Benders’ Decomposition Methods for Structured Optimization, including Stochastic Optimization

Robert M. Freund

May 2, 2001
minimize_{x,y} \quad c^T x + f^T y

\text{s.t.} \quad Ax = b

Bx + Dy = d

x \geq 0 \quad y \geq 0 \, .
minimize \( c^T x + \alpha_1 f_1^T y_1 + \alpha_2 f_2^T y_2 + \cdots + \alpha_K f_K^T y_K \)

\[ x, y_1, \ldots, y_K \]

\( s.t. \quad Ax \quad = b \)

\( B_1 x + D_1 y_1 \quad = d_1 \)

\( B_2 x + D_2 y_2 \quad = d_2 \)

\[ \vdots \quad \cdots \quad \vdots \]

\( B_K x + D_K y_K = d_K \)

\( x, y_1, y_2, \ldots, y_K \geq 0 \)
Block Ladder Structure

<table>
<thead>
<tr>
<th>Stage-1 Variables</th>
<th>Stage-2 Variables</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

State 1

State 2

State 3

State k

Objectives

©2002 Massachusetts Institute of Technology. All rights reserved.
Basic Model

Reformulation

\[ \text{VAL} = \min_{x,y} \quad c^T x + f^T y \]

s.t. \quad Ax = b

\quad Bx + Dy = d

\quad x \geq 0 \quad y \geq 0 \ ,
Basic Model

Reformulation

\[
\text{VAL} = \min_{x} \ c^T x + z(x)
\]

s.t. \quad A x = b

\[x \geq 0,\]

where:

\[
P2: \ z(x) = \min_{y} \ f^T y
\]

s.t. \quad D y = d - B x

\[y \geq 0.\]
Basic Model

Reformulation

\[
P2: \quad z(x) = \min_{y} \quad f^T y
\]

\[
s.t. \quad Dy = d - Bx
\]

\[
y \geq 0.
\]

The dual is:

\[
D2: \quad z(x) = \max_{p} \quad p^T(d - Bx)
\]

\[
s.t. \quad D^T p \leq f.
\]
Basic Model

Reformulation

\[ \text{D2 : } z(x) = \max_{p} p^T (d - Bx) \]

s.t. \[ D^T p \leq f \].

The feasible region of D2 is the set:

\[ \mathcal{D}_2 := \{ p \mid D^T p \leq f \} \].
Basic Model

The feasible region of $D_2$ is the set:

$$D_2 := \{ p \mid D^T p \leq f \} ,$$

Extreme points are:

$$p^1, \ldots, p^I$$

Extreme rays are:

$$r^1, \ldots, p^J$$
Basic Model

Reformulation

\[ D2 : \quad z(x) = \max_{p} \ p^T (d - Bx) \]

s.t. \[ D^T p \leq f. \]

If we solve D2, two cases arise:

D2 is unbounded from above, or

D2 has an optimal solution.
D2 : \( z(x) = \max_{p} p^T(d - Bx) \)

\[ \text{s.t.} \quad D^T p \leq f. \]

If D2 is unbounded from above, the algorithm will return one of the extreme rays \( \bar{r} = r^j \) for some \( j \).

\[ (r^j)^T(d - Bx) > 0 \]

\[ z(x) = +\infty. \]
If D2 has an optimal solution, the algorithm will return one of the extreme points $\bar{p} = p^i$ for some $i$. 

$$z(x) = (p^i)^T (d - Bx) = \max_{k=1,\ldots,I} (p^k)^T (d - Bx).$$
Basic Model

\[ \text{D2} : \ z(x) = \max_{p} \ p^T(d - Bx) \]
\[ \text{s.t.} \quad D^T p \leq f. \]

\[ \text{D2} : \ z(x) = \min_{z} \quad z \]
\[ \text{s.t.} \quad (p^i)^T(d - Bx) \leq z \quad i = 1, \ldots, I \]
\[ (r^j)^T(d - Bx) \leq 0 \quad j = 1, \ldots, J. \]
Basic Model

Reformulation

Full Master Problem...

\[
\text{VAL} = \min_{x} c^T x + z(x)
\]
\[
\text{s.t.} \quad Ax = b \\
x \geq 0,
\]

FMP : \[
\text{VAL} = \min_{x,z} c^T x + z
\]
\[
\text{s.t.} \quad Ax = b \\
x \geq 0 \\
(p^i)^T (d - Bx) \leq z \quad i = 1, \ldots, I \\
(r^j)^T (d - Bx) \leq 0 \quad j = 1, \ldots, J.
\]
We have eliminated the variables $y$ from the problem.
We have added a single scalar variable $z$.
We have added a generically extremely large number of constraints.

