Many of the following problems use data presented by Fossen and in *Principles of Naval Architecture*.

1. **Inertial dynamics**. The gas generator turbine of a LM-2500 marine engine rotates at $p = 10,000 rpm$ (revolutions per minute), and has a mass of approximately $200 kg$. The following question pertains to the inertial forces experienced by this spinning rigid body, with $I_{xy} = I_{xz} = I_{yz} = x_g = z_g = 0$.

   Because of high rotational speed, the turbine is exceedingly sensitive to lateral balancing. Estimate the net force that the radial bearings would have to support for a $1mm$ mass imbalance, say, $y_g = 1mm$ with respect to the spin axis.

2. **Kinematics.** Although dead-reckoning is not recommended as a method for long-term navigation, it can be quite useful for pre-planning trajectories. For example, consider a surface vessel heading profile as follows:

$$t = 0 \qquad \qquad \phi(t) = 0 \; rad$$
$$0 < t <= 60s \qquad \phi(t) = t/60 \times \pi/4 \; rad$$
$$60s < t < 240s \qquad \phi(t) = \pi/4 \; rad$$
$$240s < t < 300s \qquad \phi(t) = (300 - t)/60 \times \pi/4 \; rad$$
$$t = 300s \qquad \qquad \phi(t) = 0 \; rad$$
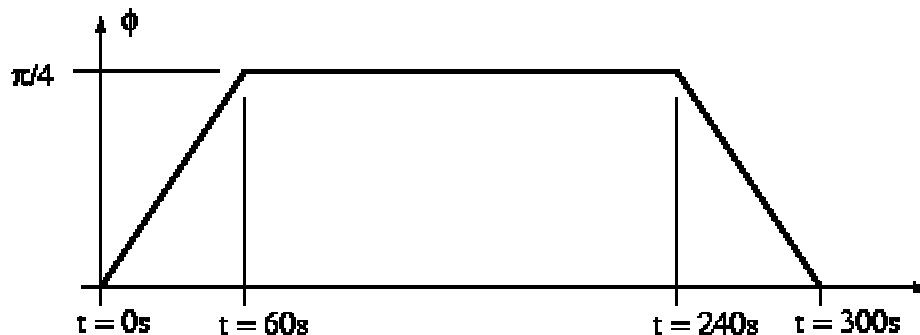
4.

5. What is the total distance traveled in the Cartesian x direction during these five minutes? You may assume that the forward speed of the vessel is $u = 1m/s$, and that there is no significant sideslip. *Hint:* Break the trajectory up into three parts, which can be evaluated separately.

6. **Coefficients and system modeling.** Consider a weather vane in a wind of velocity U_o. If θ is the angle of the vane with respect to the wind direction,
   1. Write the single-degree of freedom (N) linearized equations of motion about the fixed axis **O**.
   2. Write N_θ, $N_{\dot\theta}$, and $N_{\ddot\theta}$ in terms of N_v, N_r, $N_{\dot r}$, etc..
   3. If we consider the differential equation

$$A\ddot{y}(t) + B\dot{y}(t) + Cy(t) = 0,$$

   the condition for stability is that A, B, and C must have the same sign. Express this requirement in terms of the derivatives in the previous question. Give physical interpretations for what would make such a device stable or unstable.

4. Create a numerical model of this system, using the MATLAB ODE solver ode45. The system equation can be written as two first-order equations:

$$\frac{d}{dt}\begin{Bmatrix} \dot{\theta} \\ \theta \end{Bmatrix} = \begin{bmatrix} -B/A & -C/A \\ 1 & 0 \end{bmatrix}\begin{Bmatrix} \dot{\theta} \\ \theta \end{Bmatrix}.$$

Simulate the system response to nonzero initial conditions (e.g., $\theta(0) = 1, \dot{\theta}(0) = 0$). Discuss, using several examples, response sensitivity to B and C, which are related to the aerodynamic coefficients. For example, look at the range $\{A, B, C\} = \{1, \pm3, \pm3\}$.
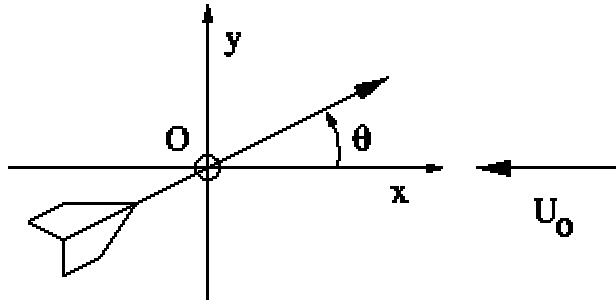
7.

