9 SLENDER-BODY THEORY

9.1 Introduction

Consider a slender body with $d << L$, that is mostly straight. The body could be asymmetric in cross-section, or even flexible, but we require that the lateral variations are small and smooth along the length. The idea of the slender-body theory, under these assumptions, is to think of the body as a longitudinal stack of thin sections, each having an easily-computed added mass. The effects are integrated along the length to approximate lift force and moment. Slender-body theory is accurate for small ratios d/L, except near the ends of the body.

As one example, if the diameter of a body of revolution is $d(s)$, then we can compute $\delta m_a(x)$, where the nominal added mass value for a cylinder is

$$\delta m_a = \rho \frac{\pi}{4} d^2 \delta x. \qquad (109)$$

The added mass is equal to the mass of the water displaced by the cylinder. The equation above turns out to be a good approximation for a number of two-dimensional shapes, including flat plates and ellipses, if d is taken as the width dimension presented to the flow. Many formulas for added mass of two-dimensional sections, as well as for simple three-dimensional bodies, can be found in the books by Newman and Blevins.

9.2 Kinematics Following the Fluid

The added mass forces and moments derive from accelerations that a fluid particles experience when they encounter the body. We use the notion of a fluid derivative for this purpose: the operator d/dt indicates a derivative taken in the frame of the passing particle, not the vehicle. Hence, this usage has an indirect connection with the derivative described in our previous discussion of rigid-body dynamics.

For the purposes of explaining the theory, we will consider the two-dimensional heave/surge problem only. The local geometry is described by the location of the centerline; it has vertical location (in body coordinates) of $z_b(x,t)$, and local angle $\alpha(x,t)$. The time-dependence indicates that the configuration is free to change with time, i.e., the body is flexible. Note that the curvilinear coordinate s is nearly equal to the body-reference (linear) coordinate x.

The velocity of a fluid particle *normal* to the body at x is $w_n(t,x)$:

$$w_n = \frac{\partial z_b}{\partial t} \cos\alpha - U \sin\alpha. \qquad (110)$$

The first component is the time derivative in the body frame, and the second due to the deflection of the particle by the inclined body. If the body reference frame is rotated to the flow, that is, if $w \neq 0$, then $\partial z_b/\partial t$ will contain w. For small angles, $\sin\alpha \simeq \tan\alpha = \partial z_b/\partial x$, and we can write

$$w_n \simeq \frac{\partial z_b}{\partial t} - U \frac{\partial z_b}{\partial x}.$$

The fluid derivative operator in action is as follows:

$$w_n = \frac{dz_b}{dt} = \left(\frac{\partial}{\partial t} - U \frac{\partial}{\partial x} \right) z_b.$$

9.3 Derivative Following the Fluid

A more formal derivation for the fluid derivative operator is quite simple. Let $\mu(x,t)$ represent some property of a fluid particle.

$$\begin{aligned} \frac{d}{dt}[\mu(x,t)] &= \lim_{\delta t \to 0} \frac{1}{\delta t} [\mu(x+\delta x, t+\delta t) - \mu(x,t)] \\ &= \left[\frac{\partial \mu}{\partial t} - U \frac{\partial \mu}{\partial x} \right]. \end{aligned}$$

The second equality can be verified using a Taylor series expansion of $\mu(x+\delta x, t+\delta t)$:

$$\mu(x+\delta x, t+\delta t) = \mu(x,t) + \frac{\partial \mu}{\partial t}\delta t + \frac{\partial \mu}{\partial x}\delta x + h.o.t.,$$

and noting that $\delta x = -U\delta t$. The fluid is convected downstream with respect to the body.

9.4 Differential Force on the Body

If the local transverse velocity is $w_n(x,t)$, then the differential inertial force on the body here is the derivative (following the fluid) of the momentum:

$$\delta F = -\frac{d}{dt}\left[m_a(x,t) w_n(x,t)\right] \delta x. \quad (111)$$

Note that we could here let the added mass vary with time also - this is the case of a changing cross-section! The lateral velocity of the point $z_b(x)$ in the body-reference frame is

$$\frac{\partial z_b}{\partial t} = w - xq, \quad (112)$$

such that

$$w_n = w - xq - U\alpha. \quad (113)$$

Taking the derivative, we have

$$\frac{\delta F}{\delta x} = -\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)\left[m_a(x,t)(w(t) - xq(t) - U\alpha(x,t))\right].$$

We now restrict ourselves to a rigid body, so that neither m_a nor α may change with time.

$$\frac{\delta F}{\delta x} = m_a(x)(-\dot{w} + x\dot{q}) + U\frac{\partial}{\partial x}\left[m_a(x)(w - xq - U\alpha)\right]. \quad (114)$$

9.5 Total Force on a Vessel

The net lift force on the body, computed with strip theory is

$$Z = \int_{x_T}^{x_N} \delta F dx \quad (115)$$

where x_T represents the coordinate of the tail, and x_N is the coordinate of the nose. Expanding, we have

$$\begin{aligned} Z &= \int_{x_T}^{x_N} m_a(x)\left[-\dot{w} + x\dot{q}\right] dx + U \int_{x_T}^{x_N} \frac{\partial}{\partial x}\left[m_a(x)(w - xq - U\alpha)\right] dx \\ &= -m_{33}\dot{w} - m_{35}\dot{q} + U m_a(x)(w - xq - U\alpha)\Big|_{x=x_T}^{x=x_N}. \end{aligned}$$

We made use here of the added mass definitions

$$\begin{aligned} m_{33} &= \int_{x_T}^{x_N} m_a(x) dx \\ m_{35} &= -\int_{x_T}^{x_N} x m_a(x) dx. \end{aligned}$$

