2 KINEMATICS OF MOVING FRAMES

2.1 Rotation of Reference Frames

We say that a vector expressed in the inertial frame has coordinates $\vec{x}$, and in a body-reference frame $\vec{x}_b$. For the moment, we assume that the origins of these frames are coincident, but that the body frame has a different angular orientation. The angular orientation has several well-known descriptions, including the Euler angles and the Euler parameters (quaternions). The former method involves successive rotations about the principle axes, and has a solid link with the intuitive notions of roll, pitch, and yaw. Quaternions present a more elegant and robust method, but with more abstraction. We will develop the equations of motion using Euler angles.
Tape three pencils together to form a right-handed three-dimensional coordinate system. Successively rotating the system about three of *its own* principle axes, it is easy to see that any possible orientation can be achieved. For example, consider the sequence of [yaw, pitch, roll]: starting from an orientation identical to some inertial frame, rotate the movable system about its yaw axis, then about the *new* pitch axis, then about the *newer still* roll axis. Needless to say, there are many valid Euler angle rotation sets possible to reach a given orientation; some of them might use the same axis twice.

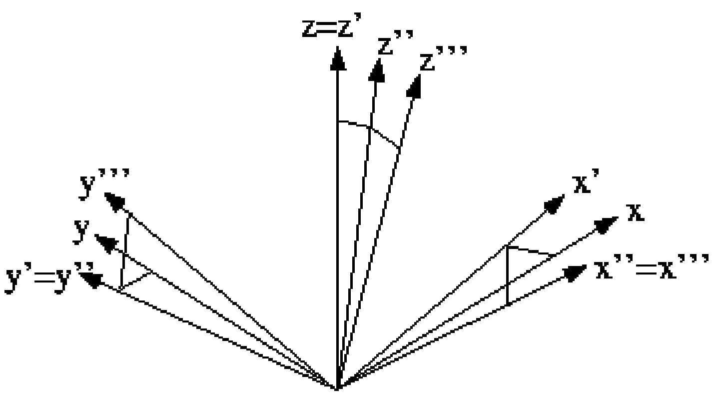

Figure 1: *Successive application of three Euler angles transforms the original coordinate frame into an arbitrary orientation.*

A first question is: what is the coordinate of a point fixed in inertial space, referenced to a rotated *body* frame? The transformation takes the form of a 3x3 matrix, which we now derive through successive rotations of the three Euler angles. Before the first rotation, the body-referenced coordinate matches that of the inertial frame: $\vec{x}_b^0 = \vec{x}$. Now rotate the movable frame yaw axis (z) through an angle ϕ. We have

$$\vec{x}_b^1 = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}_b^0 = R(\phi)\vec{x}_b^0. \qquad (1)$$

Rotation about the z-axis does not change the z-coordinate of the point; the other axes are modified according to basic trigonometry. Now apply the second rotation, pitch about the *new* y-axis by the angle θ:

$$\vec{x}_b^2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \vec{x}_b^1 = R(\theta)\vec{x}_b^1. \qquad (2)$$

Finally, rotate the body system an angle ψ about its *newest* x-axis:

$$\vec{x}_b^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi 0 \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \vec{x}_b^2 = R(\psi)\vec{x}_b^2. \qquad (3)$$

This represents the location of the original point, in the fully-transformed body-reference frame, i.e., $\vec{x}_b^3$. We will use the notation $\vec{x}_b$ instead of $\vec{x}_b^3$ from here on. The three independent rotations can be cascaded through matrix multiplication (order matters!):

$$
\begin{aligned}
\vec{x}_b &= R(\psi)R(\theta)R(\phi)\vec{x} \qquad (4)\\
&= \begin{bmatrix} c\theta c\phi & c\theta s\phi & -s\theta \\ -c\psi s\phi + s\psi s\theta c\phi & c\psi c\phi + s\psi s\theta s\phi & s\psi c\theta \\ s\psi s\phi + c\psi s\theta c\phi & -s\psi c\phi + c\psi s\theta s\phi & c\psi c\theta \end{bmatrix} \vec{x} \\
&= R(\phi,\theta,\psi)\vec{x}.
\end{aligned}
$$

All of the transformation matrices, including $R(\phi,\theta,\psi)$, are orthonormal: their inverse is equivalent to their transpose. Additionally, we should note that the rotation matrix R is universal to *all* representations of orientation, including quaternions. The roles of the trigonometric functions, as written, are specific to Euler angles, and to the order in which we performed the rotations.

