14 TRANSFER FUNCTIONS & STABILITY

The reader is referred to *Laplace Transforms* in the section *MATH FACTS* for preliminary material on the Laplace transform. Partial fractions are presented here, in the context of control systems, as the fundamental link between pole locations and stability.

14.1 Partial Fractions

Solving linear time-invariant systems by the Laplace Transform method will generally create a signal containing the (factored) form

$$Y(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}. \tag{166}$$

Although for the moment we are discussing the signal $Y(s)$, later we will see that dynamic systems are described in the same format: in that case we call the impulse response $G(s)$ a transfer function. A system transfer function is identical to its impulse response, since $L(\delta(t)) = 1$.

The constants $-z_i$ are called the zeros of the transfer function or signal, and $-p_i$ are the poles.

Viewed in the complex plane, it is clear that the magnitude of $Y(s)$ will go to zero at the zeros, and to infinity at the poles.

Partial fraction expansions alter the form of $Y(s)$ so that the simple transform pairs can be used to find the time-domain output signals. We must have $m < n$; if this is not the case, then we have to divide the numerator by the denominator as necessary to find a simple form.

14.2 Partial Fractions: Unique Poles

Under the condition $m < n$, it is a fact that $Y(s)$ is equivalent to

$$Y(s) = \frac{a_1}{s+p_1} + \frac{a_2}{s+p_2} + \cdots \frac{a_n}{s+p_n}, \tag{167}$$

in the special case that all of the poles are unique and real. The coefficient a_i is termed the *residual* associated with the i'th pole, and once all these are found it is a simple matter to go back to the transform table and look up the time-domain responses.

How to find a_i? A simple rule applies: multiply the right-hand sides of the two equations above by $(s + p_i)$, evaluate them at $s = -p_i$, and solve for a_i, the only one left.

14.3 Example: Partial Fractions with Unique Real Poles

$$G(s) = \frac{s(s+6)}{(s+4)(s-1)}e^{-2s}.$$

Since we have a pure delay and $m = n$, we can initially work with $G(s)/se^{-2s}$. We have

$$\frac{s+6}{(s+4)(s-1)} = \frac{a_1}{s+4} + \frac{a_2}{s-1}, \text{ giving}$$

$$a_1 = \left[\frac{(s+6)(s+4)}{(s+4)(s-1)}\right]_{s=-4} = -\frac{2}{5}$$

$$a_2 = \left[\frac{(s+6)(s-1)}{(s+4)(s-1)}\right]_{s=1} = \frac{7}{5}$$

Thus

$$L^{-1}(G(s)/se^{-2s}) = -\frac{2}{5}e^{-4s} + \frac{7}{5}e^t \longrightarrow$$

$$g(t) = \delta(t-2) + \frac{8}{5}e^{-4(t-2)} + \frac{7}{5}e^{t-2}.$$

The impulse response is needed to account for the step change at $t = 2$. Note that in this example, we were able to apply the derivative operator s *after* expanding the partial fractions. For cases where a second derivative must be taken, i.e., $m \geq n+1$, special care should be used when accounting for the signal *slope* discontinuity at $t = 0$. The more traditional method, exemplified by Ogata, may prove easier to work through.

The case of repeated real roots may be handled elegantly, but this condition rarely occurs in applications.

14.4 Partial Fractions: Complex-Conjugate Poles

A complex-conjugate pair of poles should be kept together, with the following procedure: employ the form

$$Y(s) = \frac{b_1 s + b_2}{(s+p_1)(s+p_2)} + \frac{a_3}{s+p_3} + \cdots, \tag{168}$$

where $p_1 = p_2^*$ (complex conjugate). As before, multiply through by $(s+p_1)(s+p_2)$, and then evaluate at $s = -p_1$.

14.5 Example: Partial Fractions with Complex Poles

$$G(s) = \frac{s+1}{s(s+j)(s-j)} = \frac{b_1 s + b_2}{(s+j)(s-j)} + \frac{a_3}{s} :$$
$$\left[\frac{s+1}{s}\right]_{s=-j} = [b_1 s + b_2]_{s=-j} \longrightarrow$$
$$1 + j = -b_1 j + b_2 \longrightarrow$$
$$b_1 = -1$$
$$b_2 = 1;\ \text{also}$$
$$\left[\frac{s+1}{(s+j)(s-j)}\right]_{s=0} = a_3 = 1.$$

Working out the inverse transforms from the table of pairs, we have simply (noting that $\zeta = 0$)
$$g(t) = -\cos t + \sin t + 1(t).$$

14.6 Stability in Linear Systems

In linear systems, *exponential stability* occurs when all the real exponents of e are strictly negative. The signals decay within an exponential envelope. If one exponent is 0, the response never decays or grows in amplitude; this is called *marginal stability.* If at least one real exponent is positive, then one element of the response grows without bound, and the system is *unstable.*

14.7 Stability $\Longleftrightarrow$ Poles in LHP

In the context of partial fraction expansions, the relationship between stability and pole locations is especially clear. The unit step function $1(t)$ has a pole at zero, the exponential e^{-at} has a pole at $-a$, and so on. All of the other pairs exhibit the same property: *A system is stable if and only if all of the poles occur in the left half of the complex plane.* Marginally stable parts correlate with a zero real part, and unstable parts to a positive real part.

14.8 General Stability

There are two definitions, which apply to systems with input $\vec{u}(t)$ and output $\vec{y}(t)$.

1. **Exponential**. If $\vec{u}(t) = \vec{0}$ and $\vec{y}(0) = \vec{y}_o$, then $|y(t)| < \alpha e^{-\gamma t}$, for finite α and $\gamma > 0$. The output asymptotically approaches zero, within a decaying exponential envelope.
2. **Bounded-Input Bounded-Output (BIBO)**. If $\vec{y}(0) = \vec{0}$, and $|f_i(t)| < \gamma, \gamma > 0$ and finite, then $|y(t)| < \alpha, \alpha > 0$ and finite.

In linear time-invariant systems, the two definitions are identical. Exponential stability is easy to check for linear systems, but for nonlinear systems, BIBO stability is usually easier to achieve.