FMP: $\text{VAL} = \min_{x,z} c^T x + z$

s.t. $Ax = b$

$x \geq 0$

$(p^i)^T(d - Bx) \leq z \quad i = 1, \ldots, I$

$(r^j)^T(d - Bx) \leq 0 \quad j = 1, \ldots, J$. 
Consider problem RMP\(^k\) composed of only \(k\) of the extreme point/extreme ray constraints from FMP:

\[
\text{RMP}^k: \quad \text{VAL}^k = \min_{x, z} \quad c^T x + z \\
\text{s.t.} \quad Ax = b \\
\quad x \geq 0 \\
\quad (p^i)^T (d - Bx) \leq z \quad i = 1, \ldots, k - l \\
\quad (r^j)^T (d - Bx) \leq 0 \quad j = 1, \ldots, l .
\]
Delayed Constraint Generation

Restricted Master Problem

Lower Bound on VAL

\[ RMP^k: \quad \text{VAL}^k = \min_{x,z} \quad c^T x + z \]

\[ \text{s.t.} \quad Ax = b \]

\[ x \geq 0 \]

\[ (p^i)^T(d - Bx) \leq z \quad i = 1, \ldots, k - l \]

\[ (r^j)^T(d - Bx) \leq 0 \quad j = 1, \ldots, l . \]

We solve this problem, obtaining \( \text{VAL}^k, \bar{x}, \bar{z} \)

\[ \text{VAL}^k \leq \text{VAL} . \]
RMP$^k$: \[ VAL^k = \min_{x, z} \quad c^T x + z \]
\[
\text{s.t.} \quad A x = b \\
\quad x \geq 0 \\
\quad (p^i)^T (d - B x) \leq z \quad i = 1, \ldots, k - l \\
\quad (r^j)^T (d - B x) \leq 0 \quad j = 1, \ldots, l.
\]

Check whether the solution $\bar{x}, \bar{z}$ is optimal for the full master problem.
Check whether $\bar{x}, \bar{z}$ violates any of the non-included constraints.
Solve the Subproblem

Finding a Violated Constraint

Solve:

\[ Q(\bar{x}) : \quad q^* = \max_p \quad p^T (d - B\bar{x}) = \min_y \quad f^T y \]

\[
\begin{align*}
\text{s.t.} \quad D^T p & \leq f \\
\text{s.t.} \quad D y & = d - B\bar{x} \\
y & \geq 0.
\end{align*}
\]

If \( Q(\bar{x}) \) is unbounded, return an extreme ray \( \bar{r} = r^j \) for some \( j \), where \( \bar{x} \) will satisfy:

\[ (r^j)^T (d - B\bar{x}) > 0. \]

Therefore \( \bar{x} \) has violated the constraint:

\[ (r^j)^T (d - Bx) \leq 0, \]

Add this constraint to RMP and re-solve RMP.
Delayed Constraint Generation

Solve the Subproblem

Upper Bound on VAL

\[ Q(\bar{x}) : \quad q^* = \max_p p^T (d - B\bar{x}) = \min_y f^T y \]
\[ \text{s.t. } D^T p \leq f \quad \text{s.t. } Dy = d - B\bar{x} \]
\[ y \geq 0 . \]

If \( Q(\bar{x}) \) has an optimal solution, return an optimal extreme point \( \bar{p} = p^i \) for some \( i \), and the optimal solution \( \bar{y} \) to the minimization problem of \( Q(\bar{x}) \). \( (\bar{x}, \bar{y}) \) is feasible for the original problem.

