at a point in the far field. Expressions similar to those above are obtained.
7.5 A Transform Relation for the Patterns of Antennas in the Presence of Partially Coherent Waves

In general, the determination of the radiation pattern of an antenna by the methods of the previous section involves integrals that are difficult to evaluate and require machine methods of computation. A method follows that allows the determination of antenna performance in a simpler manner without any serious loss in generality.

The most important assumption of the following is that the fields are stationary in space and time. This means that the correlation between the signals at two points depends only on the difference in times of observation and the difference in positions, and not upon the times or positions themselves. While in an optical system the statistics may be a function of position in the image or object plane, a measurement of the correlation between the signals arriving at two points in an antenna is unlikely to change from one part of the antenna to another.

Under this condition of stationarity of the statistics of the wave field in the vicinity of the antenna, the mutual coherence function describing the average product of the signals at two independent points of the antenna is written

\[ \Gamma_{12}(o) = \langle v(s_i) v^*(s_k) \rangle = g(s_i) g^*(s_k) \gamma(s_i - s_k) \]

where \( g(S) \) is a description of the r.m.s. average amplitude across the aperture and \( \gamma(s_1 - s_2) \) describes the normalized correlation. In the case of monochromatic coherent waves, \( g(S) \) corresponds to the amplitude taper of the aperture, and \( \gamma = 1 \). For partially coherent waves, \( g(S) \) is regarded as the time average amplitude taper, and \( \gamma(s_1 - s_2) \) as the incident wave corre-
lation in the receiving sense, or the normalized source correlation in the transmitting sense.

The expression for the mutual coherence of the far field produced by a quasi-monochromatic plane source is

$$
\Gamma(P_1, P_2, \tau) = \frac{k^2}{4\pi^2} e^{-2\pi i \omega \tau} \int \int \Gamma(S_1, S_2, \sigma) \frac{e^{i k \sigma}}{r_{12}} \, d\sigma, d\sigma
$$

where

$$
\Gamma(S_1, S_2, 0) = g(S_1) g^*(S_2) y(S_1 - S_2)
$$

The coordinates of the source points are \(x_1, y_1\), and \(x_2, y_2\), and the points \(P\) and \(P'\) are allowed to coincide. The variables \(r_1\) and \(r_2\) are written as

$$
k r_1 = kr - kx_1 \sin \theta \cos \phi - ky_1 \sin \theta \sin \phi = kr - x_1 h_x - y_1 h_y$$

$$
k r_2 = kr - kx_2 \sin \theta \cos \phi - ky_2 \sin \theta \sin \phi = kr - x_2 h_x - y_2 h_y$$

where \(r, \theta\) and \(\phi\) describe the coordinates of the observation point. The self coherence is

$$
\Gamma_n(\tau) = \frac{k^2}{4\pi^2} e^{-2\pi i \omega \tau} \int \int g(x_1, y_1) g^*(x_2, y_2) \psi(x_1 - x_2, y_1 - y_2) e^{i(x_1 - x_2) h_x + i(y_1 - y_2) h_y} dx_1 dy_1 dx_2 dy_2
$$

A change of variable for the \(\sigma_2\) integration is

$$
\xi = x_2 - x_1 \quad \eta = y_2 - y_1
$$

for which \(d\xi = dx_2\), \(d\eta = dy_2\)

Then

$$
\Gamma_n(\tau) = \frac{k^2}{4\pi^2} e^{-2\pi i \omega \tau} \int \int g(x_1, y_1) g^*(x_1 + \xi, y_1 + \eta) \psi(\xi, \eta) e^{i(x_1 + \xi) h_x + i(y_1 + \eta) h_y} d\xi d\eta dx_1 dy_1
$$

or

$$
\Gamma_n(\tau) = \frac{k^2}{4\pi^2} e^{-2\pi i \omega \tau} \int \int \psi(\xi, \eta) e^{i\xi h_x + i\eta h_y} \int g(x_1, y_1) g^*(x_1 + \xi, y_1 + \eta) dx_1 dy_1 d\xi d\eta
$$
The second integral has been previously defined as

