Benders' Decomposition Methods for Structured Optimization, including Stochastic Optimization

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Block Ladder Structure

Basic Model

$$
\begin{aligned}
\text{minimize}_{x,y} \quad & c^T x + f^T y \\
\text{s.t.} \quad & Ax = b \\
& Bx + Dy = d \\
& x \geq 0 \quad y \geq 0 \,.
\end{aligned}
$$

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Block Ladder Structure

More Complex Structure

$$
\begin{array}{rlllllll}
\underset{x, y_1, \ldots, y_K}{\text{minimize}} & c^T x & + \alpha_1 f_1^T y_1 & + \alpha_2 f_2^T y_2 & + \cdots & + \alpha_K f_K^T y_K & & \\
\text{s.t.} & Ax & & & & & = & b \\
& B_1 x & + \quad D_1 y_1 & & & & = & d_1 \\
& B_2 x & & + \quad D_2 y_2 & & & = & d_2 \\
& \vdots & & & \ddots & & & \vdots \\
& B_K x & & & & + \quad D_K y_K & = & d_K \\
& x, y_1, y_2, \ldots, y_K \geq 0 & & & & & &
\end{array}
$$

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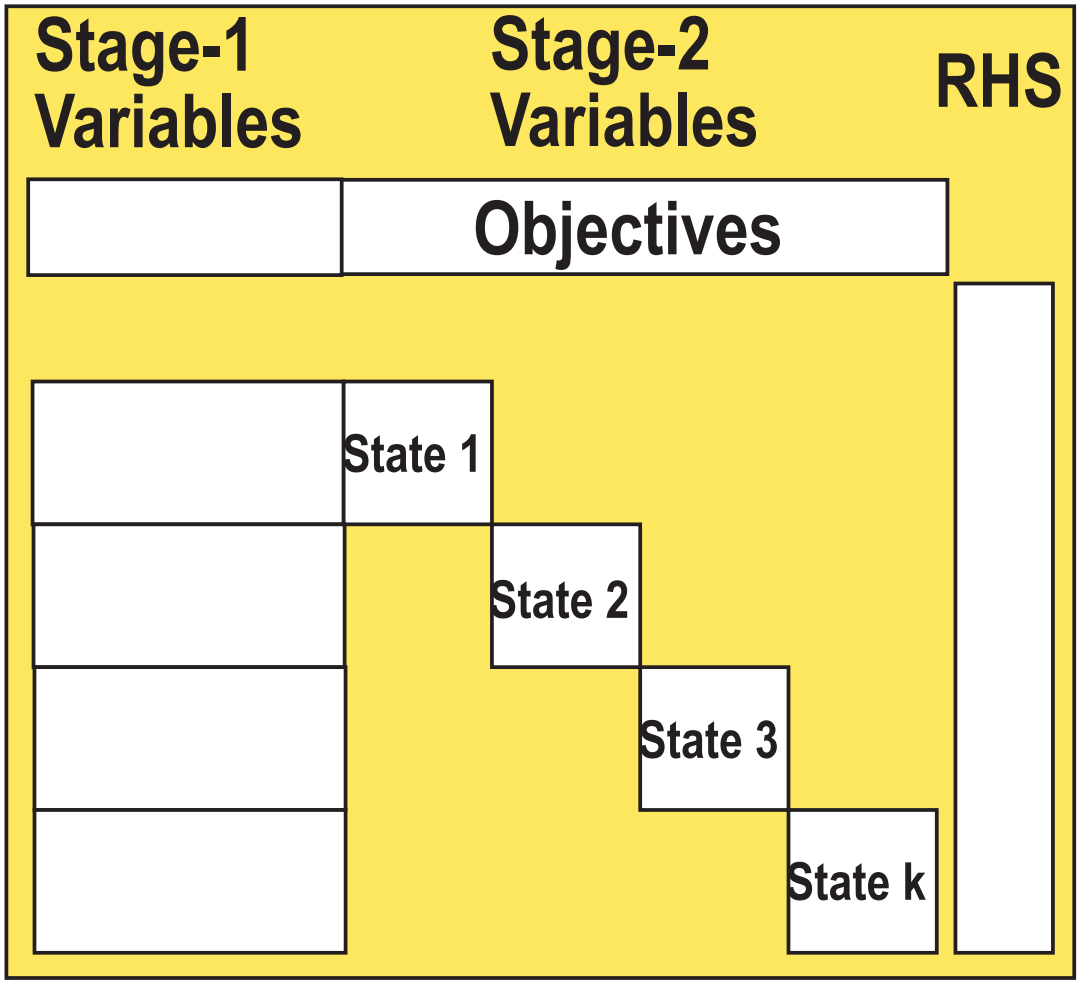
Block Ladder Structure

More Complex Structure



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Basic Model

Reformulation

$$
\begin{aligned}
\mathbf{VAL} \;= \mathbf{minimum}_{x,y} \quad & c^T x + & f^T y \\
\mathbf{s.t.} \quad & Ax & = b \\
& Bx + & Dy = d \\
& x \geq 0 & y \geq 0 \,,
\end{aligned}
$$

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Basic Model

Reformulation

$$\begin{aligned} \mathbf{VAL} \quad &= \mathbf{minimum}_x \;\; \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{z}(\boldsymbol{x}) \\ \mathbf{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \quad = \boldsymbol{b} \\ & \boldsymbol{x} \geq \mathbf{0} \;, \end{aligned}$$

where:

$$\begin{aligned} \mathbf{P2:} \;\; \boldsymbol{z}(\boldsymbol{x}) \quad &= \mathbf{minimum}_y \quad \boldsymbol{f}^T \boldsymbol{y} \\ \mathbf{s.t.} \quad & \boldsymbol{D}\boldsymbol{y} \quad = \boldsymbol{d} - \boldsymbol{B}\boldsymbol{x} \\ & \boldsymbol{y} \geq \mathbf{0} \;. \end{aligned}$$

