4 HYDRODYNAMICS: INTRODUCTION

The forces and moments on a vessel are complicated functions of many factors, including water density, viscosity, surface tension, pressure, vapor pressure, and motions of the body. The most important factors for large ocean vehicles are density and motion, and we can make simplifications to parameterize the most prominent relationships. This section pertains to the use of *hydrodynamic coefficients* for predicting hydrodynamic response.

4.1 Taylor Series and Hydrodynamic Coefficients

Recall the Taylor expansion of a function:

$$f(x) = f(x_o) + \frac{\partial f(x_o)}{\partial x}(x - x_o) + \frac{1}{2!}\frac{\partial^2 f(x_o)}{\partial x^2}(x - x_o)^2 + \cdots. \qquad (36)$$

We introduce the notation

$$\begin{aligned} f_x &= \frac{\partial f(x_o)}{\partial x} \\ f_x x &= \frac{1}{2!}\frac{\partial^2 f(x_o)}{\partial x^2}, \end{aligned}$$

and so on, so that a two-variable Taylor expansion would have the form

$$\begin{aligned} f(x, y) &= f(x_o, y_o) + \qquad (37) \\ & f_x(x - x_o) + f_y(y - y_o) + \\ & f_{xx}(x - x_o)^2 + f_{yy}(y - y_o)^2 + f_{xy}(x - x_o)(y - y_o) + \\ & f_{xxx}(x - x_o)^3 + \cdots. \qquad (38) \end{aligned}$$

Note that all of the factorials are included in the coefficients. This notation covers some instances where the formal Taylor series is meaningless, but the notation is still clear. As one example, fluid drag is often written as

$$F = \frac{1}{2}\rho C_d A u|u| = F_{u|u|}u|u|.$$

4.2 Surface Vessel Linear Model

We now discuss some of the hydrodynamic parameters which govern a ship maneuvering in the horizontal plane. The body x-axis is forward and the y-axis is to port, so positive $r$has the boat turning left. We will consider motions only in the horizontal plane, which means $\theta = \psi = p = q = w = 0$. Since the vessel is symmetric about the $x - z$plane, $y_G = 0$; $z_G$is inconsequential. We then have at the outset

$$\begin{aligned} X &= m\left(\frac{\partial u}{\partial t} - rv - x_G r^2\right) \qquad (39) \\ Y &= m\left(\frac{\partial v}{\partial t} + ru + x_G\frac{\partial r}{\partial t}\right) \\ N &= I_{zz}\frac{\partial r}{\partial t} + mx_G\left(\frac{\partial v}{\partial t} + ru\right). \end{aligned}$$

Letting $u = U + u$, where $U >> u$, and eliminating higher-order terms, this set is

$$\begin{aligned} X &= m\frac{\partial u}{\partial t} \qquad (40) \\ Y &= m\left(\frac{\partial v}{\partial t} + rU + x_G\frac{\partial r}{\partial t}\right) \\ N &= I_{zz}\frac{\partial r}{\partial t} + mx_G\left(\frac{\partial v}{\partial t} + rU\right). \end{aligned}$$

A number of coefficients can be discounted. First, in a homogeneous sea, with no current, wave, or wind effects, $\{X_x, X_y, X_\phi, Y_x, Y_y, Y_\phi, N_x, N_y, N_\phi\}$are all zero. We assume that no hydrodynamic forces depend on the position of the vessel.[2] Second, consider X_v: since this longitudinal force would have the same sign regardless of the sign of v(because of side-to-side hull symmetry), it must have zero slope with $v$at the origin. Thus $X_v = 0$. The same argument shows that

$\{X_r, X_{\dot{v}}, X_{\dot{r}}, Y_u, Y_{\dot{u}}, N_u, N_{\dot{u}}\} = 0$. Finally, since fluid particle acceleration relates linearly with pressure or force, we do not consider nonlinear acceleration terms, or higher time derivatives. It should be noted that some nonlinear terms related to those we have eliminated above are *not* zero. For instance, $Y_{uu} = 0$because of hull symmetry, but in general $X_{vv} = 0$only if the vessel is bow-stern symmetric.