Additionally, for vessels with pointed noses and flat tails, the added mass m_a at the nose is zero, so that a simpler form occurs:

$$Z = -m_{33}\dot{w} - m_{35}\dot{q} - U m_a(x_T)(w - x_T q - U\alpha(x_T)). \quad (116)$$

In terms of the linear hydrodynamic derivatives, the strip theory thus provides

$$\begin{aligned} Z_{\dot{w}} &= -m_{33} \\ Z_{\dot{q}} &= -m_{35} \\ Z_w &= -U m_a(x_T) \\ Z_q &= U x_T m_a(x_T) \\ Z_{\alpha(x_T)} &= U^2 m_a(x_T). \end{aligned}$$

It is interesting to note that both Z_w and $Z_{\alpha(x_T)}$ depend on a nonzero base area. In general, however, potential flow estimates do not create lift (or drag) forces for a smooth body, so this should come as no surprise. The two terms are clearly related, since their difference depends only on how the body coordinate system is oriented to the flow. Another noteworthy fact is that the lift force depends only on α at the tail; α could take any value(s) along the body, with no effect on Z.

9.6 Total Moment on a Vessel

A similar procedure can be applied to the moment predictions from slender body theory (again for small α):

$$
\begin{aligned}
M &= -\int_{x_T}^{x_N} x \delta F dx \\
&= \int_{x_T}^{x_N} x \left(\frac{\partial}{\partial t} - U \frac{\partial}{\partial x} \right) \left[m_a(x)(w - xq - U\alpha) \right] dx \\
&= \int_{x_T}^{x_N} x m_a(x)(\dot{w} - x\dot{q}) dx - U \int_{x_T}^{x_N} x \frac{\partial}{\partial x} \left[m_a(x)(w - xq + U\alpha) \right] dx.
\end{aligned}
$$

Then we make the further definition

$$
m_{55} = \int_{x_T}^{x_N} x^2 m_a(x) dx,
$$

(note that $m_{35} = m_{53}$) and use integration by parts to obtain

$$
\begin{aligned}
M = & -m_{35}\dot{w} - m_{55}\dot{q} - U x m_a(x)(w - xq - U\alpha)\Big|_{x=x_T}^{x=x_N} + \\
& U \int_{x_T}^{x_N} m_a(x)(w - xq - U\alpha) dx.
\end{aligned}
$$

The integral above contains the product $m_a(x)\alpha(x)$, which must be calculated if α changes along the length. For simplicity, we now assume that α is in fact constant on the length, leading to

$$
\begin{aligned}
M = & -m_{35}\dot{w} - m_{55}\dot{q} + U x_T m_a(x_T)(w - x_T q - U\alpha) + \\
& U m_{33} w + U m_{35} q - U^2 m_{33} \alpha.
\end{aligned}
$$

Finally, the linear hydrodynamic moment derivatives are

$$
\begin{aligned}
M_{\dot{w}} &= -m_{35} \\
M_{\dot{q}} &= -m_{55} \\
M_w &= U x_T m_a(x_T) + U m_{33} \\
M_q &= -U x_T^2 m_a(x_T) + U m_{35} \\
M_\alpha &= -U^2 x_T m_a(x_T) - U^2 m_{33}.
\end{aligned}
$$

The derivative M_w is closely-related to the Munk moment, whose linearization would provide $M_w = (m_{33} - m_{11})U$. The Munk moment (an exact result) may therefore be used to make a correction to the second term in the slender-body approximation above of M_w. As with the lift force, M_w and M_α are closely related, depending only on the orientation of the body frame to the flow.

9.7 Relation to Wing Lift

There is an important connection between the slender body theory terms involving added mass at the tail ($m_a(x_T)$), and low aspect-ratio wing theory. The lift force from the latter is of the form $L = -\frac{1}{2}\rho U A C_l' w$, where $A = cs$, the product of chord (long) and span (short). The lift coefficient slope is approximated by (Hoerner)

$$
C_l' \approx \frac{1}{2}\pi(AR), \tag{117}
$$

where (AR) is the aspect ratio. Inserting this approximation into the lift formula, we obtain

$$
L = -\frac{\pi}{4}\rho s^2 U w. \tag{118}
$$

Now we look at a slender body approximation of the same force: The added mass at the tail is $m_a(x_T) = \rho s^2 \pi / 4$, and using the slender-body estimate for Z_w, we calculate for lift:

$$
\begin{aligned}
Z &= -m_a(x_T) U w \\
&= -\frac{\pi}{4}\rho s^2 U w.
\end{aligned}
$$

Slender-body theory is thus able to recover exactly the lift of a low-aspect ratio wing. Where does the slender-body predict the force will act? Recalling that $M_w = U m_{33} + U x_T m_a(x_T)$, and since $m_{33} = 0$ for a front-back symmetric wing, the estimated lift force acts at the trailing edge. This location will tend to stabilize the wing, in the sense that it acts to orient the wing parallel to the incoming flow.

9.8 Convention: Hydrodynamic Mass Matrix A

Hydrodynamic derivatives that depend on accelerations are often written as components of a mass matrix A. By listing the body-referenced velocities in the order $\vec{s} = [u, v, w, p, q, r]$, we write

$(M + A)\dot{\vec{s}} = \vec{F}$, where M is the mass matrix of the *material* vessel and F is a generalized force. Therefore $A_{33} = -Z_{\dot{w}}$, $A_{5,3} = -M_{\dot{w}}$, and so on.