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Basic Model

Reformulation

$$\mathbf{P2:}\ \ z(x) \ = \mathbf{minimum}_y \quad f^T y$$

$$\mathbf{s.t.} \quad Dy \ = d - Bx$$

$$y \geq 0\ .$$

The dual is:

$$\mathbf{D2:}\ \ z(x) \ = \mathbf{maximum}_p \ p^T(d - Bx)$$

$$\mathbf{s.t.} \quad D^T p \ \leq f\ .$$

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Basic Model

Reformulation

$$\mathbf{D2:}\ \ z(x) \ = \mathbf{maximum}_p\ p^T(d - Bx)$$

$$\mathbf{s.t.} \qquad D^T p \qquad \leq f\ .$$

The feasible region of D2 is the set:

$$\mathcal{D}_2 := \left\{ p \mid D^T p \leq f \right\}\ .$$

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Basic Model

Reformulation

The feasible region of D2 is the set:

$$\mathcal{D}_2 := \left\{ \boldsymbol{p} \mid \boldsymbol{D}^T \boldsymbol{p} \leq \boldsymbol{f} \right\} \ ,$$

Extreme points are:

$$\boldsymbol{p}^1, \ldots, \boldsymbol{p}^I$$

Extreme rays are:

$$\boldsymbol{r}^1, \ldots, \boldsymbol{p}^J$$

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Basic Model

Reformulation

$$\mathbf{D2:}\ \ z(x) \ = \mathbf{maximum}_p\ p^T(d - Bx)$$

$$\mathbf{s.t.} \qquad D^T p \qquad \leq f\ .$$

If we solve D2, two case arise:

D2 is unbounded from above, or

D2 has an optimal solution.

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Basic Model

Reformulation

$$\mathbf{D2:}\ \ z(x) \ = \mathbf{maximum}_p\ p^T(d - Bx)$$

$$\mathbf{s.t.} \qquad D^T p \qquad \leq f\ .$$

If D2 is unbounded from above, the algorithm will return one of the extreme rays $\bar{r} = r^j$ for some j.

$$(r^j)^T(d - Bx) > 0$$

$$z(x) = +\infty\ .$$

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Basic Model

Reformulation

$$\mathbf{D2:}\ \ z(x) \ = \mathbf{maximum}_p\ p^T(d - Bx)$$

$$\mathbf{s.t.} \qquad D^T p \qquad \leq f\ .$$

If D2 has an optimal solution, the algorithm will return one of the extreme points $\bar{p} = p^i$ for some i.

$$z(x) = (p^i)^T(d - Bx) = \max_{k=1,\ldots,I} (p^k)^T(d - Bx)\ .$$

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Basic Model

Reformulation

$$\mathbf{D2:}\ \ z(x) \ = \mathbf{maximum}_p \ p^T(d - Bx)$$

$$\mathbf{s.t.} \qquad D^T p \qquad \leq f\ .$$

$$\mathbf{D2:}\ \ z(x) \ = \mathbf{minimum}_z \qquad z$$

$$\mathbf{s.t.}\ (p^i)^T(d - Bx) \ \leq \ z \qquad i = 1, \ldots, I$$

$$(r^j)^T(d - Bx) \ \leq \ 0 \qquad j = 1, \ldots, J\ .$$

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Basic Model

Reformulation

Full Master Problem...

$$\begin{aligned} \mathbf{VAL} \;\; &= \mathbf{minimum}_x \;\; c^T x + z(x) \\ \mathbf{s.t.} \quad & Ax = b \\ & x \geq 0 \;, \end{aligned}$$

$$\begin{aligned} \mathbf{FMP:} \quad \mathbf{VAL} \;\; &= \mathbf{minimum}_{x,z} \quad c^T x + z \\ \mathbf{s.t.} \quad & Ax = b \\ & x \geq 0 \\ & (p^i)^T (d - Bx) \leq z \qquad i = 1, \ldots, I \\ & (r^j)^T (d - Bx) \leq 0 \qquad j = 1, \ldots, J \;. \end{aligned}$$

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Basic Model

Reformulation

...Full Master Problem

$$
\begin{aligned}
\mathbf{FMP:} \quad \mathbf{VAL} &= \mathbf{minimum}_{x,z} \quad c^T x + z \\
\mathbf{s.t.} \quad & Ax = b \\
& x \geq 0 \\
& (p^i)^T (d - Bx) \leq z \qquad i = 1, \ldots, I \\
& (r^j)^T (d - Bx) \leq 0 \qquad j = 1, \ldots, J \,.
\end{aligned}
$$

We have eliminated the variables y from the problem.

We have added a single scalar variable z.

We have added a generically extremely large number of constraints.

