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UNCERTAINTY AND EXHAUSTIBLE RESOURCE MARKETS

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M.I.T. Working Paper No. MIT-EL 79-021WP

April, 1979

(Revised August, 1979)

(Second Revision: December, 1979)

Pindyck, R.S.--Uncertainty and Exhaustible Resource Markets

ABSTRACT

Demand and reserve uncertainty are included in a simple model of an exhaustible resource market by allowing the demand function and the reserve level to fluctuate via continuous-time stochastic processes. Thus, producers always know current demand and reserves, but do not know what demand and reserves will be in the future. We show that demand uncertainty has no effect on the expected dynamics of market price, while reserve uncertainty shifts the expected rate of change of price only if extraction costs are nonlinear in reserves. However if the demand function is nonlinear, both demand and reserve uncertainty affect the dynamics of production, whatever the character of extraction costs. The model is also extended to include exploration, first as a means of reducing uncertainty, and second as a means of accumulating reserves, with uncertainty over the future response of discoveries to exploratory effort.

Uncertainty and Exhaustible Resource Markets

1. Introduction

This paper examines the effects of two sources of uncertainty on the behavior of exhaustible resource markets: uncertainty over the future demand for the resource, and uncertainty over the reserve base that will ultimately be available for exploitation. These uncertainties are likely to be present in most exhaustible resource markets because of the inherent long-run dynamics involved in resource production. Our concern here is with the implications of these uncertainties for market price evolution, for the optimality of the competitive market, and for the role and value of exploration.

Our characterization of uncertainty is quite different from the usual one in which some parameter or variable is taken to be unknown. We model demand uncertainty by assuming that the market demand function shifts randomly but continuously through time according to a stochastic process. Thus, although today's demand is known exactly, future demand may be larger or smaller, and has a variance that increases with the time horizon. Similarly, we model reserve uncertainty by assuming that available reserves shift upwards or downwards, again according to a stochastic process. Thus, as exploitation proceeds over time, resource producers may find that more or less reserves are available for production than originally expected.

In such a world the observed market price will be a random process, but there are a number of questions to be asked about the behavior of the market in expected value terms. First, should the expected price behave differently in the presence of such uncertainty than in its absence? For example, should the presence of uncertainty cause producers -- in competitive or monopolistic markets -- to be more or less conservationist than they would be otherwise? Second, does the competitive market exploit the resource at a rate that is socially optimal in the presence of uncertainty? Finally, what are the implications of uncertainty for exploration, either as a means of reducing the uncertainty itself, or simply to accumulate additional reserves?

We show in this paper that with constant extraction costs and risk-neutral firms, neither demand nor reserve uncertainty (as characterized here) affect the expected price dynamics in competitive or monopolistic markets, and Hotelling's (1931) r-percent rule still applies. (The dynamics of output may, however, change in the presence of uncertainty.) We will see that when extraction costs are a function of the level of reserves, demand uncertainty still has no effect on the expected behavior of price, although reserve uncertainty will affect price.[1] In both cases, however, the rate of production in a competitive market is still socially optimal.[2]

We next extend the model to include exploration. Exploration has two functions -- to obtain information, and to actually increase reserves. We separate these two functions by introducing exploratory effort as a second policy variable in two different ways. First we treat exploratory effort as an input to the production of a stock of "knowledge," with the variance of reserve fluctuations declining as that stock increases. Here we find that exploratory effort is non-zero (and there is a value to information) only if extraction costs depend on reserves. Second, we treat exploratory effort as an input to the production of reserve discoveries, with uncertainty over the future response of discoveries to exploratory effort. We show that this uncertainty, introduced through a parameter in the discoveries function that evolves according to a stochastic process, has no effect on the expected dynamics of market price, and will affect the expected dynamics of exploratory effort only if the discoveries function is nonlinear in the stochastic parameter.

This paper can be viewed as one of a series of papers of the "cake eating" genre that have appeared over the past few years. (In this paper both the size of the cake and the consumer's appetite are changing randomly as the cake is being eaten.) In related papers, Kemp (1976), Loury (1978), Gilbert (1978), Heal (1978) and Hoel (1978) examine resource exploitation when the level of reserves is unknown. (Heal also considers the case of known reserves with an additional discovery of unknown size occurring at some discrete unknown time in the future, and Long (1975) examines the case of known reserves that may be expropriated at some time in the future.)

For most resources, however, the greatest uncertainty is over how reserves will change in the future — that is, what effective recoverable reserves will be over the lifetime of resource use. If a known reserve level changes randomly but continuously over time, then as we will see the optimal rate of resource use and the behavior of the market price will differ considerably from the case where the reserve level is simply unknown, or where it is known but discrete changes in reserves (such as a new discovery in the paper by Heal (1978) or expropriation in the paper by Long (1975))occur at discrete times.

Similarly, Dasgupta and Heal (1974) and Dasgupta and Stiglitz (1978) examine the effects of demand uncertainty on the pattern of resource use. In their models, changes in demand of discrete size occur at discrete and unknown future times as the result of an invention of a substitute for the resource. But the sudden invention and commercialization of a competitive substitute is rare, and it is more common to witness gradual changes in technologies, factor prices, other economic variables, and environmental restrictions that cause gradual changes (sometimes upwards) in the costs of substitutes, and thus gradual changes in resource demand. Again, we will see that random but continuous changes in demand over time lead to a different pattern of resource use than do discrete changes in demand.

In the next section we describe our treatment of uncertainty in more detail, and set forth the basic model. The solution to this model is obtained in Section 3, and discussed in Section 4. Section 5 examines the use of exploration as a means of reducing uncertainty, and Section 6 examines the optimal use of exploration for reserve accumulation under uncertainty. The results are summarized in the concluding section.

2. The Basic Model

Our model of the resource market includes rising extraction costs, and is straightforward except that the market demand function and the reserve level are driven by stochastic processes with independent increments (Ito processes). We first describe the dynamics of demand and the reserve level, and then state the firm's production problem, which is solved using stochastic dynamic programming.[3]

The market demand function has the form

$$p = p(q,t) = y(t)f(q) \tag{1}$$

with $f'(q) < 0$, and $y(t)$ a stochastic process of the form

$$\frac{dy}{y} = \alpha dt + \sigma_1 dz_1 = \alpha dt + \sigma_1 \varepsilon_1(t)\sqrt{dt} \tag{2}$$

where $\varepsilon_1(t)$ is a serially uncorrelated normal random variable with zero mean and unit variance (i.e., dz_1 describes a Wiener process).[4]

Equation (2) implies that uncertainty about demand grows with the time horizon,[5] and that fluctuations in demand occur continuously over time. No jumps in dy/y are possible, and $y(t)$ is continuous with probability 1 (although over any finite time period any change in dy/y of finite size is possible). We stress this point because it distinguishes our characterization of demand uncertainty. In Dasgupta and Heal (1974) and Dasgupta and Stiglitz (1978), for example, changes in demand are discontinuous (a large negative change occurs at some uncertain time).