Suppose that UB is an upper bound on VAL. Update UB:

\[
\text{If } c^T \bar{x} + f^T \bar{y} \leq UB, \text{ then BESTSOL } \leftarrow (\bar{x}, \bar{y}). \\
UB \leftarrow \min\{UB, c^T \bar{x} + f^T \bar{y}\} .
\]
Delayed Constraint Generation

Solve the Subproblem

Finding a Violated Constraint

\[ Q(\bar{x}) : \quad q^* = \max_p \quad p^T (d - B\bar{x}) = \min_y \quad f^T y \]

\[ \quad \text{s.t.} \quad D^T p \leq f \quad \quad \quad \quad \quad \text{s.t.} \quad Dy = d - B\bar{x} \]

\[ \quad y \geq 0 . \]

If \( q^* > \bar{z} \), then

\[ (p^i)^T (d - B\bar{x}) = q^* > \bar{z} . \]

We therefore have found that \( \bar{x}, \bar{z} \) has violated the constraint:

\[ (p^i)^T (d - Bx) \leq z , \]

and so we add this constraint to RMP and re-solve RMP.
Delayed Constraint Generation

Solve the Subproblem

Checking for Full Optimality

\[ Q(\bar{x}) : \quad q^* = \max_{p} \quad p^T(d - B\bar{x}) = \min_{y} \quad f^Ty \]

\[
\begin{align*}
\text{s.t.} & \quad D^Tp \leq f \\
\text{s.t.} & \quad Dy = d - B\bar{x} \\
& \quad y \geq 0.
\end{align*}
\]

If \( q^* \leq \bar{z} \), then:

\[
\max_{k=1,\ldots,I} (p^k)^T(d - B\bar{x}) = (p^i)^T(d - B\bar{x}) = q^* \leq \bar{z},
\]

\( \bar{x}, \bar{z} \) is feasible and hence optimal for FMP

\( (\bar{x}, \bar{y}) \) is optimal for the original problem.
Otherwise terminate if:

\[ UB - VAL^k \leq \varepsilon \]

for a pre-specified tolerance \( \varepsilon \).
0. Set \( LB = -\infty \) and \( UB = +\infty \).

1. A typical iteration starts with the relaxed master problem \( RMP^k \), in which only \( k \) constraints of the full master problem \( FMP \) are included. An optimal solution \( \bar{x}, \bar{z} \) to the relaxed master problem is computed. We update the lower bound on \( VAL \):

\[
LB \leftarrow VAL^k.
\]

2. Solve the subproblem \( Q(\bar{x}) \):

\[
Q(\bar{x}) : \quad q^* = \max_p p^T (d - B\bar{x}) = \min_y f^T y
\]

\[
\text{s.t. } D^T p \leq f \quad \text{s.t. } Dy = d - B\bar{x}
\]

\[y \geq 0.\]
3. If $Q(\bar{x})$ has an optimal solution, let $\bar{p}$ and $\bar{y}$ be the primal and dual solutions of $Q(\bar{x})$. If $q^* \leq \bar{z}$, then $\bar{x}, \bar{z}$ satisfies all constraints of FMP, and so $\bar{x}, \bar{z}$ is optimal for FMP. Furthermore, this implies that $(\bar{x}, \bar{y})$ is optimal for the original problem.

If $q^* > \bar{z}$, then we add the constraint:

$$\bar{p}^T (d - Bx) \leq z$$

to the restricted master problem RMP.
Delayed Constraint Generation

Update the upper bound and the best solution:

If \( c^T \bar{x} + f^T \bar{y} \leq UB \), then \( \text{BESTSOL} \leftarrow (\bar{x}, \bar{y}) \)

\[
UB \leftarrow \min\{UB, c^T \bar{x} + f^T \bar{y}\}
\]

If \( UB - LB \leq \epsilon \), then terminate. Otherwise, return to step 1.
4. If $Q(\bar{x})$ is unbounded from above, let $\bar{r}$ be the extreme ray generated by the algorithm that solves $Q(\bar{x})$. Add the constraint:

$$\bar{r}^T (d - Bx) \leq 0$$

to the restricted master problem RMP, and return to step 1.
This algorithm is known formally as *Benders’ decomposition*.

Benders’ decomposition method was developed in 1962.