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Delayed Constraint Generation

Restricted Master Problem

Consider problem RMP^k composed of only k of the extreme point/extreme ray constraints from FMP:

$$
\begin{array}{llll}
\mathbf{RMP}^k\mathbf{:} & \mathbf{VAL}^k = \mathbf{minimum}_{x,z} & c^T x + z & \\
 & \mathbf{s.t.} & Ax = b & \\
 & & x \geq 0 & \\
 & & (p^i)^T(d - Bx) \leq z & i = 1, \ldots, k - l \\
 & & (r^j)^T(d - Bx) \leq 0 & j = 1, \ldots, l\,.
\end{array}
$$

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Delayed Constraint Generation

Restricted Master Problem

Lower Bound on VAL

$$\mathbf{RMP}^k: \quad \mathbf{VAL}^k = \mathbf{minimum}_{x,z} \quad c^T x + z$$

$$\begin{aligned} \text{s.t.} \quad & Ax = b \\ & x \geq 0 \\ & (p^i)^T(d - Bx) \leq z \qquad i = 1, \ldots, k-l \\ & (r^j)^T(d - Bx) \leq 0 \qquad j = 1, \ldots, l\,. \end{aligned}$$

We solve this problem, obtaining VAL^k, $\bar{x}, \bar{z}$

$$\mathbf{VAL}^k \leq \mathbf{VAL}\,.$$

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Delayed Constraint Generation

Restricted Master Problem

$$
\begin{aligned}
\mathbf{RMP}^k: \quad \mathbf{VAL}^k \; = \mathbf{minimum}_{x,z} \quad & c^T x + z \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0 \\
& (p^i)^T (d - Bx) \leq z \qquad i = 1, \ldots, k - l \\
& (r^j)^T (d - Bx) \leq 0 \qquad j = 1, \ldots, l \,.
\end{aligned}
$$

Check whether the solution $\bar{x}, \bar{z}$ is optimal for the full master problem.

Check whether $\bar{x}, \bar{z}$ violates any of the non-included constraints.

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Delayed Constraint Generation

Solve the Subproblem

Finding a Violated Constraint

Solve:

$$\mathrm{Q}(\bar{x}): \quad q^* = \text{maximum}_p \; p^T(d - B\bar{x}) = \text{minimum}_y \quad f^T y$$

$$\text{s.t.} \; D^T p \leq f \qquad \text{s.t.} \; Dy = d - B\bar{x}$$

$$y \geq 0 \; .$$

If $\mathrm{Q}(\bar{x})$ is unbounded, return an extreme ray $\bar{r} = r^j$ for some j, where $\bar{x}$ will satisfy:

$$(r^j)^T(d - B\bar{x}) > 0 \; .$$

Therefore $\bar{x}$ has violated the constraint:

$$(r^j)^T(d - Bx) \leq 0 \; ,$$

Add this constraint to RMP and re-solve RMP.

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Delayed Constraint Generation

Solve the Subproblem

Upper Bound on VAL

$$\mathrm{Q}(\bar{x}): \quad q^* \;=\; \text{maximum}_p \; p^T(d - B\bar{x}) \;=\; \text{minimum}_y \qquad f^T y$$

$$\text{s.t.} \;\; D^T p \le f \qquad\qquad \text{s.t.} \;\; Dy = d - B\bar{x}$$

$$y \ge 0 \; .$$

If $\mathrm{Q}(\bar{x})$ has an optimal solution, return an optimal extreme point $\bar{p} = \; p^i$ for some i, and the optimal solution $\bar{y}$ to the minimization problem of $\mathrm{Q}(\bar{x})$.

$(\bar{x}, \bar{y})$ is feasible for the original problem.

Suppose that UB is an upper bound on VAL. Update UB:

$$\text{If} \;\; c^T\bar{x} + f^T\bar{y} \;\le\; \text{UB}, \;\; \text{then} \;\; \text{BESTSOL} \;\leftarrow (\bar{x}, \bar{y}) \; .$$
$$\text{UB} \leftarrow \min\{\text{UB}, c^T\bar{x} + f^T\bar{y}\} \; .$$

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Delayed Constraint Generation

Solve the Subproblem

Finding a Violated Constraint

$$\mathrm{Q}(\bar{x}): \quad q^* \;=\; \text{maximum}_p \; p^T(d - B\bar{x}) \;=\; \text{minimum}_y \qquad f^T y$$

$$\text{s.t.} \;\; D^T p \le f \qquad\qquad \text{s.t.} \;\; Dy = d - B\bar{x}$$

$$y \ge 0 \; .$$

If $q^* > \bar{z}$, then

$$(p^i)^T(d - B\bar{x}) = q^* > \bar{z} \; .$$

We therefore have found that $\bar{x}, \bar{z}$ has violated the constraint:

$$(p^i)^T(d - Bx) \le z \; ,$$

and so we add this constraint to RMP and re-solve RMP.

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Delayed Constraint Generation

Solve the Subproblem

Checking for Full Optimality

$$\mathrm{Q}(\bar{x}): \quad q^* \;=\; \text{maximum}_p \; p^T(d - B\bar{x}) \;=\; \text{minimum}_y \qquad f^T y$$

$$\text{s.t. } D^T p \le f \qquad\qquad \text{s.t. } Dy = d - B\bar{x}$$

$$y \ge 0 \; .$$

If $q^* \le \bar{z}$, then:

$$\max_{k=1,\ldots,I} (p^k)^T(d - B\bar{x}) = (p^i)^T(d - B\bar{x}) = q^* \le \bar{z} \; ,$$

$\bar{x}, \bar{z}$ is feasible and hence optimal for FMP

$(\bar{x}, \bar{y})$ is optimal for the original problem.

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Delayed Constraint Generation

Solve the Subproblem

Termination Criterion

Otherwise terminate if:

$$\mathbf{UB} - \mathbf{VAL}^k \leq \varepsilon$$

for a pre-specified tolerance ε.

