16 MODAL ANALYSIS

16.1 Introduction

The evolution of states in a linear system occurs through independent modes, which can be driven by external inputs, and observed through plant output. This section provides the basis for modal analysis of systems. Throughout, we use the state-space description of a system with D= 0:

$$\begin{aligned} \dot{\vec{x}} &= A\vec{x} + B\vec{u} \\ \vec{y} &= C\vec{x}. \end{aligned}$$

16.2 Matrix Exponential

16.2.1 Definition

In the instance of an unforced response to initial conditions, consider the system

$$\dot{\vec{x}} = A\vec{x}, \quad \vec{x}(t=0) = \vec{\chi}.$$

In the scalar case, the response is $x(t) = \chi e^{at}$, giving a decaying exponential if $a < 0$. The same notation holds for the case of a vector $\vec{x}$, and matrix A:

$$\begin{aligned} \vec{x}(t) &= e^{At}\vec{\chi}, \text{ where} \\ e^{At} &= I + At + \frac{(At)^2}{2!} + \cdots \end{aligned}$$

$e^{At}$is usually called the matrix exponential.

16.2.2 Modal Canonical Form

Introductory material on the eigenvalue problem and modal decomposition can be found in the *MATH FACTS* section. This modal decomposition of Aleads to a very useful state-space representation. Namely, since $A = V\Lambda V^{-1}$, a transformation of state variables can be made, $\vec{x} = V\vec{z}$, leading to

$$\begin{aligned} \dot{\vec{z}} &= \Lambda\vec{z} + V^{-1}B\vec{u} \qquad (173) \\ \vec{y} &= CV\vec{z}. \end{aligned}$$

This is called the modal canonical form, since the states are simply the modal amplitudes. These states are uncoupled in Λ, but may be coupled through the input ($V^{-1}B$) and output (CV) mappings. The modal form is numerically robust for computations.

16.2.3 Modal Decomposition of Response

Now we are ready to look at the matrix exponential e^{At}in terms of its constituent modes. Employing the above form for A, we find that

$$\begin{aligned} e^{At} &= I + At + \frac{(At)^2}{2!} + \cdots \\ &= V\left(I + \Lambda t + \frac{(\Lambda t)^2}{2!} + \cdots\right)W^T \\ &= Ve^{\Lambda t}W^T \\ &= \sum_{i=1}^{n} e^{\lambda_i t}\vec{v}_i\vec{w}_i^T. \end{aligned}$$

In terms of the response to an initial condition $\vec{\chi}$, we have

$$\vec{x}(t) = \sum_{i=1}^{n} e^{\lambda_i t}\vec{v}_i(\vec{w}_i^T\vec{\chi}).$$

The product $\vec{w}_i^T\vec{\chi}$is a scalar, the projection of the initial conditions onto the i'th mode. If $\vec{\chi}$is perpendicular to $\vec{w}_i$, then the product is zero and the i'th mode does not respond. Otherwise, the i 'th mode does participate in the response. The projection of the i'th mode onto the *states* $\vec{x}$is through the right eigenvector $\vec{v}_i$.

For stability of the system, the eigenvalues of A, that is, λ_i, must have negative real parts; they are in fact the poles of the equivalent transfer function description.

16.3 Forced Response and Controllability

Now consider the system with an external input $\vec{u}$:

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}, \quad \vec{x}(t=0) = \vec{\chi}.$$

Taking the Laplace transform of the system, taking into account the initial condition for the derivative, we have

$$\begin{aligned} s\vec{x}(s) - \vec{\chi} &= A\vec{x}(s) + B\vec{u}(s) \longrightarrow \\ \vec{x}(s) &= (sI - A)^{-1}\vec{\chi} + (sI - A)^{-1}B\vec{u}(s). \end{aligned}$$

Thus $(sI - A)^{-1}$ can be recognized as the Laplace transform of the matrix exponential e^{At}. In the time domain, the second term then has the form of a convolution of the matrix exponential and the net input $B\vec{u}$:

$$\begin{aligned} \vec{x}(t) &= \int_0^t e^{A(t-\tau)} B\vec{u}(\tau)d\tau \\ &= \sum_{i=1}^n \int_0^t e^{\lambda_i(t-\tau)} \vec{v}_i \vec{w}_i^T B\vec{u}(\tau)d\tau. \end{aligned}$$

Suppose now that there are m inputs, such that $B = [\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_m]$. Then some rearrangement will give

$$\vec{x}(t) = \sum_{i=1}^n \vec{v}_i \sum_{k=1}^m (\vec{w}_i^T \vec{b}_k) \int_0^t e^{\lambda_i(t-\tau)} \vec{u}_k(\tau)d\tau.$$

The product $\vec{w}_i^T \vec{b}_k$, a scalar, represents the projection of the k'th control channel onto the i'th mode. We say that the i'th mode is controllable from the k'th input if the product is nonzero. If a given mode i has $\vec{w}_i^T \vec{b}_k = 0$ for all input channels k, then the mode is uncontrollable.

In normal applications, controllability for the entire system is checked using the following test: Construct the so-called controllability matrix:

$$M_c = [B, AB, \cdots, A^{n-1}B]. \qquad (174)$$

This matrix has size $n \times (nm)$, where m is the number of input channels. If M_c has rank n, then the system is controllable, i.e., all modes are controllable.

16.4 Plant Output and Observability

We now turn to a related question: can the complete state vector of the system be observed given only the output measurements $\vec{y}$, and the known control $\vec{u}$? The response due to the external input is easy to compute deterministically, through the convolution integral. Consider the part due to initial conditions $\vec{\chi}$. We found above

$$\vec{x}(t) = \sum_{i=1}^n e^{\lambda_i t} \vec{v}_i \vec{w}_i^T \vec{\chi}.$$

The observation is $\vec{y} = C\vec{x}$ (r channels of output), and writing

$$C = \begin{bmatrix} \vec{c}_1^T \\ \cdot \\ \vec{c}_r^T \end{bmatrix}.$$

the k'th channel of the output is

$$y_k(t) = \sum_{i=1}^n (\vec{c}_k^T \vec{v}_i) e^{\lambda_i t} (\vec{w}_i^T \vec{\chi}).$$

The i'th mode is observable in the k'th output if the product $\vec{c}_k^T \vec{v}_i \neq 0$. We say that a system is observable if every mode can be seen in at least one output channel. The usual test for system observability requires computation of the observability matrix:

$$M_o = \left[C^T, A^T C^T, \cdots, (A^T)^{n-1} C^T\right]. \qquad (175)$$

This matrix has size $n \times (rn)$; the system is observable if M_o has rank n.