Benders’ decomposition method is the same as Dantzig-Wolfe decomposition applied to the dual problem.
Benders' Decomposition

\[ \text{VAL} = \text{minimize} \quad c^T x + \alpha_1 f_1^T y_1 + \alpha_2 f_2^T y_2 + \cdots + \alpha_K f_K^T y_K \]

\[ x, y_1, \ldots, y_K \]

s.t. \[ Ax = b \]

\[ B_1 x + D_1 y_1 = d_1 \]

\[ B_2 x + D_2 y_2 = d_2 \]

\[ \vdots \]

\[ B_K x + D_K y_K = d_K \]

\[ x, y_1, y_2, \ldots, y_K \geq 0 \]
Benders’ Decomposition

Block-Ladder Structure

Reformulation...

\[ \text{VAL} = \min_{x} \ c^{T} x + \sum_{\omega=1}^{K} \alpha_{\omega} z_{\omega}(x) \]

\[ \text{s.t.} \quad A x = b \]

\[ x \geq 0 , \]

where:

\[ P_{2\omega} : \quad z_{\omega}(x) = \min_{y_{\omega}} \ f_{\omega}^{T} y_{\omega} \]

\[ \text{s.t.} \quad D_{\omega} y_{\omega} = d_{\omega} - B_{\omega} x \]

\[ y_{\omega} \geq 0 . \]
Benders’ Decomposition

\[ P_{2\omega} : \quad z_\omega(x) = \min_{y_\omega} \quad f^T \omega y_\omega \]
\[ \text{s.t.} \quad D_\omega y_\omega = d_\omega - B_\omega x \]
\[ y_\omega \geq 0. \]

The dual is:

\[ D_{2\omega} : \quad z_\omega(x) = \max_{p_\omega} \quad p^T \omega (d_\omega - B_\omega x) \]
\[ \text{s.t.} \quad D^T \omega p_\omega \leq f_\omega. \]
Benders’ Decomposition

Block-Ladder Structure

...Reformulation...

\[ D_{2\omega} : \quad z_{\omega}(x) = \max_{p_{\omega}} p_{\omega}^T (d_{\omega} - B_{\omega} x) \]

\[ \text{s.t.} \quad D_{\omega}^T p_{\omega} \leq f_{\omega} . \]

The feasible region of \( D_{2\omega} \) is the set:

\[ D_{2\omega}^\omega := \left\{ p_{\omega} \mid D_{\omega}^T p_{\omega} \leq f_{\omega} \right\} , \]

Extreme points of \( D_{2\omega}^\omega \):

\[ p_{\omega}^1, \ldots, p_{\omega}^I \]

Extreme rays of \( D_{2\omega}^\omega \):

\[ r_{\omega}^1, \ldots, p_{\omega}^J \]
Benders’ Decomposition

Block-Ladder Structure

...Reformulation...

\[ D_{2\omega} : \quad z_\omega(x) = \max_{p_\omega} p_\omega^T (d_\omega - B_\omega x) \]

s.t. \[ D_{\omega}^T p_\omega \leq f_\omega. \]

If we solve \( D_{2\omega} \), two cases arise:

- \( D_{2\omega} \) is unbounded from above, or
- \( D_{2\omega} \) has an optimal solution.
Benders’ Decomposition

\[ D_2 \omega : \quad z_\omega(x) = \max_{p_\omega} p^T_\omega (d_\omega - B_\omega x) \]

\[ \text{s.t.} \quad D^T_\omega p_\omega \leq f_\omega. \]

If \( D_2 \omega \) is unbounded from above, return one of the extreme rays \( \bar{r}_\omega = r^j_\omega \) for some \( j \), with the property that

\[ (r^j_\omega)^T (d_\omega - B_\omega x) > 0 \]

in which case

\[ z_\omega(x) = +\infty. \]
Benders’ Decomposition

Block-Ladder Structure

...Reformulation...

