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Introduction to Modeling and Simulation, Spring 2002, MIT

1.992, 2.993, 3.04, 10.94, 18.996, 22.091

Introduction to Finite Element Modeling in Solid Mechanics

Franz-Josef Ulm

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Outline

1. Scales of Continuum Modeling
2. Elements of Continuum Solid Mechanics
3. Variational Principle
4. Application I
5. Application II: Water Filling of a gravity dam

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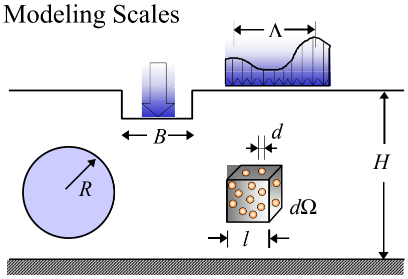
Scales of Continuum Modeling

Modeling Scales



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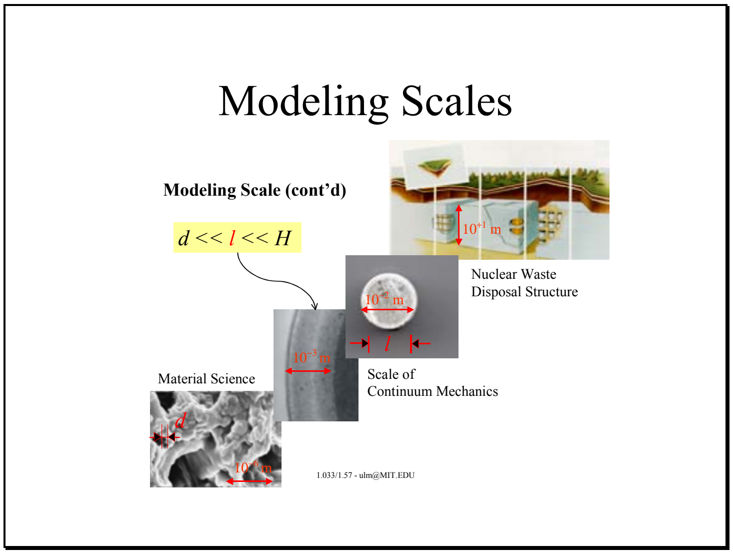
Modeling Scales
Modeling Scale (cont'd)
d << l << H
10+1 m
Nuclear Waste
Disposal Structure
10−2 m
l
10−3 m
Scale of
Continuum Mechanics
Material Science
d
10−6 m
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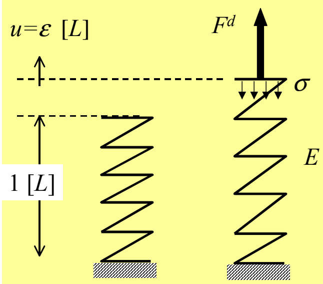
Elements of Continuum Modeling: 1. Material

- 1D-Think Model



- Force Application
- Equilibrium

$$\sigma = F^d / A$$

- Material Law

$$\sigma = E\varepsilon$$

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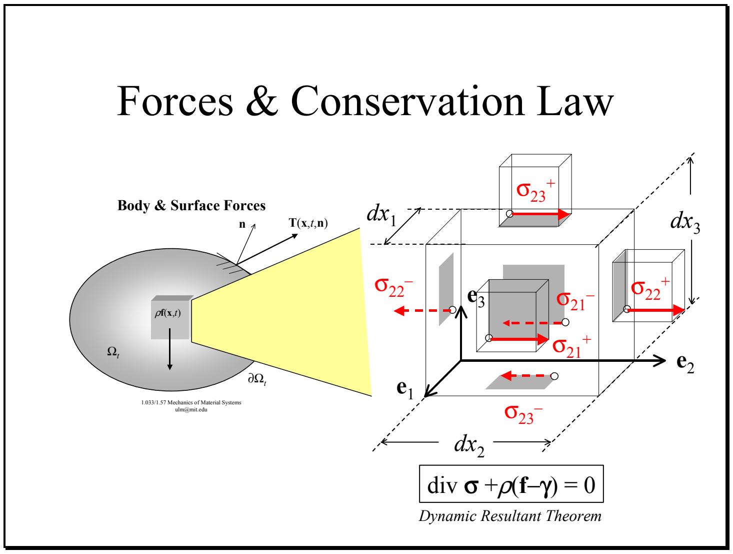
Forces & Conservation Law
Body & Surface Forces
n
T(x,t,n)
ρf(x,t)
Ω_t
∂Ω_t
dx_1
dx_2
dx_3
σ_23^+
σ_22^−
σ_21^−
σ_21^+
σ_22^+
σ_23^−
e_1
e_2
e_3
div σ + ρ(f−γ) = 0
Dynamic Resultant Theorem