8. **Hydrodynamics.** The figure below shows some characteristic fluid force curves versus a motion parameter. Give the linear hydrodynamic coefficient at two different operating conditions, origin O and A: is it zero, small, finite positive, finite negative?
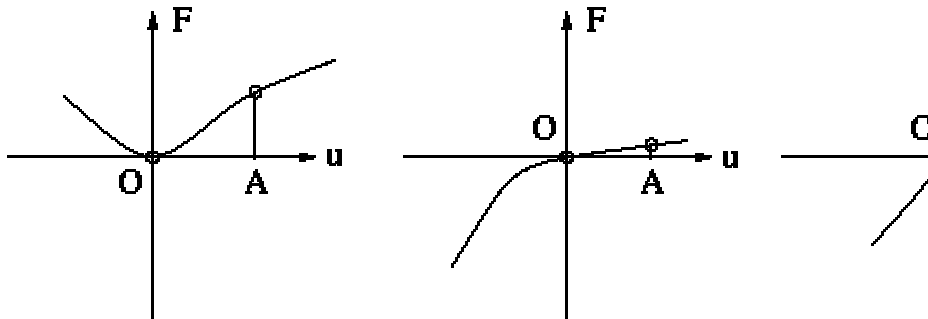
9.

10. **Second-order system.** Consider the vibration of a mass-spring-dashpot system:

$$m\ddot{x} + b\dot{x} + kx = 0,$$

wherein all the coefficients are positive and real (i.e., physical).

1. Set $x(t) = e^{st}$, and find two solutions for s in terms of m, b, and k, using the quadratic formula. The solution pairs depend on whether b exceeds a critical value: what is the critical value and how do the solutions change?
2. From the initial condition $x(0) = x_o$, and $\dot{x}(0) = 0$, find the coefficients for the general solution

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}.$$

in terms of x_o, s_1, and s_2. What do you think will happen when $s_1 = s_2$?

3. Write this $x(t)$ for sub-critical b (that is, complex s) in terms of the standard parameters

$$\omega_n = \sqrt{\frac{k}{m}} \ \text{(undamped natural frequency)}$$

$$\zeta = \frac{b}{2\sqrt{km}} \ \text{(damping ratio)}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \ \text{(damped natural frequency).}$$

Give your answer in terms of a real exponential multiplied by a single sine or cosine, and make a sketch.

4. Sketch the response $x(t)$ for supercritical b, i.e., both s are real, for the same initial conditions. What happens when the roots s are far apart vs. when they are close together?

5. Consider now the case where an input acts to drive the mass from zero initial conditions:

$$m\ddot{x} + b\dot{x} + kx = u.$$

This equation defines a system, with input u, and output x. Using the Laplace transform, write the transfer function for this system, i.e., the impulse response, both in frequency (Laplace) space, and in the time-domain. Make a sketch of the time-domain result. Be sure to cover both the sub-critical and super-critical damping cases.

6. What is the step response of this system, in both the subcritical and super-critical damping cases? Include sketches.

11. **Submarine roll dynamics and control.** A submarine has weight 1200t (tons) and the center of gravity is 0.5m above the center of buoyancy (What can you conclude?). The rolling motion can be assumed to be decoupled from the other motions. This submarine has anti-rolling fins to ensure stability. The control hydrodynamic derivative is $K_\delta = -2.8tm$ per degree of fin rotation δ, at a forward speed of 5m/s.
    1. Write the equation of motion for roll (K).
    2. If the automatic control law $\delta = k_1\psi$ is used, where ψ is the roll angle, what range of k_1 ensures stability?
    3. If the speed suddenly drops to 2.5m/s, how does the range of stabilizing k_1 change?

12. **Steering**. Draw curves of rudder angle δ vs. yaw rate r for a dynamically stable surface vessel, and then for an unstable vessel. Indicate areas where the vessel turns against the rudder action.

13. **Submarine pitch/heave dynamics.** Consider a submarine moving forward at speed U, and restricted to small motions in the vertical ($x - z$) plane. Assume that:
    o The submarine is symmetric in the $x - y$ plane and $y_G = 0$. The submarine is not symmetric in the $x - z$ plane due to the sail, etc., but assume it is *nearly* symmetric, so that you can omit certain hydrodynamic terms.
    o At rest, the submarine is neutrally buoyant and stable with the x and y axes horizontal. What does this tell us about the magnitude of the buoyant force and where it acts with respect to the center of weight?
    3. Derive the equations of motion for the submarine moving at speed U.
    4. Derive the hydrostatic, restoring pitch moment for small pitch θ.
    5. Linearize the inertial terms in the equations of motion.
    6. Expand the fluid forces and moments in terms of the motions. Omit nonlinear and memory effects, and be sure to include the hydrostatic moment. Explain your choices.

7. Write out the complete linearized equations of motion. Does surge decouple from pitch and heave? Write out the coupled equations in matrix form.
8. Can the submarine be stable in pitch without feedback control? Can the submarine be stable in depth without feedback control?