Reserves are also assumed to fluctuate randomly over time, according to the stochastic process:

$$dR = -qdt + \sigma_2 dz_2 = -qdt + \sigma_2 \varepsilon_2(t)\sqrt{dt} \tag{3}$$

where q is the rate of production. Initial reserves R_0 is known exactly (i.e. if there were no capacity constraint, a volume up to R_0 could be produced at the initial time). Similarly, actual reserves in place at any particular time is known at that time. However, when production begins we do not know what the effective reserves ultimately available for production will be. Effective reserves, given by

$$R_e = \int_0^T q(t)dt = R_0 + \sigma_2 \int_0^T dz_2 \tag{4}$$

is a random variable with mean R_0 and variance $\sigma_2^2 T$. Also, because effective reserves is unknown and because demand fluctuates randomly, the terminal time T (the end of the planning period defined as the time when average profit first becomes zero) is also a random variable.

In physical terms, reserves in this model are closest in nature to the published estimates (for oil and natural gas) of "proved reserves," as long as we remember that estimated "proved reserves" are in fact revised regularly. In the model the current reserve estimate R_0 represents the volume of resource that could be produced today if there were no capacity constraints on the rate of production. This estimate will change over time as a result of new geological surveys that extend known reservoirs, as a result of the information that comes about from exploratory activity, or as a result of the production process itself (e.g., a drop in reserves occurring as water seeps into a reservior).

We can now state the production problem as one of stochastic optimization. Producers must determine the rate of production q(t) over time so that at each point in time the expected value of the sum of discounted profits to go is maximized. Initially, at t = 0, the problem is thus

$$\max_q E_0 \int_0^T [y(t)f(q) - C_1(R)]qe^{-rt}dt = E_0 \int_0^T \Pi_d(t)dt \tag{5}$$

where $C_1(R)$ is average production cost, with $C_1'(R) < 0$. In the competitive case each producer solves the problem assuming that $f(q) = \bar{f}$ is an exogenous parameter independent of that producer's production. (In addition, we ignore the common access problem in the competitive case.) In the monopoly case the single producer recognizes that f(q) is a function of his own production.

The maximization in (5) is subject to the stochastic differential equations (2) and (3), with $R \geq 0$, and t = T when $\Pi_d(t)/q = 0$. We also make the important assumption that the stochastic processes driving demand and reserves are independent i.e., that $E[\varepsilon_1(t)\varepsilon_2(t)] = 0$ for all t.

3. Solution of the Model

We use stochastic dynamic programming to obtain the (expected) optimal price and production trajectories: Define the optimal value function:

$$J = J(y,R,t) = \max E_t \int_0^T \Pi_d(\tau)d\tau \ . \tag{6}$$

Note that J is a function of the stochastic processes y(t) and R(t), and since $E(\varepsilon_1\varepsilon_2) = 0$, the fundamental equation of optimality is:[6]

$$
\begin{aligned}
0 &= \max_q \, [\Pi_d(t) + (1/dt)E_t dJ] \\
&= \max_q \, [\Pi_d(t) + J_t - qJ_R + \alpha y J_y + \tfrac{1}{2}\sigma_1^2 y^2 J_{yy} + \tfrac{1}{2}\sigma_2^2 J_{RR}] \qquad (7)
\end{aligned}
$$

Maximization with respect to q gives:

$$\partial\Pi_d/\partial q = J_R \qquad (8)$$

i.e., the shadow price of the resource should equal the incremental profit that could be obtained by selling an additional unit. Note that in the competitive case equation (7) is linear in q. Producers will therefore produce at maximum capacity if $\Pi_d/q > J_R$, and will produce nothing if $\Pi_d/q < J_R$, so that market clearing implies $\Pi_d/q = J_R$.

Equation (8) could be substituted back into equation (7) to yield a partial differential equation for J(y,R,t). Theoretically, one could solve that equation for J and then determine the optimal production trajectory q*(t) explicitly from (8). In practice, however, the solution of such a partial differential equation is usually not feasible. Instead, we try to obtain a solution by eliminating J from the problem.

First, differentiate equation (7) with respect to R:

$$\frac{\partial\Pi_d}{\partial R} + J_{Rt} - qJ_{RR} + \alpha y J_{Ry} + \tfrac{1}{2}\sigma_1^2 y^2 J_{Ryy} + \tfrac{1}{2}\sigma_2^2 J_{RRR} = 0 \qquad (9)$$

Now using Ito's Lemma note that (9) can be re-written as:

$$\partial\Pi_d/\partial R + (1/dt)E_t d(J_R) = 0 \qquad (10)$$

We cannot differentiate both sides of equation (8) with respect to time since both Π_d and J are functions of stochastic processes, so their time derivatives do not exist. However, we can use Ito's Lemma and apply the differential operator

$(1/dt)E_t d(\ )$ to both sides of the equation:

$$(1/dt)E_t d(\partial\Pi_d/\partial q) = (1/dt)E_t d(J_R) \tag{11}$$

We can combine equations (10) and (11) to eliminate J and obtain:

$$(1/dt)E_t d(\partial\Pi_d/\partial q) = -\partial\Pi_d/\partial R \tag{12}$$

Equation (12) is a stochastic version of the well-known Euler equation from the calculus of variations.[7] The equation is easiest to interpret in its integral form (using the boundary condition that $\Pi_d(T) = 0$):

$$\frac{\partial\Pi_d(t)}{\partial q} = -E_t \int_0^T \frac{\partial\Pi_d(\tau)}{\partial R}\, d\tau \tag{12'}$$

which says that the marginal profit from selling 1 unit of reserves should just equal the expected sum of all discounted future increases in profit that would result from holding that unit in the ground (thereby reducing production costs).

