10 PRACTICAL LIFT CALCULATIONS

10.1 Characteristics of Lift-Producing Mechanisms

At a small angle of attack, a slender body experiences transverse force due to: helical body vortices, the blunt trailing end, and fins. The helical body vortices are stable and symmetric in this condition, and are convected continuously into the wake. The low pressure associated with the vortices provides the suction force, usually toward the stern of the vehicle. The blunt trailing end induces lift as a product of added mass effects, and can be accurately modeled with slender body theory. A blunt trailing edge also induces some drag, which itself is stabilizing. The fins can often be properly modeled using experimental data parametrized with aspect ratio and several other geometric quantities.

As the angle of attack becomes larger, the approximations in the fin and slender-body analysis will break down. The helical vortices can become bigger while remaining stable, but eventually will split randomly. Some of it convects downstream, and the rest peels away from the body; this shedding is nonsymmetric, and greatly increases drag by widening the wake. In the limit of a 90^o angle of attack, vorticity sheds as if from a bluff-body, and there is little axial convection.

10.2 Jorgensen's Formulas

There are some heuristic and theoretical formulas for predicting transverse force and moment on a body at various angles of attack, and we now present one of them due to Jorgensen. The formulas provide a good systematic procedure for design, and are best suited to vehicles with a blunt trailing edge. We call the area of the stern the *base area.*

Let the body have length L, and reference area A_r. This area could be the frontal projected area, the planform area, or the wetted area. The body travels at speed U, and angle of attack α. The normal force, axial force, and moment coefficients are defined as follows:

$$\begin{aligned} C_N &= \frac{F_N}{\frac{1}{2}\rho U^2 A_r} \qquad (119) \\ C_A &= \frac{F_A}{\frac{1}{2}\rho U^2 A_r} \\ C_M &= \frac{M_{x_m}}{\frac{1}{2}\rho U^2 A_r L}. \end{aligned}$$

The moment M_{x_m} is taken about a point x_m, measured back from the nose; this location is arbitrary, and appears explicitly in the formula for C_m. Jorgensen gives the coefficients as follows:

$$\begin{aligned} C_N &= \frac{A_b}{A_r}\sin 2\alpha\cos\frac{\alpha}{2} + \frac{A_p}{A_r}C_{d_n}\sin^2\alpha \qquad (120) \\ C_A &= C_{A_o}\cos^2\alpha \qquad (121) \\ C_M &= -\frac{\nabla - A_b(L-x_m)}{A_r L}\sin 2\alpha\cos\frac{\alpha}{2} - C_{d_n}\frac{A_p}{A_r}\left(\frac{x_m - x_c}{L}\right)\sin^2\alpha. \qquad (122) \end{aligned}$$

We have listed only the formulas for the special case of circular cross-section, although the complete equations do account for more complex shapes. Further, we have assumed that $L >> D$, which is also not a constraint in the complete equations. The parameters used here are

- A_b: stern base area. $A_b = 0$ for a body that tapers to a point at the stern.
- C_{d_n}: crossflow drag coefficient; equivalent to that of an infinite circular cylinder. If ``normal'' Reynolds number $Re_n = U\sin\alpha D/\nu$,
  - o $C_d \approx 1.2, Re_n < 3\times 10^5$
  - o $C_d \approx 0.3, 3\times 10^5 < Re_n < 7\times 10^5$
  - o $C_d \approx 0.6, 7\times 10^5 < Re_n$.
- A_p: planform area.
- C_{A_o}: axial drag at zero angle of attack, both frictional and form. $C_{A_o} \simeq 0.002 - 0.006$ for slender streamlined bodies, based on wetted surface area. It depends on $Re = UL/\nu$.
- ∇: body volume.
- x_c: distance from the nose backwards to the center of the planform area.

In the formula for normal force, we see that if $A_b = 0$, only drag forces act to create lift, through a $\sin^2\alpha$-dependence. Similarly, the axial force is simply the zero-α result, modified by $\cos^2\alpha$. In both cases, scaling of U^2into the body principle directions is all that is required.
There are several terms that match exactly the slender-body theory approximations for small α. These are the first term in the normal force (C_N), and the entire first term in the moment (C_M), whether or not $A_b = 0$. Finally, we note that the second term in $C_M$disappears if $x_m = x_c$, i.e., if the moment is referenced to the center of the planform area. The idea here is that the fore and aft components of crossflow drag cancel out.
The aerodynamic center (again referenced toward the stern, from the nose) can be found after the coefficients are computed:

$$x_{AC} = x_m + \frac{C_M}{C_N}L. \qquad (123)$$

As written, the moment coefficient is negative if the moment destabilizes the body, while $C_N$is always positive. Thus, the moment seeks to move the ACforward on the body, but the effect is moderated by the lift force.