14. **Submarine stability via slender body theory.**
    0. An underwater vehicle hull has the shape of a circular cylinder, except at the very ends, where it rapidly tapers down to a point. If the radius is $0.5m$, length is $17m$, and speed is $2m/s$, use slender-body theory to express the following coefficients: Z_w, M_q, and M_w, all referenced to the body midpoint.
    1. It turns out that in order to match the lift and moment of the hull in experiments, a flat tail of radius $0.4m$ has to be included in the slender-body estimate. Determine the corrected value for M_m, and in addition compute Z_w. Where is the aerodynamic center with respect to the center of the hull, and is this hull stable in pitch? Recall that the aerodynamic center is the equivalent location of the observed lift force that creates the observed moment.
    2. A fin with pre-determined lift slope of $3.6/rad$ is to be applied at the tail to make the device *just* stable in pitch. How big is the required fin area that brings the net M_w to zero, as desired?

15. **Lifting surfaces.** Using the lifting surface formulas, estimate the coefficient K_w of a submarine sail as shown below, for $U = 10m/s$.
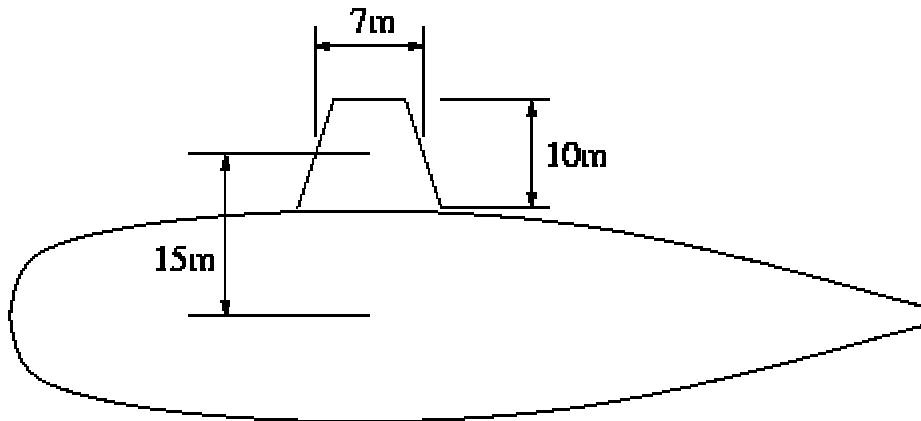

16.

17. **Slender body theory.** Consider a long ellipsoidal body of length l and diameter d.
    0. With $l/d = 7.0$, approximate the cross-body added mass, using slender-body theory, and compare it with the exact results from the table below (Blevins).
    1. Perform the slender body calculation also for a sphere, and compare again to the exact result.
    2. What is the added mass in the in-line direction?

Sphere added mass:	$2\rho\pi a^3/3$
Ellipsoid cross-body added mass:	$4\alpha\rho\pi ab^2/3$
where $a/b = 0.1$:	$\alpha = 0.075$
0.2:	0.143
0.6:	0.355
1.0:	0.500
2.0:	0.704
5.0:	0.894

7.0:	0.933
10.0:	0.960
∞:	1.000

Ellipsoid longitudinal added mass: $4\alpha\rho\pi ab^2/3$

where $a/b = 0.1$:	$\alpha = 6.148$
0.2:	3.008
0.6:	0.908
1.0:	0.500
2.0:	0.210
5.0:	0.059
7.0:	0.036
10.0:	0.021
∞:	0.000
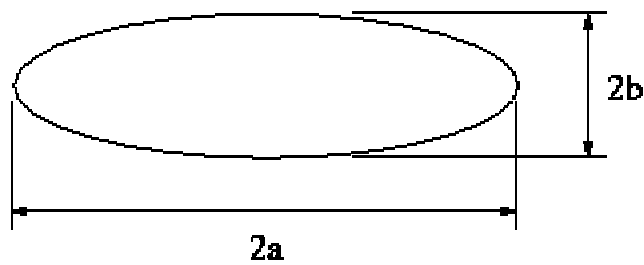
19. **Submarine pitch stability with various methods.** We will calculate the pitch derivative due to an angle of attack, M_w, for a small submarine, using several different methods.

The submarine is a rotationally symmetric ellipsoid with $L/D = 7$, $L = 35m$. We fix the body origin to the midpoint as shown below. The body is appended with two fins as shown in the figure: each has an area of $1m^2$, geometric aspect ratio of 3, and has its longitudinal center of action $1m$ forward of the body stern.