We can now use equation (12) to determine the expected price dynamics. Consider first the competitive case, for which $\partial\Pi_d/\partial q = \Pi_d/q = [p(q,t) - C_1(R)]e^{-rt}$. Substitute this into (12) and divide through by e^{-rt}:

$$-r[p-C_1(R)] + (1/dt)E_t dp - (1/dt)E_t dC_1(R) = -(\partial\Pi_d/\partial R)e^{rt} = qC_1'(R) \tag{13}$$

Now expand $dC_1(R)$ using Ito's Lemma:

$$dC_1(R) = C_1'(R)dR + \tfrac{1}{2}C_1''(R)(dR)^2 \tag{14}$$

Substituting this into (13), and using $E_t(dR) = -qdt$ and $E_t[(dR)^2] = \sigma_2^2 dt$, (from equation (3)) we obtain:

$$(1/dt)E_t dp = r[p - C_1(R)] + \tfrac{1}{2}\sigma_2^2 C_1''(R) \tag{15}$$

In the monopoly case $\partial\Pi_d/\partial q = [MR - C_1(R)]e^{-rt}$. Substituting this into (12) we find

$$(1/dt)E_t dMR = r[MR - C_1(R)] + \tfrac{1}{2}\sigma_2^2 C_1''(R) \tag{16}$$

We will also want to examine the social optimality of the competitive solution. Considering only reserve uncertainty (note that σ_1 docs not appear in (15) or (16)), and replacing (5) with

$$\max_q E_0 \int_0^T [u(q) - C_1(R)q]e^{-rt}dt = E_0 \int_0^T U_d(t)dt \quad , \tag{5'}$$

with $u' > 0$, $u'' < 0$, it is easy to show by applying (12) that the expected rate of change of marginal utility $u'(q) = \phi$ is

$$(1/dt)E_t d\phi = r[\phi - C_1(R)] + \frac{1}{2}\sigma_2^2 C_1''(R) \tag{17}$$

4. The Effects of Uncertainty

Although production and price will fluctuate stochastically in this model, we see from equation (15) that the expected rate of change of the competitive price will differ from the certainty case only if production cost $C_1(R)$ is non-linear. This deviation from the certainty case occurs for a simple reason. Suppose $C_1''(R) > 0$, and random increases and decreases in reserves occur that balance each other out, leaving effective reserves R_e unchanged. Clearly, these fluctuations will have the net effect of raising average production cost, since each decrease in R will raise $C_1(R)$ more than an equal increase in R lowers $C_1(R)$. Since fluctuations occur continuously over time, there is an incentive for producers to speed up the rate of production, thereby reducing this course of increased cost. This in indeed what occurs; the last term in equations (15), (16) and (17) is positive (when $C_1''(R) > 0$), so that relative to the fixed reserve case, price begins lower and rises more rapidly.

What is more interesting is that if average production cost is constant (or linear in R), the expected rate of change of price is the same as in the certainty case. This may seem strange at first, and contradictory to the results of other studies. It is easier to understand if we keep in mind the difference in the nature of the uncertainties in our model from those in the models of other studies.

In Kemp (1976), Gilbert (1978), Loury (1978) and Heal (1978), for example, reserve uncertainty is characterized by a level of reserves that is simply not known, so that the resource producer (or social planner) may suddenly find the stock depleted. Like the driver of a car without a gas gauge, he is likely to adopt a more "cautionary" rate of resource use. In our model, on the other hand, the current reserve level is known exactly at each moment of time, and with a finite production rate it is impossible for that level to drop to zero instantaneously. Since the stochastic component of reserves is continuous in time and the reserve level can be monitored, the rate of resource use can be continuously adapted to the changing reserve level, and thus the expected rate of change of the value of the resource is unaffected by uncertainty.

To use a distinction introduced by Merton (1973), in our model there is no "current" uncertainty, but only "future" uncertainty. Using this analogy, holding an exhaustible resource asset in our model -- whether by a competitor, a monopolist, or a social planner -- is like continually reinvesting in very short-term bonds as interest rates fluctuate stochastically, so that the return from the asset over the next "instant" is known with certainty. In the other studies cited, the return over any period, however small, is uncertain, as the resource holder can suddenly find his reserves exhausted.

It is interesting to note that even with average cost constant, the expected rate of change of production can differ from that in the certainty case -- even though the expected rate of change of price is the same as the certainty case. The expected rate of change of production is found by recognizing that the optimal rate of production is a function of the state variables y and R, i.e., $q = q^*(y,R)$, and then expanding the differentials dq and dp using Ito's Lemma. We do this is Appendix A, and show that in the competitive case (with constant average cost),

$$\frac{1}{dt}E_t dq = \frac{(r-\alpha)p - rk - \gamma(\sigma_1, \sigma_2)}{yf'(q)} \tag{18}$$

This deviates from dq/dt in the deterministic case by the factor γ,

$$\gamma(\sigma_1, \sigma_2) = \sigma_1^2 y^2 q_y f'(q) + \frac{1}{2} y f''(q)[\sigma_1^2 y^2 q_y^2 + \sigma_2^2 q_R^2] \tag{19}$$

where q_y and q_R are the derivatives of the optimal rate of production with respect to y and R.

We show in the Appendix that $q_y > 0$, so that $\gamma < 0$ if $f''(q) \leq 0$, but is of undetermined sign if $f''(q) > 0$. The effects of demand and reserve fluctuations are easiest to see by considering the two components of γ -- the term in f' and the term in f" -- separately. The term in f'(q) tends to increase the rate at which q falls, and therefore raises the initial q_0 (and lowers p_0). This occurs because given any value of α, a larger value of σ_1 causes the demand curve to rotate downwards over time (in expected value terms) at a faster rate (see footnote 4). Given the r-percent rule for the rate of growth of price, this accelerates the rate at which q falls and requires a larger initial q_0 for the terminal condition (that expected reserves and expected demand become zero simultaneously) to hold.[8]

The term in f"(q) tends to reduce the rate at which q falls and lower q_0 if $f'' > 0$. The reason is that fluctuations in p of mean zero imply a net increase (decrease) in q if f" > (<) 0. Thus equation (15) for p together with the terminal condition requires a slower rate of decline of q (and lower q_0 and higher p_0) if there are fluctuations in either demand or reserves.

The effects of uncertainty on production are similar in the case of a monopolist, except that the rate of change of q will depend on f"' as well as f".

Results for this case are given in Appendix A.