We have so far, considering only the linear hydrodynamic terms,

$$
\begin{aligned}
(m - X_{\dot{u}})\dot{u} &= X_u u + X' && (41)\\
(m - Y_{\dot{v}})\dot{v} + (mx_G - Y_{\dot{r}})\dot{r} &= Y_v v + (Y_r - mU)r + Y' && (42)\\
(mx_G - N_{\dot{v}})\dot{v} + (I_{zz} - N_{\dot{r}})\dot{r} &= N_v v - (N_r - mx_G U)r + N'. && (43)
\end{aligned}
$$

The right side here carries also the imposed forces from a thruster(s) and rudder(s) $\{X', Y', N'\}$. Note that the surge equation is *decoupled* from the sway and yaw, but that sway and yaw themselves are coupled, and therefore are of immediate interest. With the state vector $\vec{s} = \{v, r\}$ and external force/moment vector $\vec{F} = \{Y', N'\}$, a state-space representation of the sway/yaw system is

$$
\begin{aligned}
\begin{bmatrix} m - Y_{\dot{v}} & mx_G - Y_{\dot{r}} \\ mx_G - N_{\dot{v}} & I_{zz} - N_{\dot{r}} \end{bmatrix} \frac{d\vec{s}}{dt} &= \begin{bmatrix} Y_v & Y_r - mU \\ N_v & N_r - mx_G U \end{bmatrix} \vec{s} + \vec{F}, \text{ or} && (44)\\
M\dot{\vec{s}} &= P\vec{s} + \vec{F}\\
\dot{\vec{s}} &= M^{-1}P\vec{s} + M^{-1}\vec{F}\\
\dot{\vec{s}} &= A\vec{s} + B\vec{F}. && (45)
\end{aligned}
$$

The matrix $M$is a mass or inertia matrix, which is always invertible. The last form of the equation is a standard one wherein Arepresents the internal dynamics of the system, and $B$is a gain matrix for the control and disturbance inputs.

4.3 Stability of the Sway/Yaw System

Consider the homogeneous system $\dot{\vec{s}} = A\vec{s}$:

$$
\begin{aligned}
\dot{s}_1 &= A_{11}s_1 + A_{12}s_2\\
\dot{s}_2 &= A_{21}s_1 + A_{22}s_2.
\end{aligned}
$$

We can rewrite the second equation as

$$
s_2 = \left(\frac{d(\cdot)}{dt} - A_{22}\right)^{-1} A_{21}s_1 \qquad (46)
$$

and substitute into the first equation to give

$$
\ddot{s}_1 + (-A_{11} - A_{22})\dot{s}_1 + (A_{11}A_{22} - A_{12}A_{21})s_1 = 0. \qquad (47)
$$

Note that these operations are allowed because the derivative operator is linear; in the language of the Laplace transform, we would simply use s. A necessary and sufficient condition for stability of this ODE system is that each coefficient must be greater than zero:

$$
\begin{aligned}
-A_{11} - A_{22} &> 0 && (48)\\
A_{11}A_{22} - -A_{12}A_{21} &> 0
\end{aligned}
$$

