Optimization under Uncertainty

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Motivation

- Classical constrained optimization models assume all data is known with certainty.
- Inexactness of data, uncertainty, etc., is very common
- This can be incorporated in models
- But the models can become quite large.

Brief History

- 1950s, Dantzig and Beale started to work on linear optimization under uncertainty
- 1962, solution method developed by Benders
- Many and varied applications in the linear, nonlinear, and discrete form
  - Electric utility capacity planning
  - Financial Planning and Control
  - Supply chain optimization
  - Airline Planning (fleet assignment)
  - Water Resource Modeling
  - Forestry Planning
  - Many others . . .

Gemstone Tool Company

	Wrenches	Pliers	Availability (Capacity)
Steel (lbs.)	1.5	1.0	27,000 lbs./day
Molding machine (hours)	1.0	1.0	21,000 hours/day
Assembly machine (hours)	0.3	0.5	9,000 hours/day
Demand limit (tools/day)	15,000	16,000	
Contribution to earnings ($/1,000 units)	$130	$100	

Gemstone Tool Company

Linear Program

Maximize contribution = $130W + 100P$

subject to:

$$
\begin{array}{lrcrcr}
\text{Wrench demand:} & W & & & \leq & 15 \\
\text{Plier demand:} & & & P & \leq & 16 \\
\text{Steel availability:} & 1.5W & + & P & \leq & 27 \\
\text{Molding machine usage:} & W & + & P & \leq & 21 \\
\text{Assembly machine usage:} & 0.3W & + & 0.5P & \leq & 9 \\
 & W & & & \geq & 0 \\
 & & & P & \geq & 0
\end{array}
$$

Gemstone Tool Company

Optimal Solution

$\boldsymbol{W}^* = 12$, $\boldsymbol{P}^* = 9$

Contribution to earnings is \$2,460 = $130\boldsymbol{W}^* + 100\boldsymbol{P}^*$.

GTC Planning under Uncertainty

For the current quarter, GTC had contracted with a steel supplier for the delivery of 27,000 lbs. of steel per day.

Now suppose that GTC is planning for next quarter, and that they would like to determine how much steel to contract for with local suppliers for the next quarter.

The market price for such contracts is $58.00/1,000 lbs. of steel.

GTC Planning under Uncertainty

- The market price of steel is \$58.00/1,000 lbs.
- The utilization of steel, assemby machine hours, and molding machine hours in wrenches and pliers is the same as given in the original problem.
- The molding machine capacity is the same as in the original problem, namely 21,000 hours/day.
- The demand for wrenches and pliers is the same as in the original problem, namely 15,000 wrenches per day and 16,000 pliers per day.
- The unit contribution to earnings of production of pliers is the same as in the original problem, namely \$100/1,000 units.

GTC Planning under Uncertainty

- The assembly machine capacity for next quarter is uncertain: either be 8,000 hours/day (with probability 0.5) or 10,000 hours/day (with probability 0.5).
- The unit contribution to earnings of production of wrenches next quarter is uncertain: either \$90 (with probability 0.5) or \$160 (with probability 0.5).

	Wrenches	Pliers	Availability
Steel (lbs.)	1.5	1.0	S (to be determined)
Molding Machine (hours)	1.0	1.0	21,000 hours/day
Assembly Machine (hours)	0.3	0.5	either 8,000 hours/day or 10,000 hours/day
Demand Limit (tools/day)	15,000	16,000	
Contribution to Earnings (\$/1,000 units)	either \$160 or \$90	\$100	

Time Line of Events

Time	Event or Action
Today:	GTC must decide how much steel per day to contract for for the next quarter.
Soon thereafter:	• GTC will discover the actual assembly machine availability for next quarter (either 8,000 or 10,000 hours/day). • GTC will discover the actual unit earnings contribution of wrenches for next quarter (either \$160 or \$90/1,000 units).
Next quarter:	GTC must decide the production quantities of wrenches and pliers.