We also show in Appendix A that the expected rate of production under social welfare maximization is the same as with a competitive market as long as producers are (as we have so far assumed) risk-neutral. Risk-averse producers will <u>under-conserve</u>, with p beginning lower and rising more rapidly than in equation (15).[9]

5. The Use of Exploration to Reduce Uncertainty

We have seen that demand and reserve uncertainty will alter the rate of resource production, and if production cost is a nonlinear function of reserves, will alter the expected rate of change of price as well. A question that naturally arises is what expense would producers be willing to incur to reduce this uncertainty?

In this section we introduce exploration as a means of reducing stochastic fluctuations in reserves by extending the basic model in a simple way.[10] Producers now adjust two policy variables over time, production q and the level of exploratory activity w, to maximize:

$$\max_{q,w} E_0 \int_0^T [p(q)q - C_1(R)q - C_2(w)]e^{-rt}dt = E_0 \int_0^T \Pi_d(t)dt \tag{20}$$

Subject to $\quad dR = -qdt + \sigma(K)dz \qquad (21)$

and $\quad dK = g(w)dt \qquad (22)$

with $q,w,R,K \geq 0$. Here K is a stock of "knowledge" that is "produced" by exploratory activity. The value of K (if any) is that it reduces the variance of the stochastic fluctuations in R, i.e., $\sigma'(K) < 0$. C_2 is the cost of exploratory activity. We assume that $C_2' > 0$, $C_2'' \geq 0$, $g' > 0$, $g'' < 0$, that marginal discovery cost $C_2'(w)/g'(w)$ increases with w, and $C_2'(0)/g'(0) = 0$.[11]

Our objective is to determine the dynamics of exploratory activity, and in particular to determine under what conditions w will be non-zero, so that there is value to increasing K. The solution to the above optimization problem follows much the same approach as in Section 3, and is presented in Appendix B. There we show that for risk-neutral competitive producers the expected rate of change of price is again given by equation (15), and the expected rate of change of w is given by

$$\frac{1}{dt}E_t dw = \frac{rM(w) - \frac{1}{2}\sigma^2(K)w_R^2 M''(w) + \sigma'(K)\sigma(K)C_1'(R)}{M'(w)} \tag{23}$$

where $M(w) = C_2'(w)/g'(w)$, i.e., is marginal "discovery" cost.[12]

We show in Appendix B that $w = 0$ always if $C_1'(R) = 0$ (or, of course, if $\sigma'(K) = 0$). In fact, <u>a unit of knowledge K has value only if $C_1'(R) < 0$.</u> As shown in the Appendix, the value of K comes about if $C_1' < 0$ because stochastic fluctuations in R will, over the planning horizon, increase production costs. A reduction in $\sigma(K)$ through exploratory activity will reduce these costs by allowing producers to better allocate production intertemporally - i.e., to allocate more production to periods when R is relatively large. (The allocation results in a smaller expected stream of profits, of course, the less is known about the future trajectory of R.)

Note from equation (23) that if $M''(w)$ is negative or positive but small, w will (on average) <u>rise</u> over the planning horizon (falling discontinuously to zero at $t = T$). Discounted exploration costs are reduced by postponing exploration (so that the term $rM(w)$ in (23) tends to push w into the future). Further, as can be seen by equating (B.4) and (B.7), the two expressions for the shadow price of K given in Appendix B, the marginal benefit of a unit of K rises as R falls to R_{min} ($R \to R_{min}$ as $\Pi_d/q \to 0$

and $q \to 0$), so that marginal cost $M(w)$ should rise and w should rise over the horizon. (This pattern of rising w is partly counteracted by the term in $M''(w)$; w should be distributed more evenly over time if $M(w)$ rises more and more sharply with w.)[13]

6. The Use of Exploration to Accumulate Reserves

We now turn to the use of exploration as a means of discovering new reserves, but with uncertainty in the exploration-discovery relationship. Stochastic models of resource exploration have recently been developed by Arrow and Chang (1978) and Deshmukh and Pliska (1978) in which discrete increments of reserve discoveries occur stochastically (e.g., via a Poisson process) in proportion to the level of exploratory activity. Here we follow a different tack and assume that the response of discoveries to exploratory activity is known today but becomes increasingly uncertain in the future.

An earlier paper by this author (1978b) examined the linkage between resource exploration and production through a deterministic model in which reserves can be maintained or increased through exploration, and production costs vary inversely with the reserve level.[14] Here we extend that model by introducing into the discoveries function a parameter that follows a stochastic process. We examine the effects of uncertainty by comparing our results here with those of the earlier paper.

As before, producers in this model determine production q and the level of exploratory effort w. The rate of reserve discoveries depends on w, on cumulative discoveries x, and on a parameter θ that follows a stochastic process -- that is, $\dot{x} = f(w,x,\theta)$, with $f_w > 0$ and $f_x < 0$. Thus, as exploration and discovery proceed over time, it becomes more and more difficult to make new discoveries. We make no assumption now about the way θ affects the discovery rate except that f be smooth in θ, but we specify the dynamics of θ as

$$d\theta = \sigma(\theta)dz = \sigma(\theta)\varepsilon(t)\sqrt{dt} \quad , \tag{24}$$

so that $E[d\theta] = 0$. Thus, (given w and x) the rate of discoveries today is known exactly, but we cannot know what the rate will be in the future.[15]

The producer's problem is:

$$\max_{q,w} E_0\int_0^T [qp - C_1(R)q - C_2(w)]e^{-rt}dt = E_0\int_0^T \Pi_d(t)dt \tag{25}$$

subject to

$$\dot{R} = \dot{x} - q \tag{26}$$

$$\dot{x} = f(w,x,\theta) \tag{27}$$

$$d\theta = \sigma(\theta)dz \tag{28}$$

and R, q, w, x ≥ 0. We again assume that $C_1'(R) < 0$, $C_2'(w) > 0$, $C_2''(w) \geq 0$, and that marginal discovery cost $C_2'(w)/f_w$ increases with w.[16]

The solution of this problem again follows the approach used in Section 3, and is presented in Appendix C. There we show that the dynamics of price is given by:

$$(1/dt)E_t dp = rp - rC_1(R) + C_1'(R)f(w,x,\theta) \tag{29}$$

under competition, and by

$$(1/dt)E_t dMR = rMR - rC_1(R) + C_1'(R)f(w,x,\theta) \tag{30}$$

under monopoly, while the dynamics of exploratory effort is given by

$$(1/dt)E_t dw = \frac{C_2'(w)[(f_{wx}/f_w)f - f_x + r + \frac{1}{2}\sigma^2(\theta)f_{w\theta\theta}/f_w] + C_1'(R)qf_w}{C_2''(w) - C_2'(w)\ (f_{ww}/f_w)} \tag{31}$$

under both competition and monopoly.[17]

Equations (29), (30), and (31) can be compared to Equations (9), (15), and (13) in Pindyck (1978b). We can see from this that uncertainty as modeled here has no effect on the expected rate of change of market price, but will have an effect on the expected rate of change of exploratory effort, and therefore on the expected <u>level</u> of market price.