In the case that the movable (body) reference frame has a different origin than the inertial frame, we have

$$\vec{x} = \vec{x}_0 + R^T\vec{x}_b, \qquad (5)$$

where $\vec{x}_0$ is the location of the moving origin, expressed in inertial coordinates.

2.2 Differential Rotations

Now consider small rotations from one frame to another; using the small angle assumption to ignore higher-order terms gives

$$
\begin{aligned}
R &\simeq \begin{bmatrix} 1 & \delta\phi & -\delta\theta \\ -\delta\phi & 1 & \delta\psi \\ \delta\theta & -\delta\psi & 1 \end{bmatrix} \qquad (6)\\
&= \begin{bmatrix} 0 & \delta\phi & -\delta\theta \\ -\delta\phi & 0 & \delta\psi \\ \delta\theta & -\delta\psi & 0 \end{bmatrix} + I_{3\times3}.
\end{aligned}
$$

R comprises the identity plus a part equal to the (negative) cross-product operator [$-\delta\vec{E}\times$], where $\delta\vec{E} = [\delta\psi, \delta\theta, \delta\phi]$, the vector of Euler angles ordered with the axes [x, y, z]. Small rotations are completely decoupled; the order of the small rotations does not matter. Since $R^{-1} = R^T$, we have also $R^{-1} = I_{3\times3} + \delta\vec{E}\times$;

$$
\begin{aligned}
\vec{x}_b &= \vec{x} - \delta\vec{E}\times\vec{x} \qquad (7)\\
\vec{x} &= \vec{x}_b + \delta\vec{E}\times\vec{x}_b. \qquad (8)
\end{aligned}
$$

We now fix the point of interest on the *body*, instead of in inertial space, calling its location in the body frame $\vec{r}$ (radius). The differential rotations occur over a time step δt, so that we can write the location of the point before and after the rotation, with respect to the first frame as follows:

$$
\begin{aligned}
\vec{x}(t) &= \vec{r} \qquad (9)\\
\vec{x}(t+\delta t) &= R^T\vec{r} = \vec{r} + \delta\vec{E}\times\vec{r}.
\end{aligned}
$$

Dividing by the differential time step gives

$$
\begin{aligned}
\frac{\delta\vec{x}}{\delta t} &= \frac{\delta\vec{E}}{\delta t}\times\vec{r} \qquad (10)\\
&= \omega\times\vec{r},
\end{aligned}
$$

where the *rotation rate* vector $\omega \simeq d\vec{E}/dt$ because the Euler angles for this infinitesimal rotation are small and decoupled. This same cross-product relationship can be derived in the second frame as well:

$$
\begin{aligned}
\vec{x}_b(t) &= R\vec{r} = \vec{r} - \delta\vec{E}\times\vec{r} \qquad (11)\\
\vec{x}_b(t+\delta t) &= \vec{r}.
\end{aligned}
$$

such that

$$
\begin{aligned}
\frac{\delta\vec{x}_b}{\delta t} &= \frac{\delta\vec{E}}{\delta t}\times\vec{r} \qquad (12)\\
&= \omega\times\vec{r},
\end{aligned}
$$

On a rotating body whose origin point is fixed, the time rate of change of a constant radius vector is the cross-product of the rotation rate vector $\vec{\omega}$ and the radius vector itself. The resultant derivative is in the moving body frame.