GTC Planning under Uncertainty

States of the World

(Scenarios)

State of the World	Assembly Machine Capacity	Unit Earnings Contribution of Wrenches	Probability
1	8,000 hours/day	$160/1,000 units	0.25
2	10,000 hours/day	$160/1,000 units	0.25
3	8,000 hours/day	$90/1,000 units	0.25
4	10,000 hours/day	$90/1,000 units	0.25

The four possible states of the world for next quarter.

Stage-2 Variables

W_1 = number of wrenches per day to produce next quarter, in 1,000s, if state-of-the-world 1 transpires,

P_1 = number of pliers per day to produce next quarter, in 1,000s, if state-of-the-world 1 transpires,

W_2 = number of wrenches per day to produce next quarter, in 1,000s, if state-of-the-world 2 transpires,

P_2 = number of pliers per day to produce next quarter, in 1,000s, if state-of-the-world 2 transpires,

Stage-2 Variables

W_3 = number of wrenches per day to produce next quarter, in 1,000s, if state-of-the-world 3 transpires,

P_3 = number of pliers per day to produce next quarter, in 1,000s, if state-of-the-world 3 transpires,

W_4 = number of wrenches per day to produce next quarter, in 1,000s, if state-of-the-world 4 transpires,

P_4 = number of pliers per day to produce next quarter, in 1,000s, if state-of-the-world 4 transpires.

Two-Stage Model

$$
\begin{array}{ll}
\textbf{maximize} & 0.25 \cdot (160W_1 + 100P_1) + 0.25 \cdot (160W_2 + 100P_2) + \\
 & 0.25 \cdot (90W_3 + 100P_3) + 0.25 \cdot (90W_4 + 100P_4) - 58.00S \\
\textbf{subject to:} & \\
\textbf{Steel1:} & 1.5W_1 + 1.0P_1 - S \le 0 \\
\textbf{Molding1:} & 1.0W_1 + 1.0P_1 \le 21 \\
\textbf{Assembly1:} & 0.3W_1 + 0.5P_1 \le 8 \\
\textbf{W-demand1:} & W_1 \le 15 \\
\textbf{P-demand1:} & P_1 \le 16 \\
\textbf{Steel2:} & 1.5W_2 + 1.0P_2 - S \le 0 \\
\textbf{Molding2:} & 1.0W_2 + 1.0P_2 \le 21 \\
\textbf{Assembly2:} & 0.3W_2 + 0.5P_2 \le 10 \\
\textbf{W-demand2:} & W_2 \le 15 \\
\textbf{P-demand2:} & P_2 \le 16
\end{array}
$$

Two-Stage Model

$$\begin{array}{ll}
\textbf{Steel3:} & 1.5W_3 + 1.0P_3 - S \le 0 \\
\textbf{Molding3:} & 1.0W_3 + 1.0P_3 \le 21 \\
\textbf{Assembly3:} & 0.3W_3 + 0.5P_3 \le 8 \\
\textbf{W}-\textbf{demand3:} & W_3 \le 15 \\
\textbf{P}-\textbf{demand3:} & P_3 \le 16 \\
\textbf{Steel4:} & 1.5W_4 + 1.0P_4 - S \le 0 \\
\textbf{Molding4:} & 1.0W_4 + 1.0P_4 \le 21 \\
\textbf{Assembly4:} & 0.3W_4 + 0.5P_4 \le 10 \\
\textbf{W}-\textbf{demand4:} & W_4 \le 15 \\
\textbf{P}-\textbf{demand4:} & P_4 \le 16 \\
\textbf{Nonnegativity:} & S, W_1, P_1, W_2, P_2, W_3, P_3, W_4, P_4 \ge 0.
\end{array}$$

GTC Planning under Uncertainty

Two-Stage Model

Optimal Solution...

Decision Variable	Optimal Solution Value
S	27.25
W_1	15.00
P_1	4.75
W_2	15.00
P_2	4.75
W_3	12.50
P_3	8.50
W_4	5.00
P_4	16.00

The optimal solution of the linear optimization model of the GTC steel supply planning problem.