$$\int g(x_1, y_1) g^*(x_1 + \xi, y_1 + \eta) \, dx_1 dy_1 = A \, C_A(\xi, \eta)$$

where $A$ is the area of the aperture in the $x$-$y$ plane and $C_A(\xi, \eta)$ is the two dimensional aperture correlation function. Thus

$$\Gamma_{11}(r) = \frac{k^2}{4\pi^2} \frac{e^{-2\pi r}}{r} A \int \delta(\xi, \eta) C_A(\xi, \eta) e^{i\xi \psi_x + i\eta \psi_y} \, d\xi \, d\eta$$

from which it is apparent that $\Gamma_{11}$ at an observation point $P_1$ is proportional to the Fourier transform of the product of the aperture correlation function and the wave or source correlation function. When the time shift $\tau$ is zero, the self coherence function measures the intensity at $P_1$, and

$$\Gamma_{11}(0) = I(P_1) = \frac{A}{\lambda r^2} \int \delta(\xi, \eta) C_A(\xi, \eta) e^{i\xi \psi_x + i\eta \psi_y} \, d\xi \, d\eta$$

The $r$ dependence may be removed by using the power per unit solid angle

$$P(\psi_x, \psi_y) = \frac{A}{\lambda^2} \int \delta(\xi, \eta) C_A(\xi, \eta) e^{i\xi \psi_x + i\eta \psi_y} \, d\xi \, d\eta$$

This result provides a convenient and physically understandable method for the computation of power patterns of sources of partially coherent waves. The aperture correlation function is modified by the wave correlation, giving a total correlation function

$$K(\xi, \eta) = \delta(\xi, \eta) \, C_A(\xi, \eta)$$

If the wavefronts are well correlated across the aperture, $K$ will resemble the monochromatic aperture correlation, $C_A$. If the correlation is poor, so that there are many correlation lengths in the aperture, $K$ will appear similar.
to the correlation, $\gamma$. The Fourier transform of $K$ gives the far field power pattern of this particular source distribution, where the power pattern is understood to be the time average intensity per unit solid angle versus angle.

There are many functions of $\gamma$ and $C_A$ for which the Fourier transform of the product may be determined without resorting to machine methods. As examples, consider for the form of $\gamma$ the four functions used to describe wavefront correlation in scatter propagation signals. The receiving antenna is assumed to be a long line source and of uniform illumination so that $C_A(\xi)$ is nearly unity at all values of $\xi$ for which $\gamma$ is not negligible. Stated simply, the antenna is assumed to contain many correlation lengths. For these cases, $K(\xi) \approx \delta(\xi)$ and the transforms may be read from a table for the cases listed below.

**Exponential Correlation**

If the wavefront correlation is $\gamma(\xi) = e^{-|\xi|/\lambda}$

The far field average intensity pattern of the antenna is

$$I(\psi) = P_0 \frac{2l}{(l + l^2 \xi^2)}$$

**Normal Correlation**

If the wavefront correlation is

$$\gamma(\xi) = e^{-\xi^2/\lambda^2}$$

The far field average intensity pattern of the antenna is

$$I(\psi) = P_0 \frac{2}{\sqrt{\pi} \lambda} e^{-l^2 \psi^2}$$
Modified First Order Bessel Correlation

If the wavefront correlation is

$$\gamma(x) = \left| \frac{x}{2} \right| K_1 \left( \left| \frac{x}{2} \right| \right)$$

The far field average intensity pattern of the antenna is

$$I(\psi) = P_0 \frac{\pi}{(1 + \ell^2 \psi^2)^{3/2}}$$

Modified One-Third Order Bessel Correlation

If the wavefront correlation is

$$\gamma(x) = \frac{2^{\frac{2}{3}}}{\Gamma(\frac{1}{3})} \left| \frac{x}{2} \right|^{\frac{1}{3}} K_{\frac{1}{3}} \left( \left| \frac{x}{2} \right| \right)$$

The far field average intensity pattern of the antenna is

$$I(\psi) = P_0 \frac{\Gamma(\frac{2}{3})}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{3}) (1 + \ell^2 \psi^2)^{5/6}}$$

It can be readily seen that the result of Parrent and Skinner showing the far field pattern of a uniform line source with exponential correlation may be determined simply by taking the Fourier transform of

$$K(x) = \left| \frac{2a - x}{2a} \right| e^{-\frac{|x|}{2a}}$$

which may be integrated directly, yielding the result discussed previously.