0. Apply wing theory to characterize M_w due to the fins alone.
1. Apply slender body theory to estimate M_w for the body alone.
2. Linearize the Munk moment, and give a corrected slender-body value for M_w of the body.
3. Apply Jorgensen's approximate formulas to estimate M_w for the body; make linearizations where necessary.
4. Use the experimental data from Hoerner (p. 13.2) below to estimate M_w for the body.
5. Use the Hoerner result and your result from wing theory to write a net M_w for the body plus fins.
6. What is the location of the net aerodynamic center? If it has an unstable location forward of the midpoint, what increase in fin area would bring it back to the midpoint?
7. What is the diameter of a flat stern that will allow the slender body theory to give the same moment as the Hoerner (experimental) data, for the body alone?
8. What is the diameter of a flat stern that allows the approximate formulas of Jorgensen to give the same moment as the Hoerner data, for the body alone?
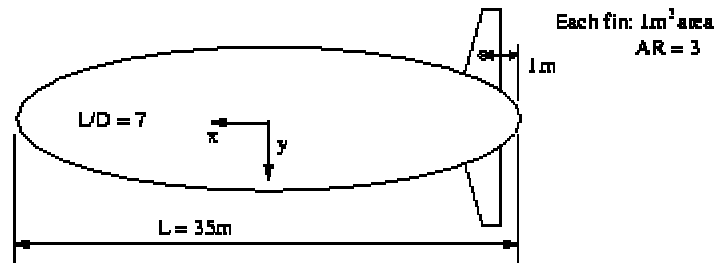
Hoerner p. 13.2:

Symmetric body of revolution with $L/D = 6.7$.

Force and moment referenced to body midpoint:

$Z = -0.5\rho U^2 D^2 C_{ydb} \tan(w/u)$, $C_y = 1.20$.

$M = 0.5\rho U^2 D^2 L C_{mb} \tan(w/u)$, $C_m = 0.53$. DESTABILIZING

Each fin: $1m^2$ area
AR = 3
1 m
L/D = 7
x
y
z
L = 35m

20. **Cable dynamics**. A towing cable is steel-jacketed, with diameter $3cm$, an effective extensibility of $E = 100 \times 10^9 Pa$, and a density of $\rho_c = 5000 kg/m^3$. The vehicle it tows is streamlined, and has negligible mass (M). If we are towing at low speed and $L = 4000m$ depth,

    0. Calculate ω_1, the first undamped natural frequency in heave, with $mL \gg M$.
    1. If the vessel heaves with a $P = 2m$ amplitude and a 4.5-second period, what is the dynamic tension amplitude at the top ($s = L$) of the cable? The formula for this dynamic tension (with $mL \gg M$) is $T = EAPk \sin(ks)/\cos(kL)$, where A is the cross-sectional area of the cable, $k = \omega\sqrt{m/EA}$, and m is the cable mass per unit length.
    2. Taking into account the static tension induced by in-water weight, does the cable unload at the surface, for the heaving conditions above?

21. **Cable mechanics (hard!)**. You are asked to assess the operational envelope of a cable/vehicle system which has been installed on a vessel. The cable is steel-jacketed, with diameter $3cm$, an effective extensibility of $100 \times 10^9 Pa$, and a density of $5000 kg/m$. The vehicle is streamlined, and has a mass (material plus added) of $100 kg$.

    0. The towing angle at the vessel cannot exceed $25 deg$ from vertical, for reasons of deck and crane geometry. If this angle is considered equal to the critical angle ϕ_c, what is the maximum speed for the system? Assume the normal drag coefficient of the cable is $C_n = 1.2$.
    1. Considering the undamped, uncoupled axial dynamics, express the natural frequency as a function of the vehicle depth, and sketch the curve. If the fastest waves that the surface vessel responds to have period $4 s$, what is the ``threshold'' operating depth?
    2. The undamped, uncoupled lateral dynamics follow the equation

$$(m + m_a)\frac{\partial^2 q}{\partial t^2} = \bar{T}\frac{\partial^2 q}{\partial s^2} + w_n \sin\bar\phi\frac{\partial q}{\partial s},$$

where m_a is the added mass of the cable per unit length, and w_n is the in-water weight of the cable. Separation of variables $q(s,t) = \bar{q}(s)\cos\omega t$ gives

$$\bar{T}\frac{\partial^2 \bar{q}}{\partial s^2} + w_n \sin\bar\phi\frac{\partial \bar{q}}{\partial s} + w_n\gamma\bar{q} = 0,$$

where $\gamma = (m + m_a)\omega^2/w_n$. Simplify this expression by writing $\bar{T} = w_n s$ (the vehicle weight effect is small), $\sin\bar\phi \simeq 1$ (the cable is nearly vertical), and then use the substitution $z = 2\sqrt{\gamma s}$ to arrive at the Bessel equation

$$z^2\frac{\partial^2 \ddot{q}}{\partial z^2} + z\frac{\partial \ddot{q}}{\partial z} + z^2\ddot{q} = 0.$$

The derivatives can be transformed from s to z coordinates via the chain rule:

$$\frac{\partial \ddot{q}}{\partial s} = \frac{\partial \ddot{q}}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$\frac{\partial^2 \ddot{q}}{\partial s^2} = \frac{\partial^2 \ddot{q}}{\partial z^2}\left(\frac{\partial z}{\partial s}\right)^2 + \frac{\partial \ddot{q}}{\partial z}\cdot\frac{\partial^2 z}{\partial z\partial s}\cdot\frac{\partial z}{\partial s}.$$

3. The equation above has a solution of the form $\ddot{q} = cJ_0(z)$, where c is a constant to be found, and $J_0(z)$ is the zero'th order Bessel function of the first kind. If $\ddot{q}(s = L) = Q$, that is, we impose a harmonic input at the top, find c in terms of Q and $z(s = L)$. What condition makes $\ddot{q}$ blow up, indicating a resonance frequency ω_n?