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Delayed Constraint Generation

Algorithm

0. Set LB$= -\infty$ and UB$= +\infty$.

1. A typical iteration starts with the relaxed master problem RMP^k, in which only k constraints of the full master problem FMP are included. An optimal solution $\bar{x}, \bar{z}$ to the relaxed master problem is computed. We update the lower bound on VAL:

$$\mathbf{LB} \leftarrow \mathbf{VAL}^k \ .$$

2. Solve the subproblem Q($\bar{x}$):

$$\begin{array}{llll} \mathrm{Q}(\bar{x}): \ q^* = \mathrm{maximum}_p \ p^T(d - B\bar{x}) & = & \mathrm{minimum}_y & f^T y \\ \qquad\qquad \mathrm{s.t.} \ D^T p \leq f & & & \mathrm{s.t.} \ Dy = d - B\bar{x} \\ & & & y \geq 0 \ . \end{array}$$

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Delayed Constraint Generation

Algorithm

3. If Q($\bar{x}$) has an optimal solution, let $\bar{p}$ and $\bar{y}$ be the primal and dual solutions of Q($\bar{x}$). If $q^* \leq \bar{z}$, then $\bar{x}, \bar{z}$ satisfies all constraints of FMP, and so $\bar{x}, \bar{z}$ is optimal for FMP. Furthermore, this implies that $(\bar{x}, \bar{y})$ is optimal for the original problem.

If $q^* > \bar{z}$, then we add the constraint:

$$\bar{p}^T(d - Bx) \leq z$$

to the restricted master problem RMP.

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Delayed Constraint Generation

Algorithm

Update the upper bound and the best solution:

$$\textbf{If } \boldsymbol{c}^T\bar{\boldsymbol{x}} + \boldsymbol{f}^T\bar{\boldsymbol{y}} \leq \textbf{UB}, \textbf{ then BESTSOL } \leftarrow (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$$

$$\textbf{UB} \leftarrow \textbf{min}\{\textbf{UB}, \boldsymbol{c}^T\bar{\boldsymbol{x}} + \boldsymbol{f}^T\bar{\boldsymbol{y}}\}$$

If UB $-$ LB $\leq \varepsilon$, then terminate. Otherwise, return to step **1.**

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Delayed Constraint Generation

Algorithm

4. If Q$(\bar{x})$ is unbounded from above, let $\bar{r}$ be the extreme ray generated by the algorithm that solves Q$(\bar{x})$. Add the constraint:

$$\bar{r}^T(d - Bx) \leq 0$$

to the restricted master problem RMP, and return to step **1.**

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Delayed Constraint Generation

Algorithm

Comments

This algorithm is known formally as *Benders' decomposition.*

Benders' decomposition method was developed in 1962.

Benders' decomposition method is the same as Dantzig-Wolfe decomposition applied to the dual problem.

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Benders' Decomposition

Block-Ladder Structure

$$
\begin{array}{rl}
\mathbf{VAL} = \underset{x, y_1, \ldots, y_K}{\mathbf{minimize}} & c^T x + \alpha_1 f_1^T y_1 + \alpha_2 f_2^T y_2 + \cdots + \alpha_K f_K^T y_K \\
\mathbf{s.t.} & Ax = b \\
& B_1 x + D_1 y_1 = d_1 \\
& B_2 x + D_2 y_2 = d_2 \\
& \vdots \qquad \ddots \qquad \vdots \\
& B_K x + D_K y_K = d_K \\
& x, y_1, y_2, \ldots, y_K \geq 0
\end{array}
$$

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Benders' Decomposition

Block-Ladder Structure

Reformulation...

$$\begin{aligned} \mathbf{VAL} \;\; &= \mathbf{minimum}_x \;\; c^T x + \sum_{\omega=1}^{K} \alpha_\omega z_\omega(x) \\ \mathbf{s.t.} \quad & \quad Ax = b \\ & \quad x \geq 0 \;, \end{aligned}$$

where:

$$\begin{aligned} \mathbf{P2}_\omega : \;\; z_\omega(x) \;\; &= \mathbf{minimum}_{y_\omega} \;\; f_\omega^T y_\omega \\ \mathbf{s.t.} \quad & \quad D_\omega y_\omega = d_\omega - B_\omega x \\ & \quad y_\omega \geq 0 \;. \end{aligned}$$

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Benders' Decomposition

Block-Ladder Structure

...Reformulation...

$$\begin{aligned} \mathbf{P2}_\omega: \quad z_\omega(x) &= \mathbf{minimum}_{y_\omega} \quad f_\omega^T y_\omega \\ \mathbf{s.t.} \quad D_\omega y_\omega &= d_\omega - B_\omega x \\ y_\omega &\geq \mathbf{0}\ . \end{aligned}$$

The dual is:

$$\begin{aligned} \mathbf{D2}_\omega: \quad z_\omega(x) &= \mathbf{maximum}_{p_\omega} \ p_\omega^T (d_\omega - B_\omega x) \\ \mathbf{s.t.} \quad D_\omega^T p_\omega &\leq f_\omega\ . \end{aligned}$$

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Benders' Decomposition

Block-Ladder Structure

...Reformulation...

$$\mathbf{D2}_\omega : \quad z_\omega(x) \quad = \mathbf{maximum}_{p_\omega} \; p_\omega^T(d_\omega - B_\omega x)$$

$$\mathbf{s.t.} \qquad D_\omega^T p_\omega \qquad \leq \; f_\omega \; .$$

The feasible region of D2$_\omega$ is the set:

$$\mathcal{D}_2^\omega := \left\{ p_\omega \mid D_\omega^T p_\omega \leq f_\omega \right\} \; ,$$

Extreme points of $\mathcal{D}_2^\omega$:

$$p_\omega{}^1, \ldots, p_\omega^{I_\omega}$$

Extreme rays of $\mathcal{D}_2^\omega$:

$$r_\omega{}^1, \ldots, p_\omega^{J_\omega}$$

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Benders' Decomposition

Block-Ladder Structure

...Reformulation...

$$\mathbf{D2}_\omega: \quad z_\omega(x) \quad = \mathbf{maximum}_{p_\omega}\ p_\omega^T(d_\omega - B_\omega x)$$

$$\mathbf{s.t.} \qquad D_\omega^T p_\omega \qquad \leq f_\omega\ .$$

If we solve $D2_\omega$, two case arise:

$D2_\omega$ is unbounded from above, or

$D2_\omega$ has an optimal solution.