The effect of uncertainty depends on the nonlinearity of f with respect to θ, and works in much the same way that reserve uncertainty affected price in our model without exploration. If $f_{w\theta\theta} > 0$ and $f_{ww} < 0$, uncertainty will make $(1/dt)E_t dw$ <u>larger</u> (and the initial value of w smaller). For example, in the certainty case, if R is initially very small, w will begin high, with $\dot{w} < 0$.[18] We see from equation (31) that if $f_{w\theta\theta} > 0$, $(1/dt)E_t dw$ will be larger, so that w will begin at a lower level and fall less rapidly. However, this does not imply a reduction in the rate of reserve accumulation. With $f_{w\theta\theta} > 0$ any increase in θ will raise the marginal physical product of exploratory effort more than an equal decrease in θ will lower it. Zero-mean fluctuations in θ will on average increase the productivity of exploratory effort, thereby <u>reducing</u> the amount of exploration currently needed in the intertemporal trade-off between the gain from postponing exploration (and discounting its cost) and the loss from higher current production costs resulting from a smaller reserve base. Similarly, in the certainty case if R is initially large, w will be initially small, with $\dot{w} > 0$ at first, and < 0 later. With $f_{w\theta\theta} > 0$, fluctuations in θ will make w still smaller at first, and $(1/dt)E_t dw$ larger (although w and q will fall to zero <u>later</u> because of the increase in productivity).

A simple example of a discovery function is $f(w,x) = Aw^{\alpha}e^{-\beta x}$.[19] We can see that linear shifts of this function, for example $f(w,x,\theta) = A\theta w^{\alpha}e^{-\beta x}$, with $d\theta = \sigma\theta dz$, will have no effect on the expected level of exploratory effort. Although the future discovery rate is unknown, the current rate is known, and producers can continuously adjust to random changes in that rate. It is only where stochastic fluctuations in θ on average raise (lower) the marginal product of exploration that the initial w is decreased (increased). For example, the initial w is reduced if $f(w,x,\theta) = Aw^{\alpha\theta}e^{-\beta x}$, or $Aw^{\alpha}e^{-\beta\theta x}$, with $d\theta = \sigma\theta dz$ in both cases.

These results also provide some insight into the measurement of resource scarcity. Resource "rent," i.e. price net of extraction cost (or marginal revenue net of extraction cost in monopolistic markets), can be shown to be a useful measure of <u>in situ</u> scarcity, but it is not clear how rent itself should be estimated.[20]

Devarajan and Fisher (1979) have raised the issue of whether marginal discovery cost can be used to measure rent if there is uncertainty.[21] In our model, marginal discovery cost differs from rent whenever the shadow price of cumulative discoveries is non-zero. As can be seen from equations (C.2), (C.3), and (C.7') in the Appendix, undiscounted rent is:

$$J_R e^{rt} = p - C_1(R) = \frac{C_2'(w)}{f_w} + e^{rt}E_t \int_t^T \frac{dC_2}{dx}\Big|_{f=f*} e^{-r\tau}d\tau \qquad (32)$$

i.e., the sum of marginal discovery cost and the undiscounted shadow price of cumulative discoveries. As long as depletion lowers the productivity of exploration this last term will be positive, so that rent must be measured by subtracting extraction costs from the observed price, or from some estimate of what the price would be in a free market.

7. Concluding Remarks

The major results of this paper are summarized for the competitive market in Table I. These results are easier to understand if we remember that our characterization of uncertainty is different from that in most other studies of resource use. Here uncertainty --- whether over demand for the resource, the reserve level, or a parameter affecting the response of discoveries to exploratory effort -- pertains to the future value of the variable in question. Producers in our model have complete information about the current status of the resource market; what they do not know is what the values of demand, reserves, etc. will be in the future. However, since stochastic fluctuations occur continuously over time, producers (or social planners) can adapt to these fluctuations continuously. As a result stochastic fluctuations alter the expected rate of change of price or exploratory activity only to the extent that the average cost of production or productivity of exploration is changed through nonlinearity in a fluctuating variable.

[Insert Table I]

Thus we find that with average production cost constant, price will rise according to Hotelling's r-percent rule. However, even with C_1 constant, the rate at which production falls, and the initial values of production and price, are affected by uncertainty. This occurs first because demand fluctuations cause rotational shifts in the demand function, and second because if demand is non-linear, zero-mean fluctuations in price imply a net change in production for market clearing.

We also examined the use of exploration, first as a means of gathering information, and second to accumulate reserves. We found that exploration should be used for information-gathering (i.e. to reduce the variance of stochastic reserve fluctuations) only if production costs vary with reserves. If $C_1'(R) < 0$, *ex ante* knowledge of the terminal time T and the distribution of R over time permit production costs to be reduced on average by allocating more production to periods when R is (known to be) larger.

We found that when exploration is used to accumulate reserves, the time-profile of exploratory activity is altered if a stochastically fluctuating parameter enters the discoveries function nonlinearly. However this occurs not because the future response of discoveries to exploration is not known, but rather because fluctuations can change the average productivity of exploratory effort, and thus shift the optimal level of exploration.

We must ask whether real-world uncertainty in resource markets can be well approximated by the continuous stochastic processes used in this paper. We have argued that the major uncertainties over demand and reserves have more to do with the *future* values of those variables, with random changes usually occurring more or less continuously over time. Of course resource markets are also affected by other types of uncertainty (several of the oil-exporting countries might suddenly cut production, for example), and our results in this paper should therefore not be taken too literally. We have only examined the effects of a particular type of uncertainty on resource markets.