This technique enables the calculation of the average far field intensity patterns of antennas emitting or receiving partially coherent waves to be performed in a considerably simpler manner than the double integral over the source coordinates. The restrictions on this method are that the signals are quasi-monochromatic and stationary in time and space in the vicinity of the antenna. These requirements are met in a great number of practical applications.
There is an extension of this analysis that leads to further understanding of the effect of wavefront correlation on antenna patterns. Consider the Fourier transform of the aperture correlation function

\[ \mathcal{P}(\psi_x, \psi_y) = A \int \mathcal{C}_A(\xi, \eta) e^{i \xi \psi_x + i \eta \psi_y} \, d\xi \, d\eta \]

which is the coherent far field power pattern, and the Fourier transform of the wave correlation function

\[ \mathcal{S}(\psi_x, \psi_y) = \int \mathcal{S}(\xi, \eta) e^{i \xi \psi_x + i \eta \psi_y} \, d\xi \, d\eta \]

which corresponds to the average angular spectrum of the waves. Applying the general convolution theorem yields

\[ \mathcal{I}(\psi_x, \psi_y) = A \int \mathcal{S}(\xi, \eta) \mathcal{C}_A(\xi, \eta) e^{i \xi \psi_x + i \eta \psi_y} \, d\xi \, d\eta = \frac{1}{(2\pi)^2} \int \mathcal{P}(\psi_x', \psi_y') \mathcal{S}(\psi_x - \psi_x', \psi_y - \psi_y') \, d\psi_x' \, d\psi_y' \]

This result states that the far field average intensity pattern is the convolution of the coherent antenna pattern and the transform of the wave correlation function. This function \( \mathcal{I}(\psi_x, \psi_y) \) corresponds to an exploration of the angular spectrum of the partially coherent waves by the antenna pattern.

The integral of the product of these two functions shifted by an amount \( \psi_x, \psi_y \) gives the value of the average intensity in that direction.

Two limiting cases illustrate this concept. If the waves are perfectly coherent, \( \mathcal{S} \) is a Dirac delta function \( \delta(\psi_x' - \psi_x, \psi_y' - \psi_y) \) and the convolution of this with \( \mathcal{P}(\psi_x', \psi_y') \) gives \( \mathcal{P}(\psi_x, \psi_y) \). If the signals at the aperture are poorly correlated, \( \mathcal{S} \) is a broad function and the convolution with \( \mathcal{P}(\psi_x, \psi_y) \) yields a broad function resembling \( \mathcal{S} \).
The expressions determined previously for the line source and the four types of wavefront correlation may be interpreted in the light of this result. The monochromatic pattern of the uniform line source of length 2a is

$$4a^2 \frac{\sin^2 \alpha}{(a \gamma)^2}$$

and the convolution of this pattern with the correlation transform for the case of a \( \gg L \) retraces the transformed correlation. The constant of proportionality, \( P_0 \), is equal to 2a, showing the linear relationship between aperture length and focal intensity for uncorrelated signals.

This technique of determining the antenna pattern is particularly amenable to graphical methods. An experimental antenna pattern for coherent waves may be placed over the transformed wave correlation function and the resultant product graphically integrated. Such a technique would be valuable in the case of antenna patterns that are not easily analytically represented.