$$D_{2\omega} : \quad z_{\omega}(x) = \max_{p_\omega} \ p_\omega^T (d_\omega - B_\omega x)$$

s.t. \quad D_{\omega}^T p_\omega \leq f_\omega.$$

If $D_{2\omega}$ has an optimal solution, return one of the extreme points $\bar{p}_\omega = p_i^\omega$ for some $i$ as well as the optimal objective function value $z_\omega(x)$

$$z_\omega(x) = (p_i^\omega)^T (d_\omega - B_\omega x) = \max_{k=1,\ldots,I_\omega} (p_k^\omega)^T (d_\omega - B_\omega x).$$
Benders’ Decomposition

\[ D_{2\omega} : \quad z_{\omega}(x) = \max_{p_{\omega}} \quad p_{\omega}^T (d_{\omega} - B_{\omega}x) \]

\[ \text{s.t.} \quad D_{\omega}^T p_{\omega} \leq f_{\omega}. \]

\[ D_{2\omega} : \quad z_{\omega}(x) = \min_{z_{\omega}} \quad z_{\omega} \]

\[ \text{s.t.} \quad (p_{\omega}^i)^T (d_{\omega} - B_{\omega}x) \leq z_{\omega} \quad i = 1, \ldots, I_{\omega} \]

\[ (r_{\omega}^j)^T (d_{\omega} - B_{\omega}x) \leq 0 \quad j = 1, \ldots, J_{\omega}. \]
Benders’ Decomposition

\[
\text{VAL } = \min_{x} \quad c^T x + \sum_{\omega=1}^{K} \alpha_{\omega} z_{\omega}(x) \\
\text{s.t. } \quad A x = b \\
\quad x \geq 0 ,
\]

FMP: \[
\text{VAL } = \min_{x, z_1, \ldots, z_K} \quad c^T x + \sum_{\omega=1}^{K} \alpha_{\omega} z_{\omega} \\
\text{s.t. } \quad A x = b \\
\quad x \geq 0 \\
(p^i_{\omega})^T (d_{\omega} - B_{\omega} x) \leq z_{\omega} \quad i = 1, \ldots, I_{\omega}, \omega = 1, \ldots, K \\
(r^j_{\omega})^T (d_{\omega} - B_{\omega} x) \leq 0 \quad j = 1, \ldots, J_{\omega}, \omega = 1, \ldots, K .
\]
Consider problem $\text{RMP}^k$ composed of only $k$ of the extreme point/extreme ray constraints from FMP:

\[
\text{RMP}^k: \quad \text{VAL}^k = \text{minimum} \quad c^T x + \sum_{\omega=1}^{K} \alpha_\omega z_\omega \\
x, z_1, \ldots, z_K \\
\text{s.t.} \\
Ax = b \\
x \geq 0 \\
(p_\omega^i)^T (d_\omega - B_\omega x) \leq z_\omega \quad \text{for some } i \text{ and } \omega \\
(r_\omega^i)^T (d_\omega - B_\omega x) \leq 0 \quad \text{for some } i \text{ and } \omega ,
\]

where there are a total of $k$ of the inequality constraints.
Block-Ladder Structure

Restricted Master Problem

Lower Bound on VAL

\[ \text{RMP}^k: \text{VAL}^k = \text{minimum} \quad c^T x + \sum_{\omega=1}^{K} \alpha_\omega z_\omega \]

\[ \begin{align*}
&x, z_1, \ldots, z_K \\
&\text{s.t.} \\
&Ax = b \\
&x \geq 0 \\
&(p_\omega^j)^T(d_\omega - B_\omega x) \leq z_\omega \quad \text{for some } i \text{ and } \omega \\
&(r_\omega^j)^T(d_\omega - B_\omega x) \leq 0 \quad \text{for some } i \text{ and } \omega ,
\end{align*} \]

Solve for VAL^k and \( x, \bar{z}_1, \ldots, \bar{z}_K \)

\[ \text{VAL}^k \leq \text{VAL} . \]
Block-Ladder Structure

Restricted Master Problem

\[ \text{RMP}^k: \text{VAL}^k = \min \quad c^T x + \sum_{\omega=1}^{K} \alpha_\omega z_\omega \]
\[ x, z_1, \ldots, z_K \]
\[ \text{s.t.} \quad A x = b \]
\[ x \geq 0 \]
\[ (p_i^\omega)^T (d_\omega - B_\omega x) \leq z_\omega \quad \text{for some } i \text{ and } \omega \]
\[ (r_j^\omega)^T (d_\omega - B_\omega x) \leq 0 \quad \text{for some } i \text{ and } \omega , \]