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Benders' Decomposition

Block-Ladder Structure

...Reformulation...

$$\mathbf{D2}_\omega : \quad z_\omega(x) \quad = \mathbf{maximum}_{p_\omega} \; p_\omega^T(d_\omega - B_\omega x)$$

$$\mathbf{s.t.} \qquad D_\omega^T p_\omega \qquad \leq \; f_\omega \; .$$

If D2$_\omega$ is unbounded from above, return one of the extreme rays $\bar{r}_\omega = r_\omega^j$ for some j, with the property that

$$(r_\omega^j)^T(d_\omega - B_\omega x) > 0$$

in which case

$$z_\omega(x) = +\infty \; .$$

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Benders' Decomposition

Block-Ladder Structure

...Reformulation...

$$\mathbf{D2}_\omega: \quad z_\omega(x) \quad = \mathbf{maximum}_{p_\omega}\ p_\omega^T(d_\omega - B_\omega x)$$

$$\mathbf{s.t.} \qquad D_\omega^T p_\omega \qquad \leq f_\omega\ .$$

If D2$_\omega$ has an optimal solution, return one of the extreme points $\bar{p}_\omega = p_\omega^i$ for some i as well as the optimal objective function value $z_\omega(x)$

$$z_\omega(x) = (p_\omega^i)^T(d_\omega - B_\omega x) = \max_{k=1,\ldots,I_\omega} (p_\omega^k)^T(d_\omega - B_\omega x)\ .$$

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Benders' Decomposition

Block-Ladder Structure

...Reformulation...

$$\mathbf{D2}_\omega: \quad z_\omega(x) \quad = \text{maximum}_{p_\omega} \; p_\omega^T(d_\omega - B_\omega x)$$

$$\text{s.t.} \qquad D_\omega^T p_\omega \qquad \leq \; f_\omega \; .$$

$$\mathbf{D2}_\omega: \quad z_\omega(x) \quad = \text{minimum}_{z_\omega} \qquad z_\omega$$

$$\text{s.t.} \; (p_\omega^i)^T(d_\omega - B_\omega x) \; \leq \; z_\omega \qquad i = 1, \ldots, I_\omega$$

$$(r_\omega^j)^T(d_\omega - B_\omega x) \; \leq \; 0 \qquad j = 1, \ldots, J_\omega \; .$$

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Benders' Decomposition

Block-Ladder Structure

...Reformulation

$$\begin{aligned} \mathbf{VAL} \;\; &= \mathbf{minimum}_x \;\; c^T x + \sum_{\omega=1}^{K} \alpha_\omega z_\omega(x) \\ \mathbf{s.t.} \quad & Ax = b \\ & x \geq 0 \,, \end{aligned}$$

$$\mathbf{FMP:}\;\; \mathbf{VAL} = \underset{x, z_1, \ldots, z_K}{\mathbf{minimum}} \;\; c^T x + \sum_{\omega=1}^{K} \alpha_\omega z_\omega$$

$$\begin{aligned} \mathbf{s.t.} \quad & Ax = b \\ & x \geq 0 \\ & (p_\omega^i)^T (d_\omega - B_\omega x) \leq z_\omega \quad i = 1, \ldots, I_\omega,\; \omega = 1, \ldots, K \\ & (r_\omega^j)^T (d_\omega - B_\omega x) \leq 0 \quad j = 1, \ldots, J_\omega,\; \omega = 1, \ldots, K \,. \end{aligned}$$

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Block-Ladder Structure

Restricted Master Problem

Consider problem RMP^k composed of only k of the extreme point/extreme ray constraints from FMP:

$$\mathbf{RMP}^k: \ \mathbf{VAL}^k = \underset{x, z_1, \ldots, z_K}{\mathbf{minimum}} \quad c^T x + \sum_{\omega=1}^{K} \alpha_\omega z_\omega$$

$$\begin{aligned} \mathbf{s.t.} \quad Ax &= b \\ x &\geq 0 \\ (p_\omega^i)^T (d_\omega - B_\omega x) &\leq z_\omega \quad \text{for some } i \text{ and } \omega \\ (r_\omega^j)^T (d_\omega - B_\omega x) &\leq 0 \quad \text{for some } i \text{ and } \omega \ , \end{aligned}$$

where there are a total of k of the inequality constraints.

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Block-Ladder Structure

Restricted Master Problem

Lower Bound on VAL

$$\mathbf{RMP}^k: \ \mathbf{VAL}^k = \underset{x, z_1, \ldots, z_K}{\text{minimum}} \quad c^T x + \sum_{\omega=1}^{K} \alpha_\omega z_\omega$$

$$\text{s.t.} \quad Ax = b$$

$$x \geq 0$$

$$(p_\omega^i)^T (d_\omega - B_\omega x) \leq z_\omega \quad \text{for some } i \text{ and } \omega$$

$$(r_\omega^j)^T (d_\omega - B_\omega x) \leq 0 \quad \text{for some } i \text{ and } \omega \ ,$$