GTC Planning under Uncertainty

Two-Stage Model

...Optimal Solution

State of the world Next Quarter	Production of Wrenches (units/day)		Production of Pliers (units/day)	
	Decision Variable	Value	Decision Variable	Value
1	W_1	15,000	P_1	4,750
2	W_2	15,000	P_2	4,750
3	W_3	12,500	P_3	8,500
4	W_4	5,000	P_4	16,000

The optimal production plan for next quarter for the GTC steel supply planning problem.

GTC Planning under Uncertainty

Two-Stage Model

Interpretation

- Today, GTC should contract for 27,250 lb./day of steel for next quarter.
- Next quarter, if assembly hour availability is 8,000 hours/day and the contribution of wrenches is $160, then GTC should produce 15,000 wrenches per day and 4,750 pliers per day.
- Next quarter, if assembly hour availability is 10,000 hours/day and the contribution of wrenches is $160, then GTC should produce 15,000 wrenches per day and 4,750 pliers per day.
- Next quarter, if assembly hour availability is 8,000 hours/day and the contribution of wrenches is $90, then GTC should produce 12,500 wrenches per day and 8,500 pliers per day.
- Next quarter, if assembly hour availability is 10,000 hours/day and the contribution of wrenches is $90, then GTC should produce 5,000 wrenches per day and 16,000 pliers per day.

Different Probabilities

We could have modeled different probabilities for different possible states of the world. For example:

$$
\begin{aligned}
P(\text{assembly machine hours = 8,000}) &= \mathbf{0.8}, \\
P(\text{assembly machine hours = 10,000}) &= \mathbf{0.2}
\end{aligned}
$$

and

$$
\begin{aligned}
P(\text{wrench contribution = \$160}) &= \mathbf{0.7}, \\
P(\text{wrench contribution = \$90}) &= \mathbf{0.3}.
\end{aligned}
$$

Different Probabilities

State of the World	Assembly Machine Capacity	Unit Earnings Contribution of Wrenches	Probability
1	8,000 hours/day	$160/1,000 units	$\mathbf{0.56 = 0.8 \times 0.7}$
2	10,000 hours/day	$160/1,000 units	$\mathbf{0.14 = 0.2 \times 0.7}$
3	8,000 hours/day	$90/1,000 units	$\mathbf{0.24 = 0.8 \times 0.3}$
4	10,000 hours/day	$90/1,000 units	$\mathbf{0.06 = 0.2 \times 0.3}$

The four possible states of the world for next quarter, with different probabilities of transpiring.