Solve for VAL^k and \( \bar{x}, \bar{z}_1, \ldots, \bar{z}_K \)
Check whether the solution \( \bar{x}, \bar{z}_1, \ldots, \bar{z}_K \) is optimal for FMP
Check whether \( \bar{x}, \bar{z}_1, \ldots, \bar{z}_K \) violates any of the non-included constraints.
Block-Ladder Structure

Restricted Master Problem

Solve the Subproblems

\[ Q_\omega(\bar{x}) : \quad q^*_\omega = \max_{p_\omega} \quad p_\omega^T (d_\omega - B_\omega \bar{x}) = \min_{y_\omega} \quad f_\omega^T y_\omega \]

\[ \text{s.t.} \quad D_\omega^T p_\omega \leq f_\omega \quad \text{s.t.} \quad D_\omega y_\omega = d_\omega - B_\omega \bar{x} \]

\[ y_\omega \geq 0. \]

If \( Q_\omega(\bar{x}) \) is unbounded, return an extreme ray \( \bar{r}_\omega = r_\omega^j \) for some \( j \), where \( \bar{x} \) will satisfy:

\[ (r_\omega^j)^T (d_\omega - B_\omega \bar{x}) > 0. \]

Therefore \( \bar{x} \) has violated the constraint:

\[ (r_\omega^j)^T (d_\omega - B_\omega x) \leq 0, \]

and so we add this constraint to RMP.

©2002 Massachusetts Institute of Technology. All rights reserved.
Block-Ladder Structure

Restricted Master Problem

Finding a Violated Constraint

\[ Q_\omega(\bar{x}) : q_\omega^* = \max_{p_\omega} \; p_\omega^T (d_\omega - B_\omega \bar{x}) = \min_{y_\omega} \; f_\omega^T y_\omega \]

\[
\begin{align*}
\text{s.t.} \quad & D_\omega^T p_\omega \leq f_\omega \\
\text{s.t.} \quad & D_\omega y_\omega = d_\omega - B_\omega \bar{x} \\
& y_\omega \geq 0.
\end{align*}
\]

If \( Q_\omega(\bar{x}) \) has an optimal solution, return an optimal extreme point \( \bar{p}_\omega = p^i_\omega \) for some \( i \), as well as the optimal solution \( \bar{y}_\omega \) of the minimization problem of \( Q_\omega(\bar{x}) \).

If \( q_\omega^* > \bar{z}_\omega \), then

\[(p^i_\omega)^T (d_\omega - B_\omega \bar{x}) = q_\omega^* > \bar{z}_\omega.
\]

Therefore \( \bar{x}, \bar{z}_\omega \) has violated the constraint:

\[(p^i_\omega)^T (d_\omega - B_\omega \bar{x}) \leq z_\omega,
\]

and so we add this constraint to RMP.
\[ Q_\omega(\bar{x}) : q^*_\omega = \max_{p_\omega} \ p_\omega^T(d_\omega - B_\omega \bar{x}) = \min_{y_\omega} \ f_\omega^T y_\omega \]

s.t. \( D_\omega^T p_\omega \leq f_\omega \)

s.t. \( D_\omega y_\omega = d_\omega - B_\omega \bar{x} \)

\[ y_\omega \geq 0. \]

If \( q^*_\omega \) is finite for all \( \omega = 1, \ldots, K \), then \( \bar{x}, \bar{y}_1, \ldots, \bar{y}_K \) satisfies all of the constraints of the original problem.

Suppose that UB is an upper bound on VAL, we update UB as follows:

If \( c^T \bar{x} + \sum_{\omega=1}^{K} \alpha_\omega f_\omega^T \bar{y}_\omega \leq UB \), then BESTSOL \( \leftarrow (\bar{x}, \bar{y}_1, \ldots, \bar{y}_K) \).

\[ UB \leftarrow \min\{UB, c^T \bar{x} + \sum_{\omega=1}^{K} \alpha_\omega f_\omega^T \bar{y}_\omega\} . \]
If \( q^*_\omega \leq \bar{z}_\omega \) for all \( \omega = 1, \ldots, K \), then we have:

\[
\max_{l=1,\ldots,I_\omega} (p^l_\omega)^T (d_\omega - B_\omega \bar{x}) = (p^i_\omega)^T (d_\omega - B_\omega \bar{x}) = q^*_\omega \leq \bar{z}_\omega \quad \text{for all } \omega = 1, \ldots, K ,
\]

\( \bar{x}, \bar{z}_1, \ldots, \bar{z}_K \) satisfies all of the constraints in FMP.