The components of $A$for the sway/yaw problem are

$$
\begin{aligned}
A_{11} &= \frac{(I_{zz} - N_{\dot{r}})Y_v + (Y_{\dot{r}} - mx_G)N_v}{(m - Y_{\dot{v}})(I_{zz} - N_{\dot{r}}) - (mx_G - Y_{\dot{r}})(mx_G - N_{\dot{v}})} && (49)\\
A_{12} &= \frac{-(I_{zz} - N_{\dot{r}})(mU - Y_r) - (Y_{\dot{r}} - mx_G)(mx_G U - N_r)}{(m - Y_{\dot{v}})(I_{zz} - N_{\dot{r}}) - (mx_G - Y_{\dot{r}})(mx_G - N_{\dot{v}})}\\
A_{21} &= \frac{(N_{\dot{v}} - mx_G)Y_v + (m - Y_{\dot{v}})N_v}{(m - Y_{\dot{v}})(I_{zz} - N_{\dot{r}}) - (mx_G - Y_{\dot{r}})(mx_G - N_{\dot{v}})}\\
A_{22} &= \frac{-(N_{\dot{v}} - mx_G)(mU - Y_r) - (m - Y_{\dot{v}})(mx_G U - N_r)}{(m - Y_{\dot{v}})(I_{zz} - N_{\dot{r}}) - (mx_G - Y_{\dot{r}})(mx_G - N_{\dot{v}})}.
\end{aligned}
$$

The denominators are identical, and can be simplified. First, let $x_G \simeq 0$; valid for many vessels with the origin is at the geometric center. If the origin is at the center of mass, $x_G = 0$. Next, if the vessel is reasonably balanced with regard to forward and aft areas with respect to the origin, the terms $\{N_{\dot v}, Y_{\dot r}, N_v, Y_r\}$ take very small values in comparison with the others. To wit, the added mass term $-Y_{\dot v}$ is of the order of the vessel's material mass m, and similarly $N_{\dot r} \simeq -I_{zz}$. Both Y_v and N_r take large negative values. Linear drag and rotational drag are significant also; these are the terms Y_v and N_r, both large and negative. The denominator for A's components reduces to $(m - Y_{\dot v})(I_{zz} - N_{\dot r})$, and

$$\begin{aligned} A_{11} &= \frac{Y_v}{m - Y_{\dot v}} < 0 \\ A_{22} &= \frac{N_r}{I_{zz} - N_{\dot r}} < 0. \end{aligned}$$

Hence the first condition for stability is met: $-A_{11} - A_{22} > 0$. For the second condition, since the denominators of the A_{ij} are identical, we have only to look at the numerators. For stability, we require

$$(I_{zz} - N_{\dot r})Y_v(m - Y_{\dot v})N_r \tag{50}$$

$$-[N_{\dot v}Y_v + (m - Y_{\dot v})N_v][-(I_{zz} - N_{\dot r})(mU - Y_r) + Y_{\dot r}N_r] > 0.$$

The first term is the product of two large negative and two large positive numbers. The second part of the second term contains mU, which has a large positive value, generally making stability critical on the (usually negative) N_v. When only the largest terms are considered for a vessel, a simpler form is common:

$$C = Y_v N_r + N_v(mU - Y_r) > 0. \tag{51}$$

C is called the vessels *stability parameter*. The terms of C compete, and yaw/sway stability depends closely on the magnitude and sign of N_v. Adding more surface area aft drives N_v more positive, increasing stability as expected. Stability can also be improved by moving the center of gravity forward. Nonzero x_G shows up as follows:

$$C = Y_v(N_r - m x_G U) + N_v(mU - Y_r) > 0. \tag{52}$$

Since N_r and Y_v are both negative, positive x_G increases the (positive) influence of C's first term.

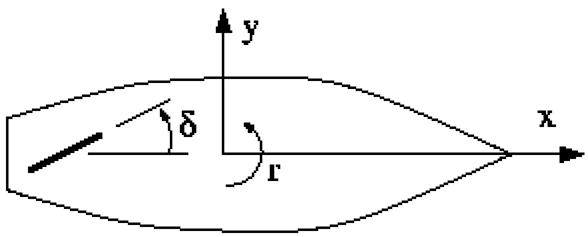

Figure 3: *Convention for positive rudder angle in the vessel reference system.*

4.4 Basic Rudder Action in the Sway/Yaw Model

Rudders are devices which develop large lift forces due to an angle of attack with respect to the oncoming fluid. As in our discussion of lift on the body, the form is as follows: $L = \frac{1}{2}\rho A U^2 C_l(\alpha)$, where α is the angle of attack. The lift coefficient C_l is normally linear with α near $\alpha = 0$, but the rudder stalls when the angle of attack reaches a critical value, and thereafter develops much less lift. We will assume that α is small enough that the linear relationship applies:

$$C_l(\alpha) = \left.\frac{\partial C_l}{\partial \alpha}\right|_{\alpha=0} \alpha. \tag{53}$$