More Scenarios

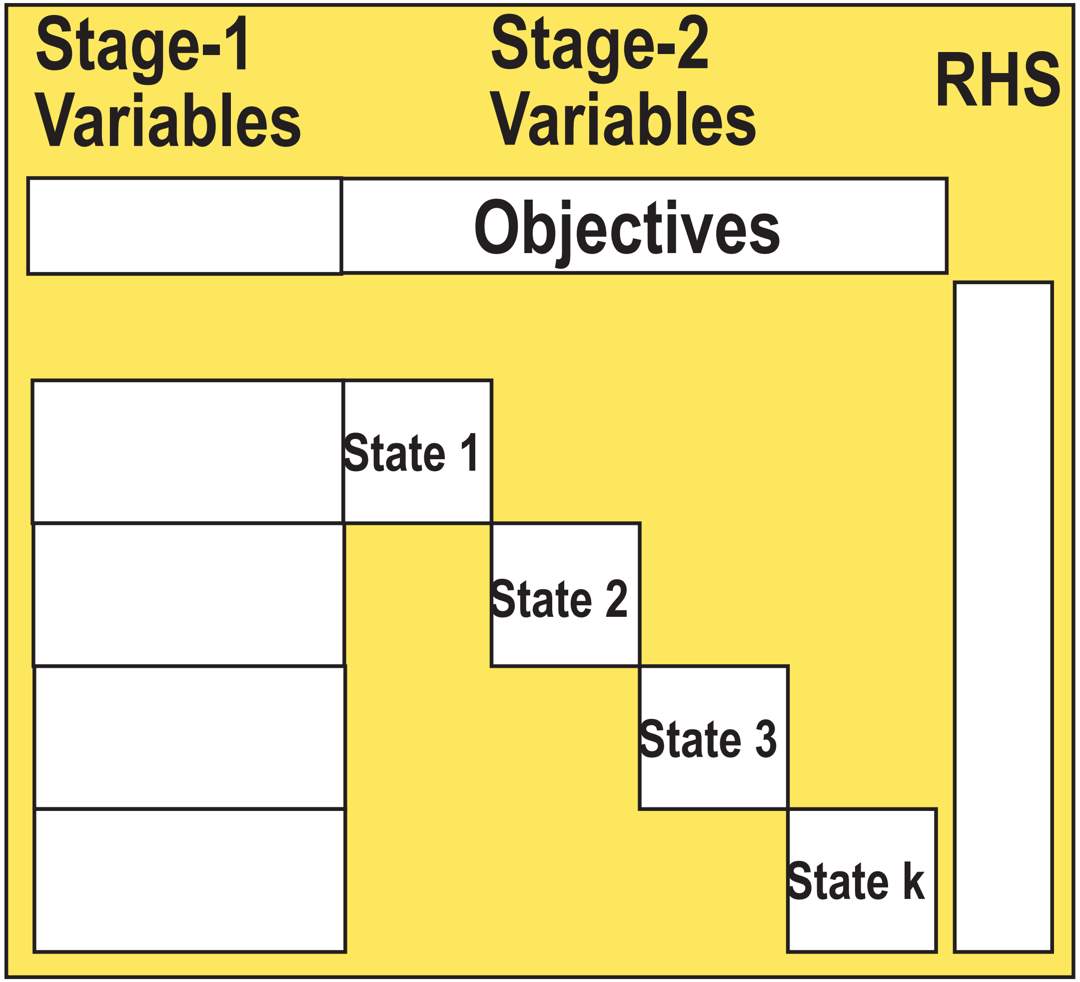
Suppose that there are seven different assembly machine capacity levels that might transpire next quarter, and that there are six different wrench unit earnings contributions levels that might transpire next quarter.

Then we would have $\mathbf{42 = 7 \times 6}$ possible states of the world, and would need $\mathbf{1 + 42 \times 2 = 85}$ decision variables, and $\mathbf{42 \times 5 = 210}$ constraints in the model.

Multi-stage Models

We might want to model more than two stages: today, next quarter, the next quarter after that, etc. The same modeling principles illustrated here would then apply, but the resulting linear optimization model can become much more complicated, as well as much larger.

Block-Ladder Structure



Block-Ladder Structure

Block ladder structure of two-stage stochastic linear optimization.

Procedure for Constructing a Two-Stage Linear Optimization Model under Uncertainty

1. Determine which decisions need to be made in stage-one (today), and which decisions need to be in stage-two (next period).

2. Enumerate the possible states of the world that might transpire next period, what the data will be in each possible state of the world, and what is the probability of each state of the world occurring.

Two-Stage Model

3. **Creating the decision variables:** Create one decision variable for each decision that must be made in stage-one. Create one decision variable for each decision that must be made in stage-two, for each possible state of the world.
4. **Constraints:** Create the necessary constraints for each possible state of the world.
5. **Objective function:** Account for the contribution of each of today's decisions in the objective function. Account for the expected value of the objective function contribution of each of next period's possible states of the world.

Matrix Form

The first-stage variables are $\boldsymbol{x}$ and are subject to constraints:

$$\boldsymbol{Ax = b} \quad , \quad \boldsymbol{x \geq 0} \; .$$

The direct contribution of the $\boldsymbol{x}$ decisions on the objective function is $\boldsymbol{c^T x}$ for some vector $\boldsymbol{c}$.

The second-stage variables are $\boldsymbol{y}$ and are subject to constraints:

$$\boldsymbol{Bx + Dy = d} \quad , \quad \boldsymbol{y \geq 0} \; .$$

Two-Stage Model

Matrix Form

The direct contribution of the $\boldsymbol{y}$ decisions on the objective function is $\boldsymbol{f}^T\boldsymbol{y}$ for some vector $\boldsymbol{f}$.