In the case that the radius vector changes with respect to the body frame, we need an additional term:

$$\frac{d\vec{r}_b}{dt} = \omega \times \vec{r} + \frac{\partial \vec{r}}{\partial t}. \tag{13}$$

Finally, allowing the origin to move as well gives

$$\frac{d\vec{r}_b}{dt} = \omega \times \vec{r} + \frac{\partial \vec{r}}{\partial t} + \frac{d\vec{r}_o}{dt}. \tag{14}$$

This result is often written in terms of body-referenced velocity $\vec{v}$:

$$\vec{v} = \omega \times \vec{r} + \frac{\partial \vec{r}}{\partial t} + \vec{v}_o, \tag{15}$$

where $\vec{v}_o$is the body-referenced velocity of the origin. The total velocity of the particle is equal to the velocity of the reference frame origin, plus a component due to rotation of this frame. The velocity equation can be generalized to *any* body-referenced vector $\vec{f}$:

$$\frac{d\vec{f}}{dt} = \frac{\partial f}{\partial t} + \vec{\omega} \times \vec{f}. \tag{16}$$

2.3 Rate of Change of Euler Angles

Only for the case of infinitesimal Euler angles is it true that the time rate of change of the Euler angles equals the body-referenced rotation rate. For example, with the sequence [yaw,pitch,roll], the Euler yaw angle (applied first) is definitely not about the final body yaw axis; the pitch and roll rotations moved the axis. An important part of any simulation is the evolution of the Euler angles. Since the physics determine rotation rate $\vec{\omega}$, we seek a mapping $\vec{\omega} \to d\vec{E}/dt$.
The idea is to consider small changes in each Euler angle, and determine the effects on the rotation vector. The first Euler angle undergoes two additional rotations, the second angle one rotation, and the final Euler angle no additional rotations:

$$\begin{aligned} \vec{\omega} &= R(\psi)R(\theta)\begin{Bmatrix} 0 \\ 0 \\ \frac{d\phi}{dt} \end{Bmatrix} + R(\psi)\begin{Bmatrix} 0 \\ \frac{d\theta}{dt} \\ 0 \end{Bmatrix} + \begin{Bmatrix} \frac{d\psi}{dt} \\ 0 \\ 0 \end{Bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\psi & \sin\psi\cos\theta \\ 0 & -\sin\psi & \cos\psi\cos\theta \end{bmatrix} \begin{Bmatrix} \frac{d\psi}{dt} \\ \frac{d\theta}{dt} \\ \frac{d\phi}{dt} \end{Bmatrix}. \end{aligned} \tag{17}$$

Taking the inverse gives

$$\begin{aligned} \frac{d\vec{E}}{dt} &= \begin{bmatrix} 1 & \sin\psi\tan\theta & \cos\psi\tan\theta \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi/\cos\theta & \cos\psi/\cos\theta \end{bmatrix} \vec{\omega} \\ &= \Gamma(\vec{E})\vec{\omega}. \end{aligned} \tag{18}$$

Singularities exist in Γat $\theta = \{\pi/2, 3\pi/2\}$, because of the division by $\cos\theta$, and hence this otherwise useful equation for propagating the angular orientation of a body fails when the vehicle rotates about the intermediate y-axis by ninety degrees. In applications where this is a real possibility, for example in orbiting satellites and robotic arms, quaternions provide a seamless mapping. For most ocean vessels, the singularity is acceptable, as long as it is not on the yaw axis!

2.4 Dead Reckoning

The measurement of heading and longitudinal speed gives rise to one of the oldest methods of navigation: dead reckoning. Quite simply, if the estimated longitudinal speed over ground is U, and the estimated heading is ϕ, ignoring the lateral velocity leads to the evolution of Cartesian coordinates:

$$\begin{aligned} \dot{x} &= U\cos\phi \\ \dot{y} &= U\sin\phi. \end{aligned}$$

Needless to say, currents and vehicle sideslip will cause this to be in error. Nonetheless, some of the most remarkable feats of navigation in history have depended on dead reckoning.