Solve for VAL^k and $\bar{x}, \bar{z}_1, \ldots, \bar{z}_K$

$$\mathbf{VAL}^k \leq \mathbf{VAL} \ .$$

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Block-Ladder Structure

Restricted Master Problem

$$\mathbf{RMP}^k: \ \mathbf{VAL}^k = \underset{x, z_1, \ldots, z_K}{\mathbf{minimum}} \quad c^T x + \sum_{\omega=1}^{K} \alpha_\omega z_\omega$$

$$\begin{aligned} \mathbf{s.t.} \quad Ax &= b \\ x &\geq 0 \\ (p_\omega^i)^T (d_\omega - B_\omega x) &\leq z_\omega \quad \text{for some } i \text{ and } \omega \\ (r_\omega^j)^T (d_\omega - B_\omega x) &\leq 0 \quad \text{for some } i \text{ and } \omega \ , \end{aligned}$$

Solve for VAL^k and $\bar{x}, \bar{z}_1, \ldots, \bar{z}_K$

Check whether the solution $\bar{x}, \bar{z}_1, \ldots, \bar{z}_K$ is optimal for FMP

Check whether $\bar{x}, \bar{z}_1, \ldots, \bar{z}_K$ violates any of the non-included constraints.

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Block-Ladder Structure

Restricted Master Problem

Solve the Subproblems

$$\mathrm{Q}_\omega(\bar{x}) : \; q_\omega^* = \mathbf{maximum}_{p_\omega} \; p_\omega^T(d_\omega - B_\omega \bar{x}) \;\; = \;\; \mathbf{minimum}_{y_\omega} \qquad f_\omega^T y_\omega$$

$$\mathbf{s.t.} \;\; D_\omega^T p_\omega \leq f_\omega \qquad\qquad \mathbf{s.t.} \;\; D_\omega y_\omega = d_\omega - B_\omega \bar{x}$$

$$y_\omega \geq 0 \; .$$

If $\mathrm{Q}_\omega(\bar{x})$ is unbounded, return an extreme ray $\bar{r}_\omega = \; r_\omega^j$ for some j, where $\bar{x}$ will satisfy:

$$(r_\omega^j)^T(d_\omega - B_\omega \bar{x}) > 0 \; .$$

Therefore $\bar{x}$ has violated the constraint:

$$(r_\omega^j)^T(d_\omega - B_\omega x) \leq 0 \; ,$$

and so we add this constraint to RMP.

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Block-Ladder Structure

Restricted Master Problem

Finding a Violated Constraint

$$\mathrm{Q}_\omega(\bar{x}): \; q_\omega^* = \text{maximum}_{p_\omega} \; p_\omega^T(d_\omega - B_\omega \bar{x}) \;=\; \text{minimum}_{y_\omega} \quad f_\omega^T y_\omega$$

$$\text{s.t.} \;\; D_\omega^T p_\omega \le f_\omega \qquad\qquad \text{s.t.} \;\; D_\omega y_\omega = d_\omega - B_\omega \bar{x}$$

$$y_\omega \ge 0 \; .$$

If $\mathrm{Q}_\omega(\bar{x})$ has an optimal solution, return an optimal extreme point $\bar{p}_\omega = p_\omega^i$ for some i, as well as the optimal solution $\bar{y}_\omega$ of the minimization problem of $\mathrm{Q}_\omega(\bar{x})$.

If $q_\omega^* > \bar{z}_\omega$, then

$$(p_\omega^i)^T(d_\omega - B_\omega \bar{x}) = q_\omega^* > \bar{z}_\omega \; .$$

Therefore $\bar{x}, \bar{z}_\omega$ has violated the constraint:

$$(p_\omega^i)^T(d_\omega - B_\omega x) \le z_\omega \; ,$$

and so we add this constraint to RMP.

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Block-Ladder Structure

Restricted Master Problem

Upper Bound on VAL

$$\mathrm{Q}_\omega(\bar{x}) : \; q_\omega^* = \text{maximum}_{p_\omega} \; p_\omega^T(d_\omega - B_\omega \bar{x}) \;=\; \text{minimum}_{y_\omega} \qquad f_\omega^T y_\omega$$

$$\text{s.t.} \;\; D_\omega^T p_\omega \le f_\omega \qquad\qquad \text{s.t.} \;\; D_\omega y_\omega = d_\omega - B_\omega \bar{x}$$

$$y_\omega \ge 0 \; .$$

If q_ω^* is finite for all $\omega = 1, \ldots, K$, then $\bar{x}, \bar{y}_1, \ldots, \bar{y}_K$ satisfies all of the constraints of the original problem.

Suppose that UB is an upper bound on VAL, we update UB as follows:

$$\text{If} \;\; c^T \bar{x} + \sum_{\omega=1}^{K} \alpha_\omega {f_\omega}^T \bar{y}_\omega \;\le\; \text{UB}, \;\; \text{then} \;\; \text{BESTSOL} \;\leftarrow (\bar{x}, \bar{y}_1, \ldots, \bar{y}_K) \; .$$

$$\text{UB} \leftarrow \min\{\text{UB}, c^T \bar{x} + \sum_{\omega=1}^{K} \alpha_\omega {f_\omega}^T \bar{y}_\omega\} \; .$$

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Block-Ladder Structure

Restricted Master Problem

Checking for Optimality

$$\mathrm{Q}_\omega(\bar{x}): \; q_\omega^* = \text{maximum}_{p_\omega} \; p_\omega^T(d_\omega - B_\omega \bar{x}) \;=\; \text{minimum}_{y_\omega} \qquad f_\omega^T y_\omega$$

$$\text{s.t.} \;\; D_\omega^T p_\omega \le f_\omega \qquad\qquad \text{s.t.} \;\; D_\omega y_\omega = d_\omega - B_\omega \bar{x}$$

$$y_\omega \ge 0 \; .$$

If $q_\omega^* \le \bar{z}_\omega$ for all $\omega = 1, \ldots, K$, then we have:

$$\max_{l=1,\ldots,I_\omega} (p_\omega^l)^T(d_\omega - B_\omega \bar{x}) = (p_\omega^i)^T(d_\omega - B_\omega \bar{x}) = q_\omega^* \le \bar{z}_\omega \quad \text{for all } \omega = 1, \ldots, K \; ,$$

$\bar{x}, \bar{z}_1, \ldots, \bar{z}_K$ satisfies all of the constraints in FMP.
$\bar{x}, \bar{z}_1, \ldots, \bar{z}_K$ is feasible and hence optimal for FMP
$(\bar{x}, \bar{y}_1, \ldots, \bar{y}_K)$ is optimal for the original problem, and we can terminate the algorithm.