If there were no uncertainty, then the problem would look like:

$$\begin{array}{rrcrcl} \textbf{minimize}_{x,y} & \boldsymbol{c}^T\boldsymbol{x} & + & \boldsymbol{f}^T\boldsymbol{y} & & \\ \textbf{s.t.} & \boldsymbol{A}\boldsymbol{x} & & & = & \boldsymbol{b} \\ & \boldsymbol{B}\boldsymbol{x} & + & \boldsymbol{D}\boldsymbol{y} & = & \boldsymbol{d} \\ & \boldsymbol{x} \geq \boldsymbol{0} & & \boldsymbol{y} \geq \boldsymbol{0} & & \end{array}$$

We assume that the values of the data B, D, d, f are uncertain.

There are $\omega = 1, \ldots, K$ possible future scenarios, with scenario ω having a probability α_ω of being realized, for $\omega = 1, \ldots, K$.

The data B, D, d, f takes on values $B_\omega, D_\omega, d_\omega, f_\omega$ with probability α_ω for $\omega = 1, \ldots, K$.

We only learn the values of this data *after* we have made our first-stage decisions x.

Once the values of $\boldsymbol{B}, \boldsymbol{D}, \boldsymbol{d}, \boldsymbol{f}$ are known, we then make our second-stage decisions $\boldsymbol{y}$.

Let $\boldsymbol{y}_\omega$ denote the decisions $\boldsymbol{y}$ under the condition that scenario $\boldsymbol{\omega}$ is realized, for $\boldsymbol{\omega = 1, \ldots, K}$.

Two-Stage Model

Matrix Form

The objective is to choose x and $y_\omega, \omega = 1, \ldots, K$, so as to solve the following optimization model:

$$
\begin{array}{rlllllr}
\underset{x, y_1, \ldots, y_K}{\text{minimize}} & c^T x & + \ \alpha_1 f_1^T y_1 & + \ \alpha_2 f_2^T y_2 & + \ \cdots & + \ \alpha_K f_K^T y_K & \\
\text{s.t.} & Ax & & & & & = b \\
& B_1 x & + \ D_1 y_1 & & & & = d_1 \\
& B_2 x & & + \ D_2 y_2 & & & = d_2 \\
& \vdots & & & \ddots & & \vdots \\
& B_K x & & & & + \ D_K y_K & = d_K \\
& \multicolumn{6}{l}{x, y_1, y_2, \ldots, y_K \geq 0}
\end{array}
$$

Powerplant Investment Problem

The newly unified nation of Timoria must invest in a system of power plants to meet its current and future demand for electrical power.

These plants are to be built for the first year only, and are expected to operate over the next 15 years.

The budget for construction of power plants is $10 billion, which is to be allocated for four different types of plants: gas turbine, coal, nuclear power, and hydroelectric.