4. Recalling that strumming occurs when $\omega_n \simeq 0.2U/d$, construct a graph of towing velocities vs. deployment depths for which the strumming would be centered. Include at least the three lowest modes in your sketch, and discuss the trends for the higher modes. $J_0(x) = 0$ for for $x \simeq \pi(n - 0.25), n = 0, 1, 2, \cdots$, especially when x is large. You should find that strumming is hard to avoid in deep towing applications!

22. **Vessel heading control.** We consider the yaw/sway dynamics of a high-speed container ship. All the calculations below are to be made in nondimensional coordinates, so you do not need to perform any transformations in this question.

   0. The nondimensional system with states $\vec{x}' = [v', r']$ evolves according to $d\vec{x}'/dt' = A\vec{x}' + B\delta$, where

$$A = \begin{bmatrix} -0.90 & -0.42 \\ -4.8 & -2.3 \end{bmatrix}, \quad B = \left\{ \begin{array}{c} -0.13 \\ 1.4 \end{array} \right\}.$$

   If the output is yaw rate r', i.e., $C = [0\ 1]$, write the transfer function $r'(s)/\delta(s) = C(sI - A)^{-1}B$. Use matrix inversion for the 2×2 matrix $(sI - A)$, perform the matrix multiplications, and then use the quadratic formula if necessary to express your answer as

$$\frac{r'(s)}{\delta(s)} = K\frac{s + z_1}{(s + p_1)(s + p_2)},$$

   where K is a constant gain, $-z_1$ is the zero (there is one real zero), and $[-p_1, -p_2]$ are the two poles of the second-order system. Are both poles stable?

   1. For the purposes of autopilot design, the transfer function $\phi(s)/\delta(s)$ is crucial. Fortunately, all you have to do is divide $r'(s)/\delta(s)$ by another factor of s to account for the fact that $d\phi/dt' = r'$. Sketch all the poles and zeros of the plant transfer function $P(s) = \phi(s)/\delta(s)$ on a root locus plot.

   2. A PID controller $C(s)$ will work to stabilize the heading angle, and brings to $P(s)C(s)$ one additional pole at the origin, and two arbitrary zeros. Demonstrate a rough PID design by devising two controller zero locations that will attract the closed-loop poles into the left-half plane. Draw the path that you expect each closed-loop pole to follow, as the control gain increases from zero to very large values.

You do *not* need to calculate the PID gains that go with your proposed zero locations.

23. **Full-state feedback of a submarine.** The coefficients governing the pitch/heave dynamics of a Deep Submergence Rescue Vehicle (DSRV) are given below. They are nondimensionalized as in the lecture notes, using the factors $\rho/2$, U, and L.

$$I'_{xx} = 0.000118 \quad M'_q = -0.0113 \quad Z'_q = -0.0175$$
$$I'_{zz} = 0.00193 \quad M'_{\dot{q}} = -0.00157 \quad Z'_{\dot{q}} = -0.000130$$
$$x'_G = 0 \quad M'_w = 0.0112 \quad Z'_w = -0.0439$$
$$m' = 0.0364 \quad M'_{\dot{w}} = -0.000146 \quad Z'_{\dot{w}} = -0.0315$$
$$U = 2.0m/s \quad M'_{\delta} = -0.0128 \quad Z'_{\delta} = -0.0277$$
$$L = 15.0m \quad M'_{\theta} = -0.156/U^2$$

24.

0. Write the linearized (nondimensional) dynamics in the matrix form $\dot{x} = Ax + Bu$, where the input $u = \delta$, and the state vector is $x = [w', q', \theta, Z']$. Z'here is the elevation of the vehicle in inertial coordinates; your approximation for $\dot{Z}'$should include both $w'$and θ.

1. Is the DSRV vehicle open-loop stable, that is, without control action δ? You can assess this either by finding the eigenvalues of your A-matrix above, or by computing the stability parameter C'.

2. In preparation for controller design, create a simulation using Matlab. Make a graph of the open-loop step response, showing all the dimensional state variables versus dimensional time.

3. Assuming that all the states can be measured accurately, a full-state controller $\delta = -Kx$can be used, where $K$is a 1×4gain matrix. Under this control, the system dynamics are governed by $\dot{x} = (A - BK)x$, so that if $A - BK$has all negative eigenvalues, the system is stable.