\( \bar{x}, \bar{z}_1, \ldots, \bar{z}_K \) is feasible and hence optimal for FMP

\( (\bar{x}, \bar{y}_1, \ldots, \bar{y}_K) \) is optimal for the original problem, and we can terminate the algorithm.
Else, we can also terminate the algorithm whenever:

$$\text{UB} - \text{VAL}^k \leq \varepsilon$$

for a pre-specified tolerance $\varepsilon$. 
Benders’ Decomposition

0. Set \( LB = -\infty \) and \( UB = +\infty \).

1. A typical iteration starts with the relaxed master problem \( RMP^k \), in which only \( k \) constraints of the full master problem \( FMP \) are included. An optimal solution \( \bar{x}, \bar{z}_1, \ldots, \bar{z}_K \) to the relaxed master problem is computed. We update the lower bound on VAL:

\[
LB \leftarrow VAL^k.
\]

2. For \( \omega = 1, \ldots, K \), solve the subproblem \( Q_\omega(\bar{x}) \):

\[
Q_\omega(\bar{x}) : q^*_\omega = \max_{p_\omega} p_\omega^T (d_\omega - B_\omega \bar{x}) = \min_{y_\omega} f_\omega^T y_\omega \\
\text{s.t. } D_\omega^T p_\omega \leq f_\omega \quad \text{s.t. } D_\omega y_\omega = d_\omega - B_\omega \bar{x} \\
y_\omega \geq 0.
\]
If $Q_\omega(\bar{x})$ is unbounded from above, let $\bar{r}_\omega$ be the extreme ray generated by the algorithm that solves $Q(\bar{x})$. Add the constraint:

$$\bar{r}_\omega^T(d_\omega - B_\omega x) \leq 0$$

to the restricted master problem RMP.
Benders' Decomposition

If \( Q_\omega(\bar{x}) \) has an optimal solution, let \( \bar{p}_\omega \) and \( \bar{y}_\omega \) be the primal and dual solutions of \( Q_\omega(\bar{x}) \). If \( q^*_\omega > \bar{z}_\omega \), then we add the constraint:

\[
\bar{p}_\omega^T (d_\omega - B_\omega x) \leq z_\omega
\]

to the restricted master problem RMP.
3. If $q^*_\omega \leq z_\omega$ for all $\omega = 1, \ldots, K$, then $(\bar{x}, \bar{y}_1, \ldots, \bar{y}_K)$ is optimal for the original problem, and the algorithm terminates.

4. If $Q_\omega(\bar{x})$ has an optimal solution for all $\omega = 1, \ldots, K$, then update the upper bound on VAL as follows:

$$
\text{If } c^T \bar{x} + \sum_{\omega=1}^{K} \alpha_\omega f_\omega^T \bar{y}_\omega \leq UB, \text{ then } \text{BESTSOL} \leftarrow (\bar{x}, \bar{y}_1, \ldots, \bar{y}_K).
$$

$$
UB \leftarrow \min\{UB, c^T \bar{x} + \sum_{\omega=1}^{K} \alpha_\omega f_\omega^T \bar{y}_\omega\}.
$$

