

20 LOOP TRANSFER RECOVERY

20.1 Introduction

The Linear Quadratic Regulator (LQR) and Kalman Filter (KF) provide practical solutions to the full-state feedback and state estimation problems, respectively. If the sensor noise and disturbance properties of the plant are indeed well-known, then an LQG design approach, that is, combining the LQR and KF into an output feedback compensator, may yield good results. The LQR tuning matrices Q and R would be picked heuristically to give a reasonable closed-loop response.

There are two reasons to avoid this kind of direct LQG design procedure, however. First, although the LQR and KF each possess good robustness properties, there do exist plants for which there is *no* robustness guarantee for an LQG compensator. Even if one could steer clear of such pathological cases, a second problem is that this design technique has no clear equivalent in frequency space. It cannot be directly mapped to the intuitive ideas of loopshaping and the Nyquist plot, which are at the root of feedback control.

We now reconsider just the feedback loop of the Kalman filter. The KF has open-loop transfer function $L(s) = C\phi(s)H$, where $\phi(s) = (sI - A)^{-1}$. This follows from the estimator evolution equation

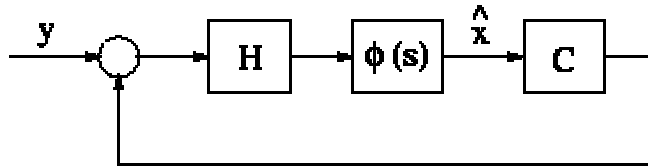
$$\dot{\hat{x}} = A\hat{x} + Bu + H(y - C\hat{x})$$

and the figure. Note that we have not included the factor Bu as part of the figure, since it does not affect the error dynamics of the filter.

As noted previously, the KF loop has good robustness properties, specifically to perturbations at the output \hat{y} , and further is amenable to output tracking. In short, the KF loop is an ideal candidate for a loopshaping design. Supposing that we have an estimator gain H which creates an attractive loop function $L(s)$, we would like to find the compensator $C(s)$ that establishes

$$\begin{aligned} P(s)C(s) &\approx C\phi(s)H, \text{ or} \\ C\phi(s)BC(s) &\approx C\phi(s)H. \end{aligned} \quad (215)$$

It will turn out that the LQR can be set up so that the an LQG-type compensator achieves exactly this result. The procedure is termed Loop Transfer Recovery (LTR), and has two main parts. First, one carries out a KF design for H , so that the Kalman filter loop itself has good performance and robustness properties. In this regard, the KF loop has sensitivity function $S(s) = (I + C\phi(s)H)^{-1}$ and complementary sensitivity $T(s) = (I + C\phi(s)H)^{-1}C\phi(s)H$. The condition $\bar{\sigma}(W_1(s)S(s)) + \bar{\sigma}(W_2(s)T(s)) < 1$ is sufficient for robust performance with multiplicative plant uncertainty at the output. Secondly, we pick suitable parameters of the LQR design, so that the LQG compensator satisfies the approximation of Equation 215.



LTR is useful as a SISO control technique, but has a much larger role in multivariable control.

20.2 A Special Property of the LQR Solution

Letting $Q = C^T C$ and $R = \rho I$, where I is the identity matrix, we will show (roughly) that

$$\lim_{\rho \rightarrow 0} (\sqrt{\rho} K) = WC,$$

where K is the LQR gain matrix, and W is an orthonormal matrix, for which $W^T W = I$.

First recall the gain and Riccati equations for the LQR:

$$K = R^{-1} B^T P$$

$$0 = Q + PA + A^T P - PBR^{-1} B^T P.$$

Now $Q = C^T C = C^T W^T W C = (WC)^T W C$. The Riccati equation becomes

$$0 = \rho (WC)^T W C + \rho P A + \rho A^T P - P B B^T P = 0.$$

In the limit as $\rho \rightarrow 0$, it must be the case that $P \rightarrow 0$ also, and so in this limit

$$\begin{aligned}
\rho(WC)^T WC &\approx PBB^T P \\
&= (B^T P)^T B^T P \\
&= (R^{-1} B^T P)^T R R (R^{-1} B^T P) \\
&= \rho^2 K^T K \rightarrow \\
WC &\approx \sqrt{\rho} K. \quad \blacksquare
\end{aligned}$$

Note that another orthonormal matrix W' could be used in separating K^T from K in the last line. This matrix may be absorbed into W through a matrix inverse, however, and so does not need to be written. The result of the last line establishes that the plant must be square: the number of inputs (i.e., rows of K) is equal to the number of outputs (i.e., rows of C).

Finally, we note that the above property is true only for LQR designs with minimum-phase plants, i.e., those with only stable zeros (Kwakernaak and Sivan).

20.3 The Loop Transfer Recovery Result

The theorem is stated as: If $\lim_{\rho \rightarrow 0} (\sqrt{\rho} K) = WC$ (the above result), with W an orthonormal matrix, then the limiting LQG controller $C(s)$ satisfies

$$\lim_{\rho \rightarrow 0} P(s)C(s) = C\phi(s)H.$$

The LTR method is limited by two conditions:

The plant has an equal number of inputs and outputs.

The design plant has no unstable zeros. The LTR method can be in fact be applied in the presence of unstable plant zeros, but the recovery is not to the Kalman filter loop transfer function. Instead, the recovered function will exhibit reasonable limitations inherent to unstable zeros. See Athans for more details and references on this topic.