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Elements of Continuum Modeling: 2. Field = Structure

- **Heat Transfer**
  1. Conservation of Energy (Field Equation)

     $div\,\underline{q} + r = 0$

  2. Heat Flux Boundary Condition

     $\underline{q} \cdot \underline{n} = q^{d}$

  3. Heat Flux Constitutive Law (Fourrier - Linear)

     $\underline{q} = -k\nabla T$

- **Continuum Mechanics**
  1. Conservation of Momentum

     $div\,\underline{\underline{\sigma}} + \rho\left(\underline{f} + \frac{d^{2}\underline{u}}{dt^{2}}\right) = 0$

  2. Stress Vector (Surface Traction) B. Condition

     $\underline{\underline{\sigma}} \cdot \underline{n} = \underline{T}^{d}$

  3. Stress – Strain Constitutive Law (linear)

     $\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} : {}^{s}\nabla\underline{u}$

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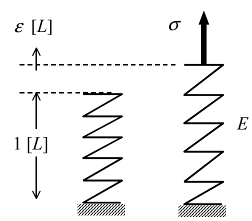
Example: Material Law

Elasticity Potential



• 1st + 2nd Law:

$$\varphi dt = \sigma\, d\varepsilon - d\psi = 0$$

Work — Stored Energy = Helmholtz Energy

• 1D-Model:

$$\psi = 1/2 E\varepsilon^2 \rightarrow \sigma = \frac{\partial \psi}{\partial \varepsilon}$$

• 3D-Model:

$$\psi = \psi(\varepsilon_{ij}) \rightarrow \sigma_{ij} = \frac{\partial \psi}{\partial \varepsilon_{ij}}$$

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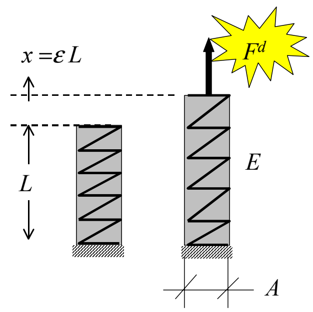
Stationary of Potential Energy



1-Parameter System

- Potential Energy

$$E_{pot} = W(\varepsilon) - \Phi(x)$$

Internal Energy

$$W(\varepsilon) = \frac{1}{2} E \varepsilon^2 AL$$

External Work

$$\Phi(x) = F^d x$$

- Stationary of Potential

$$dE_{pot} = E\varepsilon AL \frac{dx}{L} - F^d dx = 0$$

$$\Rightarrow \forall dx; \; x = \varepsilon L = \frac{F^d L}{EA}$$

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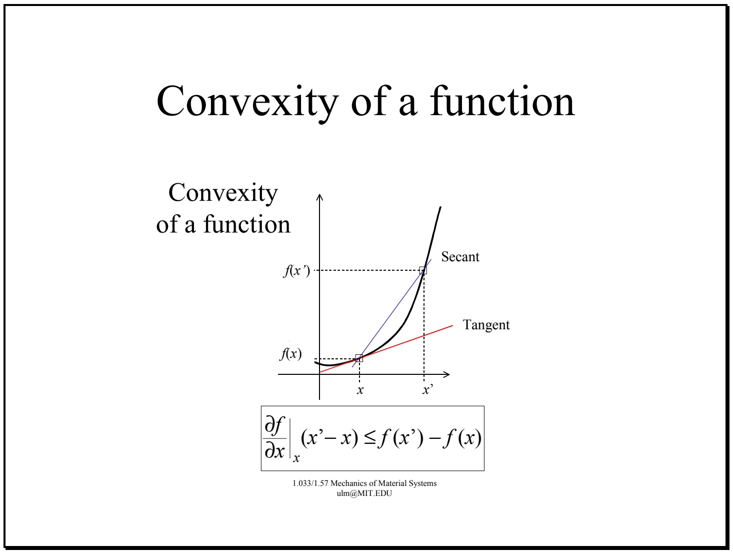

Convexity of a function
Convexity
of a function
Secant
Tangent
f(x')
f(x)
x
x'
$\left.\frac{\partial f}{\partial x}\right|_x (x'-x) \leq f(x') - f(x)$
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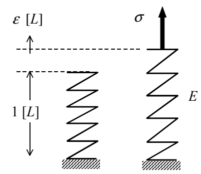
Convexity of Free Energy

Convexity: Applied to Free Energy



σ

State Equation

$$\left.\frac{\partial\psi}{\partial\varepsilon}\right|_{\varepsilon}(\varepsilon' - \varepsilon) \leq \psi(\varepsilon') - \psi(\varepsilon)$$

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Variational Method (1D)