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Block-Ladder Structure

Restricted Master Problem

Termination Criterion

Else, we can also terminate the algorithm whenever:

$$\mathbf{UB} - \mathbf{VAL}^k \leq \varepsilon$$

for a pre-specified tolerance ε.

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Benders' Decomposition

Block-Ladder Structure

Algorithm...

0. Set LB$= -\infty$ and UB$= +\infty$.

1. A typical iteration starts with the relaxed master problem RMP^k, in which only k constraints of the full master problem FMP are included. An optimal solution $\bar{x}, \bar{z}_1, \ldots, \bar{z}_K$ to the relaxed master problem is computed. We update the lower bound on VAL:

$$\mathbf{LB} \leftarrow \mathbf{VAL}^k \ .$$

2. For $\omega = 1, \ldots, K$, solve the subproblem Q$_\omega(\bar{x})$:

$$\mathrm{Q}_\omega(\bar{x}) : \ q_\omega^* = \mathbf{maximum}_{p_\omega} \ p_\omega^T(d_\omega - B_\omega \bar{x}) \ = \ \mathbf{minimum}_{y_\omega} \qquad f_\omega^T y_\omega$$

$$\mathbf{s.t.} \ D_\omega^T p_\omega \leq f_\omega \qquad\qquad \mathbf{s.t.} \ D_\omega y_\omega = d_\omega - B_\omega \bar{x}$$

$$y_\omega \geq \mathbf{0} \ .$$

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Benders' Decomposition

Block-Ladder Structure

...Algorithm...

- If $Q_\omega(\bar{x})$ is unbounded from above, let $\bar{r}_\omega$ be the extreme ray generated by the algorithm that solves $Q(\bar{x})$. Add the constraint:

$$\bar{r}_\omega^T(d_\omega - B_\omega x) \leq 0$$

to the restricted master problem RMP.

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Benders' Decomposition

Block-Ladder Structure

...Algorithm...

- If $Q_\omega(\bar{x})$ has an optimal solution, let $\bar{p}_\omega$ and $\bar{y}_\omega$ be the primal and dual solutions of $Q_\omega(\bar{x})$. If $q^*_\omega > \bar{z}_\omega$, then we add the constraint:

$$\bar{p}_\omega^T(d_\omega - B_\omega x) \leq z_\omega$$

to the restricted master problem RMP.

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Benders' Decomposition

Block-Ladder Structure

...Algorithm

3. If $q_\omega^* \leq \bar{z}_\omega$ for all $\omega = 1, \ldots, K$, then $(\bar{x}, \bar{y}_1, \ldots, \bar{y}_K)$ is optimal for the original problem, and the algorithm terminates.

4. If $Q_\omega(\bar{x})$ has an optimal solution for all $\omega = 1, \ldots, K$, then update the upper bound on VAL as follows:

$$\textbf{If } \; c^T\bar{x} + \sum_{\omega=1}^{K} \alpha_\omega {f_\omega}^T \bar{y}_\omega \; \leq \; \textbf{UB}, \; \textbf{then BESTSOL} \; \leftarrow (\bar{x}, \bar{y}_1, \ldots, \bar{y}_K) \; .$$

$$\textbf{UB} \leftarrow \min\{\textbf{UB}, c^T\bar{x} + \sum_{\omega=1}^{K} \alpha_\omega {f_\omega}^T \bar{y}_\omega\} \; .$$

If UB $-$ LB $\leq \varepsilon$, then terminate. Otherwise, add all of the new constraints to RMP and return to step **1.**

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Benders' Decomposition

Block-Ladder Structure

Comment

In this version of Benders' decomposition, we might add as many as K new constraints per major iteration.

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The Power Plant Investment Model

$$\min_{x,y} \quad \sum_{i=1}^{4} c_i x_i + \sum_{\omega=1}^{125} \alpha_\omega \sum_{i=1}^{5} \sum_{j=1}^{5} \sum_{k=1}^{15} 0.01 f_i(\omega) h_j y_{ijk\omega}$$

$$\text{s.t.} \quad \sum_{i=1}^{4} c_i x_i \leq 10,000 \qquad \text{(Budget constraint)}$$

$$x_4 \leq 5.0 \qquad \text{(Hydroelectric constraint)}$$

$$y_{ijk\omega} \leq x_i \quad \text{for } i = 1, \ldots, 4, \text{ all } j, k, \omega \qquad \text{(Capacity constraints)}$$

$$\sum_{i=1}^{5} y_{ijk\omega} \geq D_{jk\omega} \quad \text{for all } j, k, \omega \qquad \text{(Demand constraints)}$$

$$x \geq 0, \quad y \geq 0$$

α_ω is the probability of scenario ω
$D_{jk\omega}$ is the power demand in block j and year k under scenario ω.

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The Power Plant Investment Model

Size of Model

4 stage-1 variables

$46,875$ stage-2 variables

2 stage-1 constraints

375 constraints for each of the 125 scenarios in stage-2.