   Use the Matlab function `place` to put the four closed-loop poles at $g(-0.95 \pm 0.31i)$and $g(-0.59 \pm 0.81i)$, $g = 1$. Demonstrate that your closed-loop design is stable against nonzero initial conditions. What are the effects of increasing or decreasing g?

   Note that the pole locations suggested above are with respect to the *dimensional* system, i.e., using dimensional time.

25. **Nyquist stability.** The inverted pendulum shown below is often used as a simple model for rocket flight, and can also illustrate the dynamic behavior of an unstable ocean vessel which is propelled from the stern, e.g., a barge being pushed by a tugboat.

For this problem, we assume that *all* of the mass (m) is concentrated at the distal end of an arm of length l; a high-performance positioning system sets the horizontal position of the cart u_0.
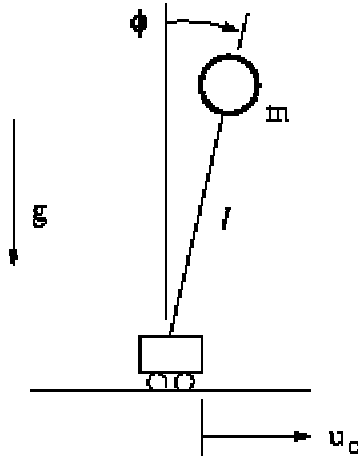
A Lagrangian derivation of the dynamics gives:

$$\ddot{\phi} = \frac{g}{l}\sin(\phi) - \frac{1}{l}\cos(\phi)\ddot{u}_o.$$

Note that the *acceleration* of the cart is now considered to be the input to the plant, e.g., $u = \ddot{u}_o$.

0. If the observation is taken to be the angle of the bar ϕ, i.e., $y = \phi$, write the state-space representation of the dynamics linearized about the point $\phi = 0$.
1. Show that this plant is fully state observable and controllable.
2. Now make the assumption that $l = 1$, and $g = 1$ (it can be achieved through a proper nondimensionalization) What is the transfer function of the plant, $P(s)$? Use this plant model for the rest of this question.
3. Where are the plant poles and zeros, if any?
4. A stabilizing controller is not hard to find, and a suggested one is:

$$C(s) = \frac{-4(2s+1)}{s+2}.$$

Create a rough Nyquist plot of the loci of $P(s)C(s)$, for the frequencies $\omega = [0, 2.2, \infty]\, rad/s$. These three frequencies are all that you need to sketch the overall shape; there are no hidden loops or other features. In the case of $\omega = 2.2$, for which you will need to make explicit calculations, you may find the following identity useful:

$$\frac{a+jb}{c+jd} = \frac{(a+jb)(c-jd)}{(c+jd)(c-jd)} = \frac{(ac+bd)+j(bc-ad)}{c^2+d^2}.$$

Be sure to include the complex-conjugate points for $-\omega$ on your plot.

5. Invoke the Nyquist stability criterion to confirm that the closed-loop system is stable.
6. About how many degrees of phase margin does this design provide? What reduction in low-frequency gain can be tolerated? Is it a reasonable design?
26. **Root locus and loopshaping.** The parameters governing the surface maneuvering of a high-speed container ship are given below for reference:

$$m' \ 0.00792 \quad I'_{zz} \ 0.000456 \quad x'_G \ -0.05$$
$$L \ 175.0m \quad U \ 8m/s \quad Y'_{\dot{v}} \ -0.00705$$
$$Y'_{\dot{r}} \ 0.0000 \quad Y'_v \ -0.0116 \quad Y'_r \ 0.00242$$
$$Y'_\delta \ -0.00258 \quad N'_{\dot{v}} \ 0.0000 \quad N'_{\dot{r}} \ -0.000419$$
$$N'_v \ -0.00385 \quad N'_r \ -0.00222 \quad N'_\delta \ 0.00126$$

27. Note that the center of vessel mass is located *aft* of the origin; for this model, the origin coincides with the center of added mass, so that $Y_{\dot{r}} = N_{\dot{v}} = 0$. The nondimensional system with states $\vec{x} = [v', r']$ evolves according to $d\vec{x}'/dt' = A\vec{x}' + B\delta$, where

28.
$$A = \begin{bmatrix} -0.90 & -0.42 \\ -4.8 & -2.3 \end{bmatrix}, \quad B = \left\{ \begin{array}{c} -0.13 \\ 1.4 \end{array} \right\}.$$

29. The relevant output is yaw rate: $C = [0 \ 1]$ and $D = 0$. For the purposes of autopilot design, however, the transfer function $\phi(s)/\delta(s)$ is needed.

30. The following steps create two heading autopilots, using the root-locus and loopshaping techniques. In addition to the Matlab commands listed below, you will find very useful the convolution function `conv()` which can be used to combine systems, e.g., for numerators, `numPC = conv(numP,numC);`. Also, be sure that you equalize axis scaling for your plots in the complex-plane, by using `axis('equal');`.