Since the rudder develops force (and a small moment) far away from the body origin, say a distance l aft, the moment equation is quite simple. We have

$$Y_\alpha = \frac{1}{2}\rho A \left.\frac{\partial C_l}{\partial \alpha}\right|_{\alpha=0} U^2 \tag{54}$$

$$N_\alpha = -\frac{1}{2}\rho A \left.\frac{\partial C_l}{\partial \alpha}\right|_{\alpha=0} l U^2. \tag{55}$$

Note the difference between the rudder angle expressed in the body frame, δ, and the total angle of attack α. Angle of attack is influenced by δ, as well as v/U and lr. Thus, in tank testing with $v = 0$, $\delta = \alpha$ and $N_\delta = N_\alpha$, etc., but in real conditions, other hydrodynamic derivatives are augmented to capture the necessary effects, for example N_v and N_r. Generally speaking, the hydrodynamic characteristics of the vessel depend strongly on the rudder, even when $\delta = 0$. In this case the rudder still opposes yaw and sway perturbations and acts to stabilize the vessel.

A positive rudder deflection (defined to have the same sense as the yaw angle) causes a negative yaw perturbation, and a very small positive sway perturbation.

4.4.1 Adding Yaw Damping through Feedback

The stability coefficient C resulting from the addition of a control law $\delta = k_r r$, where $k_r > 0$ is a feedback gain, is

$$C = Y_v(N_r - m x_G U + k_r N_\delta) + N_v(mU - Y_r - k_r Y_\delta). \quad (56)$$

Y_δ is small positive, but N_δ is large and negative. Hence C becomes more positive, since Y_v is negative.

Control system limitations and the stalling of rudders make obvious the fact that even a very large control gain k_r cannot completely solve stability problems of a poorly-designed vessel with an inadequate rudder. On the other hand, a vessel which is overly stable ($C >> 0$ with no rudder action) is unmaneuverable. A properly-balanced vessel just achieves stability with zero rudder action, so that a reasonable amount of control will provide good maneuvering capabilities.

4.4.2 Heading Control in the Sway/Yaw Model

Considering just the yaw equation of motion, i.e., $v = 0$, with a rudder, we have

$$(I_{zz} - N_{\dot{r}})\ddot{\phi} + (m x_G U - N_r)\dot{\phi} = N_\delta \delta. \quad (57)$$

Employing the control law $\delta = k_\phi \phi$, the system equation becomes a homogeneous, second-order ODE:

$$(I_{zz} - N_{\dot{r}})\ddot{\phi} + (m x_G U - N_r)\dot{\phi} - N_\delta k_\phi \phi = 0. \quad (58)$$

Since all coefficients are positive (recall $N_\delta < 0$), the equation gives a stable θ response, settling under second-order dynamics to $\theta(\infty) = 0$. The control law $\delta = k_\phi(\phi - \phi_{desired}) + k_r r$ is the basis for heading autopilots, which are used to track $\phi_{desired}$. This use of an error signal to drive an actuator is in fact the essence of feedback control. In this case, we require *sensors* to obtain r and ϕ, a *controller* to calculate δ, and an *actuator* to implement the corrective action.