Total of $46,877$ constraints.

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The Power Plant Investment Model

Subproblem

Consider a fixed x which is feasible (i.e., $x \geq 0$ and satisfies the budget constraint and hydroelectric constraint).

$$
\begin{aligned}
z_\omega(x) := \min_y \quad & \sum_{i=1}^{5} \sum_{j=1}^{5} \sum_{k=1}^{15} 0.01 f_i(\omega) h_j y_{ijk\omega} \\
\text{s.t.} \quad & y_{ijk\omega} \leq x_i \qquad \text{for } i = 1, \ldots, 4, \text{ all } j, k, \quad \text{(Capacity constraints)} \\
& \sum_{i=1}^{5} y_{ijk\omega} \geq D_{jk\omega} \qquad \text{for all } j, k \qquad \text{(Demand constraints)} \\
& y \geq 0.
\end{aligned}
$$

Each subproblem has only 375 variables and 375 constraints.

(Furthermore, the structure of this problem lends itself to a simple greedy optimal solution.)

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The Power Plant Investment Model

Dual of Subproblem

The dual of the above problem is:

$$
\begin{aligned}
z_\omega(x) := \max_{p,q} \quad & \sum_{i=1}^{4}\sum_{j=1}^{5}\sum_{k=1}^{15} x_i p_{ijk} + \sum_{j=1}^{5}\sum_{k=1}^{15} D_{jk\omega} q_{jk} \\
\text{s.t.} \quad & p_{ijk} + q_{jk} \le 0.01 f_i(\omega) h_j \qquad i = 1, \ldots, 4, \text{ for all } j, k \\
& p_{ijk} \le 0 \\
& q_{jk} \ge 0.
\end{aligned}
$$

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Computational Results

Original Model

Iterations

All models were solved using CPLEX, called from the AMPL programming environment on a 450 MHz Pentium II processor.

	Original Model Iterations
Simplex	41,339
Interior-point	51

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Computational Results

Benders' Decomposition

Iterations and Constraints

Benders' duality gap tolerance is $\varepsilon = 10^{-2}$.

	Benders' Decomposition	
	Basic (1 Block)	Block-Ladder (125 Blocks)
Master Iterations	39	15
Generated Constraints	42	1,726

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Computational Results

CPU Time

		Benders' Decomposition	
	Original Model	Basic (1 Block)	Block-Ladder (125 Blocks)
Simplex	7:50	4:55	2:35
Interior-point	1:00	6:20	3:00

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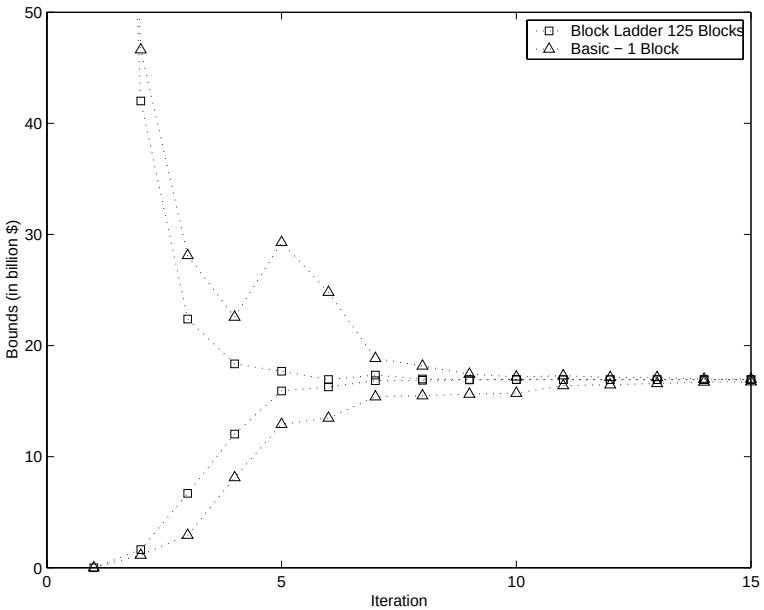
Computational Results

Upper and Lower Bounds



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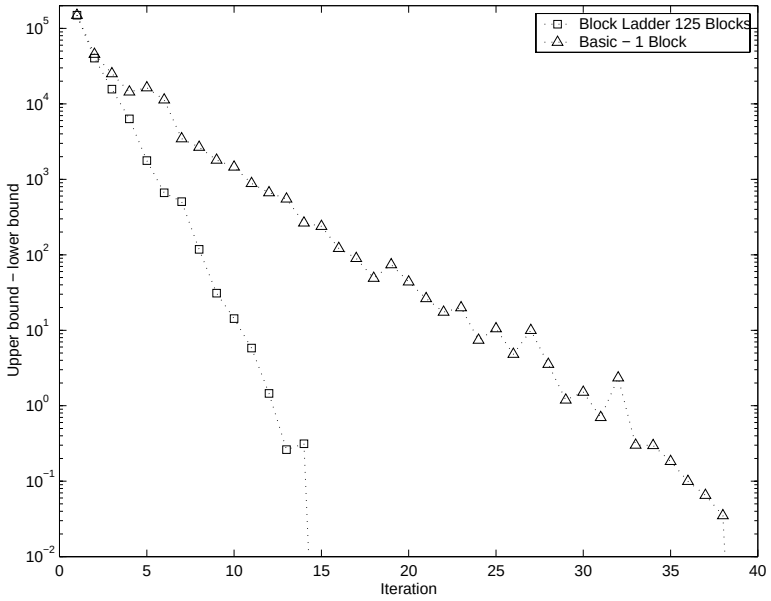
Computational Results

Duality Gap



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