10.3 Hoerner's Data: Notation

An excellent reference for experimental data is the two-volume set by S. Hoerner. It contains a large amount of aerodynamic data from many different types of vehicles, wings, and other common engineering shapes. A few notations are used throughout the books, and are described here.
First, dynamic pressure is given as $q = \frac{1}{2}\rho U^2$, such that two typical body lift coefficients are:

$$C_Y = \frac{Y}{DLq}$$

$$C_{Y_d} = \frac{Y}{D^2 q}.$$

The first version uses the *rectangular planform area* as a reference, while the second uses the *square frontal area*. Hence, $C_Y = C_{Y_d} D/L$. Two moment coefficients are:

$$C_N = \frac{N}{LD^2 q}$$

$$C_{N_1} = \frac{N_1}{LD^2 q},$$

where $N$is the moment taken about the body mid-point, and $N_1$is taken about the nose. Note that the reference area for moment is the *square* frontal projection, and the reference length is body length L. The following relation holds for these definitions

$$C_{N_1} = C_N + \frac{1}{2}C_{Y_d}.$$

The lift and moment coefficients are strongly dependent on angle of attack; Hoerner uses the notation

$$C_{nb} = \frac{\partial C_N}{\partial b}$$

$$C_{n\cdot b} = \frac{\partial C_{N_1}}{\partial b}$$

$$C_{yb} = \frac{\partial C_y}{\partial b},$$

and so on, where $b$is the angle of attack, usually in degrees. It follows from above that $C_{n\cdot b} = C_{nb} + C_{ydb}/2$.

10.4 Slender-Body Theory vs. Experiment

In an experiment, the net moment is measured, comprising both the destabilizing part due to the potential flow, and the stabilizing part due to vortex shedding or a blunt tail. Comparison of the measurements and the theory allows us to place the action point of the suction force. This section gives the formula for this location in Hoerner's notation, and gives two further examples of how well the slender-body theory matches experiments.

For $L/D > 6$, the slender-body (pure added mass) estimates give $\tilde{C}_{n\delta} \simeq -0.015/deg$, acting to destabilize the vehicle. The value compares well with $-0.027/deg$ for a long cylinder and $-0.018/deg$ for a long ellipsoid; it also reduces to -0.009 for $L/D = 4$. Note that the negative sign here is consistent with Hoerner's convention that destabilizing moments have negative sign.
The experimental lift force is typically given by $C_{Y\delta} \simeq 0.003/deg$; this acts at a point on the latter half of the vehicle, stabilizing the angle. Because this coefficient scales roughly with wetted area, proportional to LD, it changes little with L/D. It can be compared with a low-aspect ratio wing, which achieves an equivalent lift of $\pi(AR)/2 = 0.0027$ for $(AR) = 10 \simeq D/L$.
The point at which the viscous forces act can then be estimated as the following distance aft of the nose:

$$\frac{x}{L} = \frac{C_{n \cdot \delta} - \tilde{C}_{n\delta}}{C_{y d\delta}} \qquad (124)$$

The calculation uses experimental values of $C_{yd\delta}$ and $C_{n \cdot \delta}$, the moment slope referenced to the nose. In the table following are values from Hoerner (p. 13-2, Figure 2) for a symmetric and a blunt-ended body.

	symmetric	blunt
L/D	6.7	6.7
$\tilde{C}_{n\delta}$	-0.012	-0.012
$C_{y\delta}$	0.0031	0.0037
$C_{yd\delta}$	0.021	0.025
$C_{n \cdot \delta}$	0.0012 (stable)	0.0031 (stable)
$C_{n\delta}$	-0.0093 (unstable)	-0.0094 (unstable)
x/L	0.63	0.60

In comparing the two body shapes, we see that the moment at the nose is much more stable (positive) for the body with a blunt trailing edge. At the body midpoint, however, both vehicles are equally unstable. The blunt-tailed geometry has a much larger lift force, but it acts too close to the midpoint to add any stability there.
The lift force dependence on the blunt tail is not difficult to see, using slender-body theory. Consider a body, with trailing edge radius r. The slender-body lift force associated with this end is simply the product of speed U and local added mass $m_a(x_T)$ (in our previous notation). It comes out to be

$$Z = \frac{1}{2}\rho U^2(\pi r^2)(2\alpha), \qquad (125)$$

such that the first term in parentheses is an effective area, and the second is a lift coefficient. With respect to the area πr^2, the lift curve slope is therefore $2/rad$. Expressed in terms of the Hoerner reference area D^2, the equivalent lift coefficient is $C_{yd\delta} = 0.0044/deg$, where we made the assumption here that $2r/D = 0.4$ for the data in the table. This lift difference, due solely to the blunt end condition, is consistent with measurements.

10.5 Slender-Body Approximation for Fin Lift

Let us now consider two fins of span s each, acting at the tail end of the vehicle. This is the case if the vehicle body tapers to a point where the fins have their trailing edge. The slender-body approximation of lift as a result of blunt-end conditions is

$$Z = \pi s^2 \rho U^2 \alpha. \qquad (126)$$

Letting the aspect ratio be $(AR) = (2s)^2/A_f$, where A_f is the total area of the fin pair, substitution will give a lift curve slope of

$$C_l' = \frac{\pi}{2}(AR).$$

This is known as Jones' formula, and is quite accurate for $(AR) \simeq 1$. It is inadequate for higher-aspect ratio wings however, overestimating the lift by about 30% when $(AR) = 2$, and worsening further as (AR) grows.