4.5 Response of the Vessel to Step Rudder Input

4.5.1 Phase 1: Accelerations Dominate

When the rudder first moves, acceleration terms dominate, since the velocities are zero. The equation looks like this:

$$\begin{bmatrix} m - Y_{\dot{v}} & m x_G - Y_{\dot{r}} \\ m x_G - N_{\dot{v}} & I_{zz} - N_{\dot{r}} \end{bmatrix} \begin{Bmatrix} \dot{v} \\ \dot{r} \end{Bmatrix} = \begin{Bmatrix} Y_\delta \\ N_\delta \end{Bmatrix} \delta. \quad (59)$$

Since $Y_{\dot{r}}$ and $m x_G$ are comparatively small in the first row, we have

$$\dot{v}(0) = \frac{Y_\delta \delta}{m - Y_{\dot{v}}}, \quad (60)$$

and the vessel moves to the left, the positive v-direction. The initial yaw is in the negative r-direction, since $N_\delta < 0$:

$$\dot{r}(0) = \frac{N_\delta \delta}{I_{zz} - N_{\dot{r}}}. \quad (61)$$

The first phase is followed by a period (**Phase 2**), in which many terms are competing and contributing to the transient response.

4.5.2 Phase 3: Steady State

When the transients have decayed, the vessel is in a steady turning condition, and the accelerations are zero. The system equations reduce to

$$\begin{Bmatrix} v \\ r \end{Bmatrix} = \frac{\delta}{C} \begin{Bmatrix} (m x_G U - N_r)Y_\delta + (Y_r - mU)N_\delta \\ N_v Y_\delta - Y_v N_\delta \end{Bmatrix}. \quad (62)$$

Note that the denominator is the stability parameter. The steady turning rate is thus approximated by

$$r = -\frac{Y_v N_\delta}{C}\delta. \qquad (63)$$

With $C > 0$, the steady-state yaw rate is negative. If the vessel is unstable ($C < 0$), it turns in the opposite direction than expected. This turning rate equation can also be used to estimate turning radius R:

$$R = \frac{U}{r} = \frac{UC}{-Y_v N_\delta \delta}. \qquad (64)$$

The radius goes up directly with C, indicating that too stable a ship has poor turning performance. We see also that increasing the rudder area increases N_δ, decreasing R as desired. Increasing the deflection δ to reduce R works only to the point of stalling.

4.6 Summary of the Linear Maneuvering Model

We conclude our discussion of the yaw/sway model by noting that

1. The linearized sway/yaw dynamics of a surface vessel are strongly coupled, and they are independent of the longitudinal dynamics.
2. The design parameter C should be slightly greater than zero for easy turning, and `` hands-off'' stability. The case $C < 0$ should only be considered under active feedback control.
3. The analysis is valid only up to small angles of attack and turning rates. Very tight maneuvering requires the nonlinear inertial components and hydrodynamic terms. Among other effects, the nonlinear equations couple surge to the other motions, and the actual vessel loses forward speed during maneuvering.

4.7 Stability in the Vertical Plane

Stability in the horizontal plane changes very little as a function of speed, because drag and lift effects generally scale with U^2. This fact is *not* true in the vertical plane, for which the dimensional weight/buoyancy forces and moments are invariant with speed. For example, consider the case of heave and pitch, with $x_G = 0$ and no actuation:

$$m\left(\frac{\partial w}{\partial t} - Uq\right) = Z_{\dot{w}}\dot{w} + Z_w w + Z_{\dot{q}}\dot{q} + Z_q q + (B - W) \qquad (65)$$

$$I_{yy}\frac{dq}{dt} = M_{\dot{w}}\dot{w} + M_w w + M_{\dot{q}}\dot{q} + M_q q - Bl_b \sin\theta. \qquad (66)$$

The last term in each equation is a hydrostatic effect induced by opposing net buoyancy B and weight W. l_b denotes the vertical separation of the center of gravity and the center of buoyancy, creating the so-called righting moment which nearly all underwater vehicles possess. Because buoyancy effects do not change with speed, the dynamic properties and hence stability of the vehicle may change with speed.

[2] Note that the linearized heave/pitch dynamics of a submarine do depend on the pitch angle; this topic will be discussed later.