The proof of the LTR result depends on some easy lemmas, given at the end of this section. First, we develop $C(s)$, with the definitions $\phi(s) = (sI - A)^{-1}$ and $X(s) = (\phi^{-1}(s) + HC)^{-1} = (sI - A + HC)^{-1}$

$$\begin{aligned}
C(s) &= K(sI - A + BK + HC)^{-1}H \\
&= K(X^{-1}(s) + BK)^{-1}H, \text{ then use Lemma 2 } \rightarrow \\
&= K(X(s) - X(s)B(I + KX(s)B)^{-1}KX(s))H \\
&= KX(s)H - KX(s)B(I + KX(s)B)^{-1}KX(s)H \\
&= (I - KX(s)B(I + KX(s)B)^{-1})KX(s)H, \text{ then use Lemma 3 } \rightarrow \\
&= (I + KX(s)B)^{-1}KX(s)H \\
&= (\sqrt{\rho}I + \sqrt{\rho}KX(s)B)^{-1}\sqrt{\rho}KX(s)H.
\end{aligned}$$

Next we invoke the result from the LQR design, with $\rho \rightarrow 0$, to eliminate $\sqrt{\rho}K$:

$$\begin{aligned}
\lim_{\rho \rightarrow 0} C(s) &= (WCX(s)B)^{-1}WCX(s)H \\
&= (CX(s)B)^{-1}CX(s)H.
\end{aligned}$$

In the last expression, we used the assumption that W is square and invertible, both properties of orthonormal matrices. Now we look at the product $CX(s)$.

$$\begin{aligned}
CX(s) &= C(SI - A + HC)^{-1} \\
&= C(\phi^{-1}(s) + HC)^{-1}, \text{ then use Lemma 2 } \rightarrow \\
&= C(\phi(s) - \phi(s)H(I + C\phi(s)H)^{-1}C\phi(s)) \\
&= (I - C\phi(s)H(I + C\phi(s)H)^{-1})C\phi(s), \text{ then use Lemma 3 } \rightarrow \\
&= (I + C\phi(s)H)^{-1}C\phi(s).
\end{aligned}$$

This result, reintroduced into the limiting compensator, gives

$$\begin{aligned}\lim_{\rho \rightarrow 0} C(s) &= ((I + C\phi(s)H)^{-1}C\phi(s)B)^{-1}(I + C\phi(s)H)^{-1}C\phi(s)H \\ &= (C\phi(s)B)^{-1}C\phi(s)H \\ &= P^{-1}(s)C\phi(s)H.\end{aligned}$$

Finally it follows that $\lim_{\rho \rightarrow 0} P(s)C(s) = C\phi(s)H$, as desired.

20.4 Usage of the Loop Transfer Recovery

The idea of LTR is to "recover" a Kalman filter loop transfer function $L(s) = C\phi(s)H$, by using the limiting cheap-control LQR design, with $Q = \rho^2 C$ and $R = \rho I$. The LQR design step is thus trivial.

Some specific techniques are useful.

Scale the plant outputs (and references), so that one unit of error in one channel is as undesirable as one unit of error in another channel. For example, in depth and pitch control of a large submarine, one meter of depth error cannot be compared directly with one radian of pitch error.

Scale the plant inputs in the same way. One Newton of propeller thrust cannot be compared with one radian of rudder angle.

Design for crossover frequency. The bandwidth of the controller is roughly equal to the frequency at which the (recovered) loop transfer function crosses over $0dB$. Often, the bandwidth of is a more intuitive design parameter than is, for example, the high-frequency multiplicative weighting W_2 . Quantitative uncertainty models are usually at the cost of a lengthy identification effort.

Integrators should be part of the KF loop transfer function, if no steady-state error is to be allowed. Since the Kalman filter loop has only as many poles as the plant, the plant input channels must be augmented with the necessary additional poles (at the origin). Then, once the KF design is completed, and the compensator $C(s)$ is constructed, the integrators are moved from the plant over to the input side of the compensator. The tracking errors will accrue as desired.

20.5 Three Lemmas

Lemma 1: Matrix Inversion

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}.$$

Proof:

$$\begin{aligned}(A + BCD)(A + BCD)^{-1} &= I \\ A(A + BCD)^{-1} &= I - BCD(A + BCD)^{-1} \\ (A + BCD)^{-1} &= A^{-1} - A^{-1}BCD(A + BCD)^{-1} \\ &= A^{-1} - A^{-1}BCD(I + A^{-1}BCD)^{-1}A^{-1} \\ &= A^{-1} - A^{-1}B(D^{-1}C^{-1} + A^{-1}B)^{-1}A^{-1} \\ &= A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}. \quad \blacksquare\end{aligned}$$

Lemma 2: Short Form of Lemma 1

$$(X^{-1} + BD)^{-1} = X - XB(I + DXB)^{-1}DX$$

Proof: substitute $A = X^{-1}$ and $C = I$ into Lemma 1.

Lemma 3

$$I - A(I + A)^{-1} = (I + A)^{-1}$$

Proof:

$$\begin{aligned}I - A(I + A)^{-1} &= (I + A)(I + A)^{-1} - A(I + A)^{-1} \\ &= (I + A - A)(I + A)^{-1} \\ &= (I + A)^{-1}. \quad \blacksquare\end{aligned}$$