APPENDIX

A. Dynamics of Production in the Basic Model

To obtain the expected dynamics of production under competition, monopoly, and utility maximization, remember that $q = q^*(y,R)$ along the optimal trajectory. Now expand the differential dq using Ito's Lemma:

$$dq = q_y dy + q_R dR + \frac{1}{2}q_{yy}(dy)^2 + \frac{1}{2}q_{RR}(dR)^2 + q_{yR}dydR \qquad (A.1)$$

Use equations (2) and (3) for dy and dR, and recall that $E(dz_1dz_2) = 0$ by assumption, so that

$$E_t[(dq)^2] = \sigma_1^2 y^2 q_y^2 dt + \sigma_2^2 q_R^2 dt + o(t) \qquad (A.2)$$

where o(t) represents terms that vanish as $dt \to 0$. Also note that

$$E_t[dqdy] = \sigma_1^2 y^2 q_y dt + o(t) \qquad (A.3)$$

Now, to determine the dynamics of production in the competitive case (with constant average production cost), expand $dp = d[y(t)f(q)]$:

$$dp = yf'(q)dq + f(q)dy + \frac{1}{2}yf''(q)(dq)^2 + f'(q)dqdy \qquad (A.4)$$

Take expectations, divide through by dt, substitute in equations (15) (for $E_t dp$), (A.2) and (A.3) and re-arrange to yield equations (18) and (19).

The sign of $\gamma(\sigma_1,\sigma_2)$ in equation (19) depends in part on the sign of q_y. Consider an increase in y, so that the demand curve rotates to the right. Now suppose $q_y \leq 0$. p rises according to equation (15), so as long as $C_1 > 0$, q will fall to zero before reserves are exhausted. Thus with $C_1 >$ (=) 0, the terminal conditions can be satisfied only if $q_y >$ (=) 0.

To determine the dynamics of production in the monopoly case, expand $dMR = d(yf + qyf'(q))$, take expectations, divide through by dt, substitute in equations (16), (A.2) and (A.3), and re-arrange, to yield:

$$\frac{1}{dt}E_t dq = \frac{(r-\alpha)MR - rk - \beta(\sigma_1,\sigma_2)}{MR'(q)} \qquad (A.5)$$

with $\beta(\sigma_1,\sigma_2) = \sigma_1^2 y q_y MR'(q) + \frac{1}{2}y[f''(q) + qf'''(q)][\sigma_1^2 y^2 q_y^2 + \sigma_2^2 q_R^2]$ (A.6)

The behavior of q can be somewhat more complicated than in the competitive case insofar as $f'''(q)$ might change sign as q falls. Thus q might at first fall more rapidly, but later more slowly than in the certainty case.

The dynamics of production under social welfare maximization is found by expanding $d[u'(q)]$ and then following the same steps as in the competitive and monopoly cases above. The reader can easily demonstrate that equations (18) and (19) will again apply, so that the competitive market exploits the resource at the socially optimal rate.

This is not the case, however, if the competitive producers are risk-averse. If the integrand in equation (5) is replaced by $U_d = U(\Pi)e^{-rt}$, with $\Pi = (p-C_1)q$, and $U' > 0$ and $U'' < 0$, then the dynamics of price is found by replacing Π_d with U_d in the stochastic Euler equation (12). Doing this (assuming for simplicity C_1 constant and only reserve uncertainty) yields:

$$\frac{1}{dt}E_t dp = \frac{r(p-C_1)U'(\Pi) - \sigma_2^2 q_R^2 [p'(q)]^2 U''(\Pi)}{U'(\Pi) + (p-C_1)U''(\Pi)} \tag{A.7}$$

Since $U'' < 0$, p will rise faster and begin lower the greater is σ_2, so that producers under-conserve.

B. Exploration to Reduce Uncertainty

Here we show that equations (15) and (23) describe the dynamics of price and exploration for the model of Section 5. As usual we define the optimal value function

$$J = J(R,K,t) = \max_{q,w} E_t \int_0^T \Pi_d(\tau)d\tau \tag{B.1}$$

The fundamental equation of optimality is:

$$0 = \max_{q,w} [\Pi_d(t) + J_t - qJ_R + g(w)J_K + \tfrac{1}{2}\sigma^2(K)J_{RR}] \tag{B.2}$$

Maximization with respect to q implies:

$$\partial\Pi_d/\partial q \;=\; J_R \tag{B.3}$$

as before. Maximization with respect to w implies

$$\partial\Pi_d/\partial w = -g'(w)J_K \tag{B.4}$$

so that J_K, the shadow price of a unit of "knowledge", is equal to $e^{-rt}C_2'(w)/g'(w)$, the discounted marginal cost of "finding" that unit.

To see that equation (15) again describes the dynamics of price, note that equation (10) results from differentiating (B.2) with respect to R and re-writing using Ito's Lemma, and equation (11) again results from applying the differential operator $(1/dt)E_t d(\;)$ to both sides of (B.3). Combining equations (10) and (11) again gives equation (12), and (15) follows from this.

To determine the dynamics of exploration, differentiate (B.2) with respect to K, and, noting from (B.3) that $J_{RR} = \partial^2\Pi_d/\partial R\partial q$, re-write as:

$$(1/dt)E_t d(J_K) + \sigma'(K)\sigma(K)\partial^2\Pi_d/\partial R\partial q = 0 \tag{B.5}$$

Now apply the differential operator to both sides of (B.4) and then combine with (B.5) to eliminate J_K and yield:

$$(1/dt)E_t d[e^{-rt}C_2'(w)/g'(w)] = \sigma'(K)\sigma(K)C_1'(R)e^{-rt} \tag{B.6}$$

Equation (23) follows by expanding the left-hand side of (B.6) using Ito's Lemma and noting that $w = w^*(R,K)$, so that $E_t(dw)^2 = \sigma^2 w_R^2 dt$.

Besides yielding equation (23), equation (B.6) can be interpreted to show the value of information. Using (B.4) for the shadow price of information J_K, write (B.6) in integral form as follows:

$$J_K = \frac{\partial}{\partial K} E_t \int_t^T \frac{1}{2}\sigma^2(K)C_1'(R)e^{-r\tau}d\tau \tag{B.7}$$

Note that K has value only if $C_1' < 0$. If $C_1'(R) < 0$ and the distribution of reserves over time is known for all time, production costs can on average be reduced by allocating more production to periods when R is relatively large.[22] Stochastic fluctuations add to production costs by reducing the ability to make this optimal allocation of production over time. If $C_1'(R) = 0$ (or if $\sigma'(K) = 0$), $J_K = 0$, $C_2'(w)/g'(w) = 0$, and therefore $w = 0$.