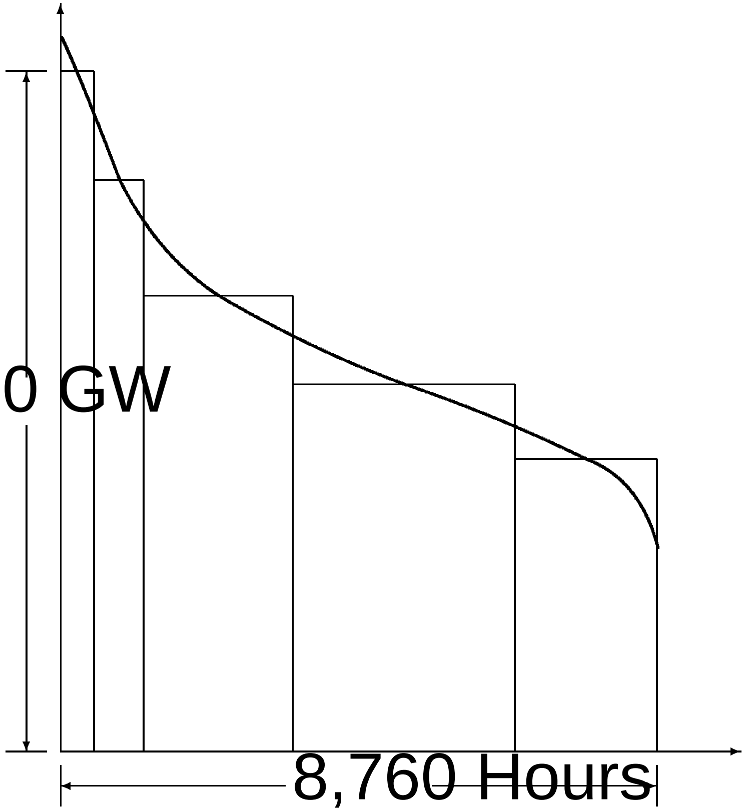
Investment Costs

Plant	Cost per GW capacity
Gas Turbine	$110 million
Coal	$180 million
Nuclear	$450 million
Hydroelectric	$950 million

Investment cost per GW of capacity.

Since hydroelectric energy depends on the availability of rivers which may be dammed, the geography of the country constrains the hydroelectric power capacity. In the case of Timoria, no more than 5.0 GW of power may be produced by hydroelectric plants.

Load Duration Curve



Projected load duration curve for the first year of demand.

Discretized Load Curve

Demand Block	Demand (GW)	Duration (hours)
#1	10.0	490
#2	8.4	730
#3	6.7	2,190
#4	5.4	3,260
#5	4.3	2,090

Power demand in the first year.

Demand Growth Rate

Growth	Probability
-1%	20%
1%	20%
3%	20%
5%	20%
7%	20%

Projected yearly growth in power demand

Plant Operating Costs

Plant	Cost per KWH
Gas Turbine*	3.92 ¢
Coal*	2.44 ¢
Nuclear	1.40 ¢
Hydroelectric	0.40 ¢
External Source	15.0 ¢

Operating cost of power generation. *Gas turbine and coal plant operating costs are expected values.

Gas Turbine Costs

Cost per KWH	Probability
3.1 ¢	10%
3.3 ¢	20%
3.9 ¢	40%
4.5 ¢	20%
4.9 ¢	10%

Probability distribution of gas turbine operating costs.

Coal Plant Costs

Cost per KWH	Probability
1.7 ¢	10%
2.1 ¢	20%
2.4 ¢	40%
2.9 ¢	20%
3.1 ¢	10%

Probability distribution of coal plant operating costs.

Stochastic Programming Formulation

Stage-1 Variables

$x = (x_1, x_2, x_3, x_4)$ is the gigawatts of capacity to be built for each type of plant.

(1. Gas turbine, 2. Coal, 3. Nuclear, 4. Hydroelectric)

The cost vector c is also a four-dimensional vector representing the investment costs shown earlier.

Stage-2 Variables

y_{ijk} denotes the amount of electricity capacity used to produce electricity by power plant type i for demand block j in year k, for $i = 1, \ldots, 5$, $j = 1, \ldots, 5$, and $k = 1, \ldots, 15$.

y_{5jk} is the amount of electricity capacity purchased from the external source.

The units of the y_{ijk} variables are in GW.

For example, y_{312} represents the amount of nuclear power used at peak demand time during the second year.

Stage-2 Variables

It is evident that the optimal value of the second-stage variables depends on the stochastic problem data.

Each possible value of the problem data is referred to as a **scenario**, which will be indexed by ω.

There are five different possible demand growth rates, five possible operating costs for gas turbines, and five possible operating costs for coal plants.

$K = 5 \times 5 \times 5 = 125$ scenarios.

$\omega = 1, \ldots, 125$.

The second-stage variable is then written as a function of the scenario, as $y_{ijk\omega}$.

Expected Cost Function

$f_i(\omega)$ is the operating cost for the various electricity sources, in cents/KWH, for $i = 1, \ldots, 5$ and $\omega = 1, \ldots, 125$.