These results are directly applicable to the determination of the radiation pattern of a partially coherent source. The wave coherence function \( \chi(\xi, \eta) \) has been assumed to be independent of the direction of the observation point, as it well must be. In the case of the reception of partially coherent waves, the wave coherence function is not, in general, independent of direction. Previously it has been shown that for the case of quasi-monochromatic waves the wave correlation along a line oriented at an angle \( \theta \) to the direction of propagation is

$$\gamma(s, s') = \gamma \left[ (s - s') \cos \theta \right] e^{-ik(s - s') \sin \theta}$$

if the field is spatially stationary as discussed before. If the coordinates of the antenna aperture are \( x \) and \( y \) and the angle of incidence of the incoming waves is \( \phi \), then

$$\delta(x_1, x_2) = \gamma \left[ (x_1 - x_2) \cos(\theta \cos \phi), (y_1 - y_2) \cos(\theta \sin \phi) \right] e^{-ik(S_1 - S_2) \sin \theta}$$
Assuming that the aperture phase is constant and that the amplitude taper is 
\( g(x, y) \), the collected intensity is proportional to

\[
I_F \sim \int g(x, y) g(x', y') \delta \left[ (x, y) \cos(\theta \cos \phi), (y, y') \cos(\theta \sin \phi) \right] e^{i k (x - x') \sin \theta \cos \phi + i k (y - y') \sin \theta \sin \phi} dx \, dx', dy \, dy'.
\]

or

\[
I_F \sim \int g(x, y) g(x', y') \delta \left[ (x, y) \cos(\theta \cos \phi), (y, y') \cos(\theta \sin \phi) \right] e^{i (x - x') \psi_x + i (y - y') \psi_y} dx \, dx', dy \, dy'.
\]

Introducing the variables \( \xi = x_2 - x_1 \), \( \eta = y_2 - y_1 \) and defining the aperture correlation function as before:

\[
C_A(\xi, \eta) = \frac{1}{A} \int g(x_1, y_1) g^*(x_1 + \xi, y_1 + \eta) \, dx_1 \, dy_1
\]

the result is

\[
I_F \sim A \int C_A(\xi, \eta) \delta \left[ \xi \cos(\theta \cos \phi), \eta \cos(\theta \sin \phi) \right] e^{i \psi_x \xi + i \psi_y \eta} d\xi \, d\eta
\]

which differs from the case of the partially coherent source by the expression for \( \delta' \). The reason for this difference is that the wave correlation across the aperture of the partially coherent source is always the same, while the wave correlation that appears across a receiving aperture in the presence of partially coherent radiation is dependent upon the angle between the aperture normal and the direction of radiation. If the antenna is sufficiently directive and the wavefronts fairly well correlated, \( (ka \gg 1, k\ell \gg 1) \), little energy will be focussed for appreciable values of the angle of radiation and the techniques for dealing with the radiating antenna may be carried over to that of the receiving antenna. However, if these conditions are not satisfied, the above integral must be evaluated as it stands without the benefits of the Fourier transform method.
An example of the application of this method is the determination of the radiation pattern of a very large circular two-dimensional source of uniform illumination and a wave correlation given by

\[ \gamma(x, y) = e^{-\frac{x^2 + y^2}{4l^2}} \]

The monochromatic pattern of this aperture is

\[ P(x, y) = \left( \frac{a^2}{a^2} \right)^2 \left( \frac{2J_0(a\psi)}{(a\psi)} \right)^2 \]

where \( a \) is the aperture radius and \( \psi^2 = x^2 + y^2 \). The transform of the wave correlation

\[ \gamma(x, y) = e^{-\frac{x^2 + y^2}{4l^2}} \]

is

\[ S(\psi) = 4\pi l^2 e^{-l^2\psi^2} \]

Performing the convolution for \( a \gg l \) gives

\[ I(\psi) = a^2 l^2 e^{-l^2\psi^2} \]

as the power pattern of the antenna for a gaussian wave correlation across the aperture.

If the wave correlation across the circular aperture described above is

\[ \gamma(x, y) = e^{-\sqrt{x^2 + y^2}} \]

corresponding to the two-dimensional exponential correlation, the power pattern for \( a \gg l \) is

\[ I(\psi) = \frac{a^2 l^2}{2(1 + l^2\psi^2)^{3/2}} \]

These relations illustrate the determination of the average intensity pattern of a two-dimensional antenna in the presence of partially coherent waves.
VIII

CONCLUSIONS

The multiple plate antenna is a very high resolution radio telescope that can be constructed at an economical cost per unit area. The aperture is divided into a number of smaller independently positioned antennas that are relatively simple and easy to build. This technique of zoning compels narrow band operation for fixed settings, but the adjustability of the plates permits operation over a wide frequency range and rapid and flexible beam steering.