C. Exploration to Accumulate Reserves

Here we show that equations (29) and (31) describe the dynamics of price and exploration for the model of Section 6 under competition, and (30) and (31) describe the dynamics under monopoly. The fundamental equation of optimality is now:

$$0 = \max_{q,w} [\Pi_d(t) + J_t + [f(w,x,\theta) - q]J_R + f(w,x,\theta)J_x + \tfrac{1}{2}\sigma^2(\theta)J_{\theta\theta}] \qquad \text{(C.1)}$$

Maximization with respect to q (and market clearing in the competitive case) again gives:

$$\partial\Pi_d/\partial q = J_R \qquad \text{(C.2)}$$

and maximization with respect to w gives:

$$\partial\Pi_d/\partial w = -f_w(J_R + J_x) \qquad \text{(C.3)}$$

Note that $(J_R + J_x)$ is the "net" shadow price of a unit of reserve discoveries, i.e., the value of increasing the reserve base by 1 unit plus the (negative) value of increasing cumulative discoveries by 1 unit, thereby increasing the cost of all future discoveries. Since $\partial\Pi_d/\partial w = -C_2'(w)e^{-rt}$, this net shadow price is just equal to the discounted marginal cost of a unit of discoveries (an in equation (B.4) previously).

Differentiate (C.1) with respect to R, and re-write as:

$$\partial\Pi_d/\partial R + (1/dt)E_t d(J_R) = 0 \tag{C.4}$$

Apply the differential operator $(1/dt)E_t d(\ )$ to both sides of (C.2), and combine the resulting equation with (C.4) to yield:

$$(1/dt)E_t d(\partial\Pi_d/\partial q) = -\ \partial\Pi_d/\partial R \tag{C.5}$$

Note that this is the same as equation (12) in Section 3, and is used to obtain the equilibrium price trajectory.

Now differentiate (C.1) with respect to x, and noting that $\partial\Pi_d/\partial x = 0$, use Ito's Lemma to re-write the resulting equation as:

$$(1/dt)E_t d(J_x) = -\ f_x(J_R + J_x) \tag{C.6}$$

Combine this with equation (C.3) to yield:

$$(1/dt)E_t d(J_x) = (f_x/f_w)\partial\Pi_d/\partial w \tag{C.7}$$

If we write (C.7) in integral form we see that it boils down to the definition of the shadow price of cumulative discoveries, J_x:

$$-J_x = E_t \int_t^T (f_x/f_w)C_2'(w)e^{-r\tau}d\tau = E_t \int_t^T \frac{dC_2}{dx}\Big|_{f=f*} e^{-r\tau}d\tau \tag{C.7'}$$

i.e., $-J_x$ is just the sum of all discounted future increases in discovery costs (evaluated at the optimal discovery rate) brought about by a 1-unit increase in cumulative discoveries.

Now apply the differential operator $(1/dt)E_t d(\ )$ to both sides of (C.3):

$$\frac{1}{dt} E_t d(\frac{\partial\Pi_d}{\partial w}) = -\ f_w \frac{1}{dt}E_t d(J_R) - f_w \frac{1}{dt}E_t d(J_x) - (J_R+J_x)\frac{1}{dt}E_t d(f_x) \tag{C.8}$$

Substitute equations (C.3), (C.4) and (C.7) into (C.8) to eliminate the derivatives of J:

$$\frac{1}{dt} E_t d(\frac{\partial\Pi_d}{\partial w}) = f_w \frac{\partial\Pi_d}{\partial R} - f_x\frac{\partial\Pi_d}{\partial w} + (1/f_w)(\frac{\partial\Pi_d}{\partial w})\frac{1}{dt} E_t d(f_w). \tag{C.9}$$

The dynamics of price and exploratory effort can now be obtained from Equations (C.5) and (C.9) respectively. For risk-neutral competitive producers, $\partial\Pi_d/\partial q = [p - C_1(R)]e^{-rt}$, while in the monopolistic market, $\partial\Pi_d/\partial q = [MR - C_1(R)]e^{-rt}$. Substitution into (C.5) yields equations (29) and (30) respectively. To obtain equation (31) substitute $\partial\Pi_d/\partial w = - C_2'(w)e^{-rt}$ into (C.9), expand the differential operators, and re-organize.

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FOOTNOTES

*Research leading to this paper was supported by the National Science Foundation under Grant No. SOC77-24573, and that support is gratefully acknowledged. The author also wishes to thank John Cox, Tracy Lewis, Robert Merton, Richard Schmalensee, and an anonymous referee for helpful comments and suggestions.

[1]Of course, when extraction costs are not constant, the r-percent rule no longer applies even in the deterministic case. See Levhari and Liviatan (1977) and Pindyck (1978a).

[2]Weinstein and Zeckhauser (1975) examined demand uncertainty using a discrete-time model similar to the one in this paper, where current demand is known but demands in future periods are unknown (but are revealed at the beginning of each period). They also found that (with zero extraction costs and risk-neutral firms) the expected competitive market price will rise at the rate of interest, and the competitive market equilibrium is socially optimal. Here we extend the Weinstein-Zeckhauser results to the continuous-time case.

[3]The paper makes use of Ito's differentiation rule for functions of stochastic processes as well as stochastic dynamic programming. For a brief introduction to these techniques, see Chow (1979) or the Appendix of Fischer (1975). Kushner (1967) provides a detailed treatment. Applications to problems in economics can be found in Merton (1969, 1971, 1973) and Fischer (1975).

[4]Equation (2) is the limiting form as $h \to 0$ of the discrete-time difference equation $\frac{y(t+h) - y(t)}{y(t)} = \alpha h + \sigma_1 \varepsilon_1(t)\sqrt{h}$, and $E[dy/y] = \alpha dt$, and $Var[dy/y] = \sigma_1^2 dt$. Note that $y(t)$ is log-normally distributed, with $E_o[\log\frac{y(t)}{y(0)}] = (\alpha - \frac{1}{2}\sigma_1^2)t$, so that the expected value of demand remains stationary if $\alpha = \sigma_1^2/2$. In general we would expect $\alpha > \sigma_1^2/2$ so that demand has some positive deterministic

drift as a result, say, of economic growth. For an introduction to stochastic processes of the form of (2), see Karlin and Taylor (1975).