The duration in hours of each demand block is the scalar h_i for $i = 1, \ldots, 5$

The total expected cost (in \$ million) is then written as:

$$\sum_{i=1}^{4} c_i x_i + E\Big(\sum_{i=1}^{5} \sum_{j=1}^{5} \sum_{k=1}^{15} (10^6 \text{ KW/GW}) \cdot (10^{-8} \text{ million \$/¢}) \cdot (f_i(\omega) \text{ ¢/KWH}) \cdot (h_j \text{ hours}) \cdot (y_{ijk\omega} \text{ GW}) \Big)$$

Stochastic Programming Formulation

The Full Model

$$\min_{x,y} \quad \sum_{i=1}^{4} c_i x_i + \sum_{\omega=1}^{125} \alpha_\omega \sum_{i=1}^{5} \sum_{j=1}^{5} \sum_{k=1}^{15} 0.01 f_i(\omega) h_j y_{ijk\omega}$$

$$\text{s.t.} \quad \sum_{i=1}^{4} c_i x_i \leq 10,000 \qquad \text{(Budget constraint)}$$

$$x_4 \leq 5.0 \qquad \text{(Hydroelectric constraint)}$$

$$y_{ijk\omega} \leq x_i \quad \text{for } i = 1, \ldots, 4, \text{ all } j, k, \omega \qquad \text{(Capacity constraints)}$$

$$\sum_{i=1}^{5} y_{ijk\omega} \geq D_{jk\omega} \quad \text{for all } j, k, \omega \qquad \text{(Demand constraints)}$$

$$x \geq 0, \quad y \geq 0$$

α_ω is the probability of scenario ω

$D_{jk\omega}$ is the power demand in block j and year k under scenario ω.

Size of Problem

There are $\mathbf{4}$ first-stage variables and $\mathbf{5} \times \mathbf{5} \times \mathbf{15} \times \mathbf{125}$ second-stage variables

Total of $\mathbf{46,879}$ variables.

There is one budget constraint, one hydroelectric constraint, $\mathbf{4} \times \mathbf{5} \times \mathbf{15} \times \mathbf{125}$ capacity constraints and $\mathbf{5} \times \mathbf{15} \times \mathbf{125}$ demand constraints.

Total is $\mathbf{46,877}$ constraints.

Solution

Plant	Optimal Construction Decision based on stochastic data	Optimal Construction Decision based on expected demand and costs
Gas Turbine	4.66 GW	1.92 GW
Coal	4.57 GW	3.33 GW
Nuclear	4.68 GW	4.0 GW
Hydroelectric	5.0 GW	5.0 GW
Expected Cost	$16.933 billion	$17.794 billion

Power plant capacity construction decisions.

Increase in expected cost would be 5.1% under decisions obtained from expected data.

Solution

Operating Decisions under Optimal Investment

Plant	Demand Block				
	#1	#2	#3	#4	#5
Gas Turbine	4.66	4.66	3.02	0.00	0.00
Coal	4.57	4.57	4.57	4.24	1.40
Nuclear	4.68	4.68	4.68	4.68	4.68
Hydroelectric	5.00	5.00	5.00	5.00	5.00
External Source	6.87	2.74	0.00	0.00	0.00
Cost (million $)	689.6	575.2	685.4	610.5	249.0

GW of power during 15th year, given the optimal capacity construction decisions, assuming a 7% growth rate, 3.9¢ gas turbine and 2.4¢ coal operating cost.

The total operating cost for this year and scenario is $2.81 billion for the optimal capacity allocation.

Solution

Operating Decisions under Average-Case Investment

Plant	Demand Block #1	#2	#3	#4	#5
Gas Turbine	1.92	1.92	1.92	1.59	0.00
Coal	3.33	3.33	3.33	3.33	2.08
Nuclear	4.00	4.00	4.00	4.00	4.00
Hydroelectric	5.00	5.00	5.00	5.00	5.00
External Source	11.53	7.40	3.02	0.00	0.00
Cost (million $)	960.5	978.8	1497.5	710.5	263.2

GW of power during 15th year, given the capacity construction decisions based on expected demand and costs, assuming a 7% growth rate, 3.9¢ gas turbine and 2.4¢ coal operating cost.

The total operating cost for this year and scenario is $4.41 billion with the sub-optimal capacity construction decisions.

References

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