The specification of the antenna requires a determination of the amplitude distribution across the aperture and the positions of the individual elements. It is desirable to omit as large a number of plates from the aperture as is consistent with the design objectives. The calculation of the radiation pattern of incompletely filled apertures can be done easily for small numbers of elements, but requires a description of the plate distribution that ignores the minute detail and presents the gross, pertinent effects when large numbers of elements are considered.

The purpose of constructing antennas of this type is to obtain surveys of the distribution of cosmic sources. The antenna scans the sky and produces an image of the source distribution. The analysis of the extent to which the antenna system reproduces the source variations leads to the use of spatial frequencies and the antenna transfer function, which measures the weighting that the antenna applies to the source angular spectrum.

No antenna can exactly reproduce an incoherent source distribution that is not a constant, as the transfer function cannot be flat over its
entire range of response. The antenna output is, therefore, subjected to restoration, and as much of the coloration inserted by the antenna is removed as is possible in the presence of errors. The mean square difference between the restored output and the source distribution is a measure of the fidelity of the image. The error in the image formation is related to the antenna characteristics, and these can be varied to find the optimum antenna for the intended use. Application of the calculus of variations and constraints intrinsic to antennas results in an integral equation that defines the condition for the optimum antenna. This equation can be used for radiative systems as the Wiener-Hopf equation is used for linear time domain systems. The antenna integral equation is of the form of a self-consistent equation and is amenable to iterative techniques. It is formulated for continuous apertures, where the answer is given as the amplitude distribution across the aperture, and for arrays, where the result is a density function that measures the average number of elements per unit area across the aperture. This equation allows the determination of the arrangement of plates that best satisfies the performance objectives of the antenna system.

For the determination of the errors present in radio astronomy measurements it is necessary to consider the effect of the ground upon the antenna performance. The earth beneath and adjacent to a multiple plate antenna contributes unwanted noise at the output, partly by reflection of the sky noise into the feed and partly by direct radiation from ground.

Another source of deterioration of antenna performance is the lack
of correlation of the signal across a wavefront from effects such as a turbulent atmosphere or ionospheric refraction. An analysis based on the properties of the mutual coherence function provides a technique for evaluating antenna properties in the presence of partially coherent fields. A relation is derived for the normal gain of an antenna, defined as the maximum possible gain for a linear antenna in a partially coherent field. In general, the degradations due to partial coherence are very small for decimeter antennas of the present time, but correlation effects do impose a limit on the possible size of effective apertures within the atmosphere.
REFERENCES


BIOGRAPHY

Allan Carter Schell was born in New Bedford, Massachusetts on April 14, 1934. He graduated from Dartmouth High School in June 1951 and entered the Massachusetts Institute of Technology in the following September. As a cooperative course student in the Department of Electrical Engineering, he was affiliated with the Bell Telephone Laboratories during the undergraduate portion of the program. In 1955 he received a National Science Foundation Fellowship for graduate study. Mr. Schell was awarded the S.B. and S.M. degrees in June 1956. His thesis is entitled "An Analysis of Phase Distortion in Amplitude Limiters". The following year he attended the Technical University of Delft, Holland, on a Fulbright Award.

He was called to active duty as a lieutenant in the U. S. Air Force and was assigned to the Electromagnetic Radiation Laboratory, Air Force Cambridge Research Center, where he worked on the analysis and design of antennas. After completion of his two year tour of duty, Mr. Schell returned to MIT in February, 1960 to undertake a doctoral program as a research assistant in the Research Laboratory for Electronics. The following year he studied as a National Science Foundation Fellow under a program jointly sponsored by the Air Force Cambridge Research Laboratories.

Mr. Schell is an associate member of Sigma Xi, a member of Tau Beta Pi and RBSA, and a member of the Institute of Radio Engineers.