[5]Note that $Var[\log(y(t)/y(0))] = \sigma_1^2 t$.

[6]We use the notation $J_R = \partial J/\partial R$, etc. $(1/dt)E_t d(\ )$ is Ito's differential operator. For a discussion, see Dreyfus (1965), Kushner (1967), Merton (1971) or Chow (1979).

[7]A clear discussion of the connection between deterministic dynamic programming and the calculus of variations, and a clear derivation of the fundamental equation or optimality in stochastic dynamic programming are provided by Dreyfus (1965).

[8]Thus the presence of the term in $f'(q)$ results because the stochastic component of demand is multiplicative (and log-normally distributed) rather than additive. This term would not be present were the demand function of the form $p(q,t) = f(q) + y(t)$.

[9]Weinstein and Zeckhauser (1975) obtain the same result for their discrete-time characterization of demand uncertainty.

[10]We could also introduce a "market research" activity to reduce demand uncertainty, but this would complicate the analysis without adding marginally valuable insights.

[11]Note that we could have written $g = g(w,K)$, with $g_K < 0$. Again, the additional algebra that results outweighs the insights obtained. Note also that we are ignoring problems associated with common access, and most important, the appropriability of the stock of "knowledge."

[12]The monopoly case leads to similar results and is not presented here.

[13]The reader can show (by expanding $dq = d[q^*(R,K)]$ to obtain $(1/dt)E_t dq$) that in this context exploratory activity has no effect on the dynamics of production. The expected rate of change of q is again given by equations (18) and (19) (but with $\sigma_1 = 0$).

[14]That paper shows that if the reserve level is initially very small, the equilibrium price trajectory will be U-shaped in both competitive and monopolistic markets.

[15]While this characterization of uncertainty may not seem realistic for an individual

(small) producer, it is quite reasonable as a way of thinking about a resource market as a whole. Taking oil exploration in the United States as an example, the aggregate discoveries likely to result from a given total level of exploratory activity this year can be assessed with limited uncertainty. The uncertainty becomes much greater, however, as we try to assess the discoveries likely to result in future years.

[16]Note that there is no demand uncertainty in the model. Demand uncertainty, as specified in Section 2, is easily shown to have no effect on the dynamics of price and exploration as long at it is uncorrelated with fluctuations in θ. We therefore ignore it for simplicity.

[17]This does not mean that the expected pattern of exploratory activity is the same in the competitive and monopoly cases. q will be initially lower for the monopolist, so that $(1/dt)E_t dw$ will be larger, since $C_1'(R)$ is negative. Thus, the monopolist will initially undertake less, but later more exploratory activity than the competitive industry.

[18]See Pindyck (1978b).

[19]See Uhler (1976) and the Appendix of Pindyck (1978b) for a discussion of the empirical supportability of this function.

[20]A measure of resource scarcity should reflect the present value of all current and future sacrifices required to obtain a unit of the resource. Rent provides such a measure in an in situ context, and is independent of such things that affect prices as technology-based changes in extraction costs, most taxes, government price controls, etc. For a discussion of this issue, see Fisher(1979), Brown and Field (1978), and Pindyck (1978b).

[21]Devarajan and Fisher work with a two-period model in which there is current uncertainty over the returns from exploration. In the context of that model they show that rent can deviate from marginal discovery cost even if the shadow price of cumulative discoveries is zero.

[22] To interpret (B.7), consider, for any reserve level R, the expected rate of additions to marginal production cost resulting only from random fluctuations (as opposed to extraction itself). This is found by setting $q = 0$, expanding dC_1 using Ito's Lemma as in equation (14), thereby obtaining: $(1/dt)E_t dC_1 = \frac{1}{2}\sigma^2(K)C_1''(R)$ This is the expected rate of flow of additions to marginal cost. The expected rate of flow of additions to total production cost is found by integrating this equation over all reserves from 0 to R. This yields the integrand in (B.7). Thus J_K is just the expected reduction in the discounted sum of these additional costs resulting from 1 extra unit of K.

Table I - Summary of Results

	Expected Price Dynamics	Expected Production Dynamics	Eqn. Reference
1. Demand Uncertainty, No Exploration			
(a) Extraction cost constant or linear in R.	No effect on rate of change of price for linear or nonlinear demand.	For linear demand, increases rate at which q falls (and raises initial q_o). For concave demand, can increase or decrease rate at which q falls.	(15), (18), (19)
(b) Extraction cost a nonlinear function of R.	No effect on rate of change of price for linear or nonlinear demand.	Same as above.	(15), (18), (19)
2. Reserve Uncertainty, No Exploration			
(a) Extraction cost constant or linear in R.	No effect on rate of change of price for linear or nonlinear demand.	No effect on rate of change of q for linear demand. For concave demand, reduces rate at which q falls (and lowers initial q_o).	(15), (18), (19)
(b) Extraction cost a concave function of R.	Reduces rate of increase of price, and raises initial price.	For linear demand, reduces initial production level and rate of decrease of production. For concave demand, reduces rate of decrease of q, but net effect on initial q_o ambiguous.	(15), (18), (19)

Table I - Summary of Results (continued)

3. Exploration to Reduce Reserve Uncertainty	Expected Price Dynamics	Expected Dynamics of Exploratory Effort	Eqn.Ref.
(a) Extraction Cost constant.	Reserve uncertainty has no effect on expected rate of change of price.	No exploratory effort.	(15),(23)
(b) Extraction cost rising as R falls.	Reserve uncertainty has no effect on expected rate of change of price if extraction cost is linear function of R. Price rises more slowly if cost is concave function of R.	Expected level of exploratory effort generally rising over time, falling abruptly when production ceases.	(15),(23)
4. Exploration to Accumulate Reserves			
(a) Discovery function linear in random parameter.	Uncertainty has no effect on expected price dynamics for any extraction cost function.	Uncertainty has no effect on expected dynamics of exploratory effort.	(29),(31)
(b) Discovery function nonlinear in random parameter.	Uncertainty has no effect on expected rate of change of price, but will raise (lower) entire expected price trajectory if marginal physical product of exploration is a concave (convex) function of the random parameter.	Uncertainty will reduce (increase) expected rate of change of exploratory effort and reduce (increase) initial level of exploratory effort if MPP of exploration is a concave (convex) function of the random parameter.	(29),(31)