- Let x be solution of problem
- Let x' be any other approx. solution

External Work = Internal Work

$$F^d x = \sigma \varepsilon A L$$

$$F^d x' = \sigma \varepsilon' A L$$

$$F^d (x' - x) = \sigma(\varepsilon' - \varepsilon) A L$$

$$\sigma = \frac{\partial \psi}{\partial \varepsilon}$$

Convexity

$$\frac{\partial \psi}{\partial \varepsilon}(\varepsilon - \varepsilon') A L \leq [\psi(\varepsilon) - \psi(\varepsilon')]$$

Theorem of Minimum Potential Energy

$$\underbrace{W(\varepsilon) - \Phi(x)}_{\substack{E_{pot}(x) \\ SOLUTION}} \leq \underbrace{W(\varepsilon') - \Phi(x')}_{\substack{E_{pot}(x') \\ APPROXIMATION}}$$

Elements of 3D Variational Methods in Linear Elasticity

- Starting Point: Displacement Field $\underline{u}$'
- Calculate Potential Energy
- Minimize Potential Energy

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Application to Finite Elements

- Displacement Field: $\underline{u} = \sum_{i=1}^{i=N} q_i N_i(x_j)$
- Calculate Potential Energy: $E_{pot} = W(q_i, q_j) - \Phi(q_i)$

$W(q_i, q_j) = \frac{1}{2} k_{ij} q_i q_j$ $\Phi(q_i) = F_i q_i$

- Minimize Potential Energy

$$dE_{pot} = 0 : k_{ij} q_j - F_i = 0$$

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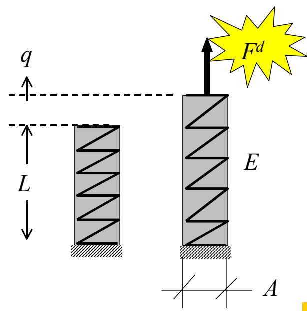
Application I



1-Parameter System

- Displacement Field (DOF)

$$u' = q\frac{x}{L}$$

- Potential Energy

$$E_{pot} = \frac{1}{2}kq - F^d q \qquad k = \frac{EA}{L}$$

- Minimization with regard to DOF

$$dE_{pot} = \frac{\partial E_{pot}}{\partial q}dq = 0 \rightarrow kq - F^d = 0$$

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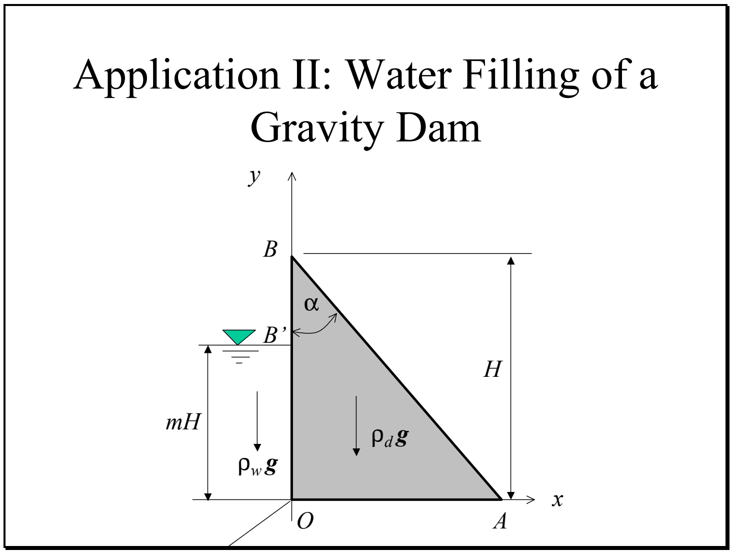

Application II: Water Filling of a Gravity Dam
y
B
α
B'
H
mH
ρd g
ρw g
x
O
A

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From 1D to 3D

- 1D
  - Strain-Displacement

$$\varepsilon = \frac{\Delta l}{L}$$

  - Free Energy (p.vol)

$$\psi = \frac{1}{2} E \varepsilon^2$$

  - External Work

$$\Phi = F^d u$$

- 3D
  - Strain-Displacement

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + {}^t\nabla \underline{u}\right)$$

  - Free Energy (p.vol)

$$\psi = \frac{1}{2}\lambda\left(tr\underline{\underline{\varepsilon}}\right)^2 + 2\mu\left(\tfrac{1}{2} tr\left(\underline{\underline{\varepsilon}} \cdot \underline{\underline{\varepsilon}}\right)\right)$$

  - External Work

$$\Phi = \int_\Omega \rho \underline{f} \cdot \underline{u} d\Omega + \int_{\partial\Omega} \underline{T}^d \cdot \underline{u} da$$

(Volume Forces) (Surface Forces)