   0. Use the Matlab command `tf()`, or `ss()`, to create a system model of the open-loop transfer function $P(s)C(s)$, using the plant above and a PID-type controller:

   $$C(s) = k_p \left(1 + \tau_d s + \frac{1}{\tau_i s} \right).$$

   The actual numerical values for k_p, τ_d, and τ_i are to be found in the next step.

   1. Using $\tau_d = 2$ and $\tau_i = 6$ as suggested values, use the Matlab command `rlocus()` and then `rlocfind()` to select a controller gain k_p, that puts the three slow poles in the following sector: 1) minimum undamped frequency (nondimensional) of 0.3, 2) maximum frequency of 0.5, and 3) minimum damping ratio 0.7. Give a root locus plot, with your pole locations clearly marked on top of the trajectories taken as k_p varies. You don't need to show the fourth, fast pole, which will be quite far to the left.

   2. Apply the k_p you selected to $P(s)C(s)$, and then use the Matlab functions `feedback()` to create the resulting feedback system, and `step()` to plot the closed-loop system response to a step input in desired heading.

   3. Use the Matlab command `nyquist()` to make a Nyquist plot of $P(s)C(s)$ for your design. Make a visual estimate of the gain and phase margins.

   4. An alternate approach for controller design of this stable plant is loopshaping: For the open-loop function $L(s) = \omega_c/s$, where $\omega_c = 2.0$, invert the plant to come up with a compensator: $C(s) = L(s)/P(s)$. This design has infinite gain margin and 90 degrees phase margin.

   5. As above, create the feedback system, and plot the closed-loop step response.

   6. The loopshaping control is *not quite* a PID-controller; how does it differ, and what would $L(s)$ have to contain to make it a PID?

   7. The controllers you just designed are in nondimensional time coordinates; give the P,I, and D gains for use on a real time scale, for the root-locus design.

31. **LQR**. Consider the state-space system and LQR design:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
$$B = [1 \ 0]^T$$
$$C = [0 \ 1]$$
$$D = 0$$
$$Q = C^T C$$
$$R = \rho.$$

The plant is an undamped oscillator with undamped poles at $\pm j$. Note that the plant output is *position* for this problem.

0. What is the control gain K in terms of ρ? **Hints:** There are two solutions for P_{12}; choose the positive one. Also, the expression for P_{22} is messy; luckily, you won't need to use it.
1. Determine the limiting approximations for K with ρ very small and very large - these are the cheap control and expensive control problems.
2. Derive the limiting closed-loop pole locations for $\rho \to 0$, giving the frequency and damping ratio of the Butterworth pattern in terms of ρ. You can get the characteristic equation for the poles as $det(sI - (A - BK)) = 0$, and then make it fit the form $s^2 + 2\zeta\omega_n + \omega_n^2 = 0$.

32. **LQR.** Consider the first-order, unstable system governed by $\dot{x} = x + u$, for which the output is $y = x$. The state-space matrices are $A = B = C = 1$.
    0. Solve the LQR Riccati equation for P, for the case $Q = C^T C$ and $R = \rho$. Your answer should give a positive P as a function of ρ alone.
    1. Consider the cheap-control problem, with $\rho \to 0$. Write the leading-term approximations for P and then K, and compute the eigenvalue (there is only one) of the closed-loop system. Note that $eig(M) = M$, if M is a scalar instead of a matrix.
    2. Now look at the expensive-control problem, with $\rho \to \infty$. Again, write the approximations for P and K, and find the closed-loop system eigenvalue.
33. **LQG/LTR.** For the inverted pendulum plant (see previous question) with state-space matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = [0 \ 1],$$

the KF Riccati equation yields $H = [100 \ 14]^T$, for the choices $V_1 = [100 \ 0 ; 0 \ 0]$, and $V_2 = 0.01$.

0. Compute the open-loop transfer function for the Kalman filter loop:
$$L(s) = C(sI - A)^{-1}H.$$
1. Even though this $L(s)$ is unstable, its magnitude plot confirms that it is a reasonable loopshaping design. What is the low-frequency gain for this $L(s)$, and what kind of tracking performance should we expect from the associated LTR design?
2. Now consider the closed-loop transfer function, which you can write as $S(s) = L(s)/(1 + L(s))$ (the sensitivity). What is the characteristic equation for the closed-loop system, and about what damping ratio has the Kalman filter provided?
34. **LQG/LTR design.** The parameters for the linearized sway/yaw motions of a swimmer delivery vehicle are given below.
35. % Parameters, all nondimensional except [U,L]
36.
37. U =   4.0 ; % m/s
38.
39. L =   5.3 ; % m
40.
41. Izz =   0.006326 ;

```
42.
43.  m =                 0.1415 ;
44.
45.  xg =                0. ;
46.
47.  Yv =                -0.1 ;
48.
49.  Yr =                0.03 ;
50.
51.  Nv =                -0.0074 ;
52.
53.  Nr =                -0.016 ;
54.
55.  Ydelta =            0.027 ;
56.
57.  Yvdot =             -0.055 ;
58.
59.  Yrdot =             0. ;
60.
61.  Nvdot =             0. ;
62.
63.  Nrdot =             -0.0034 ;
64.
65.  Ndelta =            -0.013 ;
66.
```

You are asked to develop an LQG/LTR controller for this plant, and it is suggested that you compose a single Matlab script to perform the steps in sequence. Please make sure you answer all the questions, and include a listing of your code. This entire design is made in nondimensional coordinates.