If $UB - LB \leq \varepsilon$, then terminate. Otherwise, add all of the new constraints to RMP and return to step 1.
In this version of Benders’ decomposition, we might add as many as $K$ new constraints per major iteration.
The Power Plant Investment Model

\[
\begin{align*}
\min_{x,y} & \quad \sum_{i=1}^{4} c_i x_i + \sum_{\omega=1}^{125} \alpha_\omega \sum_{i=1}^{5} \sum_{j=1}^{5} \sum_{k=1}^{15} 0.01 f_i(\omega) h_j y_{ijk}\omega \\
\text{s.t.} & \quad \sum_{i=1}^{4} c_i x_i \leq 10,000 \quad \text{(Budget constraint)} \\
& \quad x_4 \leq 5.0 \quad \text{(Hydroelectric constraint)} \\
& \quad y_{ijk\omega} \leq x_i \quad \text{for } i = 1, \ldots, 4, \text{ all } j, k, \omega \quad \text{(Capacity constraints)} \\
& \quad \sum_{i=1}^{5} y_{ijk\omega} \geq D_{jk\omega} \quad \text{for all } j, k, \omega \quad \text{(Demand constraints)} \\
& \quad x \geq 0, \quad y \geq 0
\end{align*}
\]

\(\alpha_\omega\) is the probability of scenario \(\omega\)

\(D_{jk\omega}\) is the power demand in block \(j\) and year \(k\) under scenario \(\omega\).
The Power Plant Investment Model

- **Size of Model**
  - 4 stage-1 variables
  - 46,875 stage-2 variables
  - 2 stage-1 constraints
  - 375 constraints for each of the 125 scenarios in stage-2.
  - Total of 46,877 constraints.
Consider a fixed $x$ which is feasible (i.e., $x \geq 0$ and satisfies the budget constraint and hydroelectric constraint).

$$z_\omega(x) := \min_y \sum_{i=1}^5 \sum_{j=1}^5 \sum_{k=1}^{15} 0.01 f_i(\omega) h_j y_{ijk\omega}$$

s.t. $y_{ijk\omega} \leq x_i$ for $i = 1, \ldots, 4$, all $j, k$, (Capacity constraints)

$$\sum_{i=1}^5 y_{ijk\omega} \geq D_{jk\omega} \quad \text{for all } j, k \quad \text{(Demand constraints)}$$

$y \geq 0$.

Each subproblem has only 375 variables and 375 constraints.

(Furthermore, the structure of this problem lends itself to a simple greedy optimal solution.)
The dual of the above problem is:

\[
  z_\omega(x) := \max_{p,q} \sum_{i=1}^{4} \sum_{j=1}^{5} \sum_{k=1}^{15} x_i p_{ijk} + \sum_{j=1}^{5} \sum_{k=1}^{15} D_{jkw} q_{jk}
\]

s.t.  
\[
p_{ijk} + q_{jk} \leq 0.01 f_i(\omega) h_j \quad \text{for all } i = 1, \ldots, 4, \text{ for all } j, k
\]

\[
p_{ijk} \leq 0
\]

\[
q_{jk} \geq 0.
\]
Computational Results

All models were solved using CPLEX, called from the AMPL programming environment on a 450 MHz Pentium II processor.

<table>
<thead>
<tr>
<th></th>
<th>Original Model Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simplex</td>
<td>41,339</td>
</tr>
<tr>
<td>Interior-point</td>
<td>51</td>
</tr>
</tbody>
</table>
Benders’ duality gap tolerance is $\varepsilon = 10^{-2}$.

<table>
<thead>
<tr>
<th></th>
<th>Benders’ Decomposition</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Basic (1 Block)</td>
<td>Block-Ladder (125 Blocks)</td>
</tr>
<tr>
<td>Master Iterations</td>
<td>39</td>
<td>15</td>
</tr>
<tr>
<td>Generated Constraints</td>
<td>42</td>
<td>1,726</td>
</tr>
</tbody>
</table>
### Computational Results

<table>
<thead>
<tr>
<th>Benders’ Decomposition</th>
<th>Original Model</th>
<th>Basic (1 Block)</th>
<th>Block-Ladder (125 Blocks)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simplex</td>
<td>7:50</td>
<td>4:55</td>
<td>2:35</td>
</tr>
<tr>
<td>Interior-point</td>
<td>1:00</td>
<td>6:20</td>
<td>3:00</td>
</tr>
</tbody>
</table>
Computational Results

Upper and Lower Bounds

![Graph showing computational results with iterations and bounds in billion $]
Computational Results

Duality Gap

![Graph showing computational results]

©2002 Massachusetts Institute of Technology. All rights reserved.
References