0. *Plant Modeling and Characteristics*
    1. Construct a state-space plant model, to take rudder angle δ as an input and give heading angle ϕ as an output. Please provide the *numerical* values for the A, B, C matrices. There should be three states in your model, with one input channel and one output channel.
    2. Compute and list the controllability and observability matrices; is the plant state-controllable and state-observable?
    3. Where are the poles of your plant model? Is this model stable?
    4. Show a plot of your plant's step response.
1. *LQR and KF Designs*
    1. Using the Matlab command `lqr()`, you can compute the LQR feedback gain K, for given A, B, Q, and R matrices. With the choices $Q = C^T C$, and $R = \rho$, list K and plot the closed-loop step responses for the choices $\rho = [0.1, 0.001, 0.00001]$. How do the gains and step responses change as you make ρ smaller and smaller?

        Note that the fundamental closed-loop LQR system is

        $$\dot{\vec{x}} = (A - BK)\vec{x} + BK\vec{x}_{desired}$$
        $$y = \vec{x},$$

        i.e., the input to the closed-loop system is $\vec{x}_{desired}$ and the output is $\vec{x}$. Your plot should show specifically the output ϕ, for an input of $\vec{x}_{desired} = [v_{desired} = 0, r_{desired} = 0, \phi_{desired} = 1]$. This compression can be achieved in one step by premultiplying the system by C^T, and post-multiplying it by C:

        $$\dot{\vec{x}} = (A - BK)\vec{x} + BKC^T y_{desired}$$
        $$y = C\vec{x},$$

2. The Matlab command `lqe()` can be used to generate the Kal-man filter gain H, given design matrices A, C, V_1, and V_2. For the choices $V_1 = I_{3\times3}$ and $V_2 = 0.01$, compute H, and make a plot of the closed-loop step response. Be sure to give the numerical values of H.

   Note that the `lqe()` command asks for a disturbance gain matrix G; you should set this to $I_{3\times3}$. The closed-loop KF system is as follows:

   $$\dot{\hat{x}} = (A - HC)\hat{x} + Hy$$
   $$\dot{\hat{y}} = C\hat{x},$$

   i.e., the input is the measurement y and the output is an estimated version of it, $\hat{y}$.

2. *Loop Transfer Recovery*

   The LQG compensator is a combination of the KF and LQR designs above. With normal negative feedback, the compensator $C(s)$ has the following state space representation:

   $$\dot{\vec{z}} = (A - BK - HC)\vec{z} + He$$
   $$u = K\vec{z},$$

   so that the input to the compensator is the tracking error $e = r - y$, and its output u is the control action to be applied as input to the plant. The total open-loop transfer function is the $P(s)C(s)$; in Matlab, you may simply multiply the systems, e.g., `sysPC = sysP * sysC ;`.

   1. Make a log(magnitude) plot of the KF open-loop transfer function $L(s) = C(sI - A)^{-1}H$, versus log(frequency). You may find the Matlab command `freqresp()` helpful. $|L(s)|$ should be large at low frequencies, and small at high frequencies, consistent with the rules of loopshaping.
   2. As $\rho \to 0$, the product $P(s)C(s) \to L(s)$. Demonstrate this by computing $P(s)C(s)$ for the three different values of ρ above, and overlaying the respective $|P(s)C(s)|$ over the plot of part 3a).
   3. Make a closed-loop step response plot for the smallest value of ρ. How does it compare with the KF step response of part 2b)?

   In real LTR applications, the particular values of V_1 and V_2 can be picked to control the low-frequency gain, and crossover frequency of the open-loop KF system $L(s) = C(sI - A)^{-1}H$.

67. **Inertial navigation.** A triaxial accelerometer package is aligned with the [x,y,z] (fwd, port, up) axes of an underwater vehicle. At a particular instant in time, the three raw accelerations from the strain guages are $[\ddot{x} = -0.96, \ddot{y} = 0.80, \ddot{z} = 9.73]$ meters/second. Note that under the influence of gravity alone, this sensor will always report that the vehicle is accelerating *upwards*.
    0. Ignoring the coupling of Euler small angles, what estimates can you give for the roll and pitch angles? I suggest that you draw the measurement vector in the coordinates of the vehicle frame, and then consider the orientation of the body which would lead to it.
    1. Given that the acceleration due to gravity is 9.81m/s, how do we know that these are *reasonable* (but not foolproof!) angle calculations to make?