Electromagnetically Induced Transparency and Electron Spin Dynamics using Superconducting Quantum Circuits

by
Kota Murali

Submitted to the Department of Electrical Engineering and Computer Science in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Electrical Engineering at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY

February 2006

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Author .................................................................
Department of Electrical Engineering and Computer Science
January, 2006

Certified by ............................................................
Terry P. Orlando
Professor of Electrical Engineering
Thesis Supervisor

Certified by ............................................................
David G. Cory
Professor of Nuclear Science and Engineering
Thesis Supervisor

Read by ..............................................................
Mildred S. Dresselhaus
Institute Professor of Physics and Electrical Engineering
Thesis Reader

Accepted by ..........................................................
Arthur C. Smith
Chairman, Department Committee on Graduate Students
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Abstract

This thesis is an exploration on superconducting devices, quantum optics, and magnetic resonance. Superconductive quantum circuits (SQC) comprising mesoscopic Josephson junctions can exhibit quantum coherence amongst their macroscopically large degrees of freedom. They feature quantized flux and/or charge states depending on their fabrication parameters, and the resultant quantized energy levels are analogous to the quantized internal levels of an atom. This thesis builds on the SQC-atom analogy to quantum optical effect associated with atoms, known as Electromagnetically Induced Transparency (EIT). An EIT (denoted as S-EIT) based technique has been proposed to demonstrate microwave transparency using a superconductive quantum circuit exhibiting two metastable states (e.g., a qubit) and a third, shorter-lived state (e.g., the readout state). This technique is shown to be a sensitive probe of decoherence, besides leading to the prospects of observing other interesting quantum optical effects like AC-stark effect in SQCs. The second part of this thesis concerns a novel technique for sensitive detection of magnetic resonance using SQC-based resonance circuits. Superconducting quantum circuits are also known to sensitive detectors of magnetic fields. In particular, the effect of electron spin resonance signal on a Superconducting QUantum Interference Device (SQUID) based non-linear resonant circuit is derived. The electron spin resonance signal propagates as a non-linear behavior of the SQUID voltage, that can sensitively detect extremely small (less than $10^{-5}\Phi_0$) electron spin resonance signals.

Thesis Supervisor: Terry P. Orlando
Title: Professor of Electrical Engineering

Thesis Supervisor: David G. Cory
Title: Professor of Nuclear Science and Engineering
Dedicated to my parents and my sister
Acknowledgments

Writing the acknowledgements section is as important as writing the thesis itself as this section holds closest to my heart. It gives me an opportunity to thank everyone who has made this thesis possible.

First of all, I would like to thank both my thesis supervisors, Profs. Terry Orlando and David Cory. I would like to thank Prof. Orlando for giving me the freedom to venture into new areas of research that has made working with him very rewarding. His encouragement to seek new topics to work on, and also infusing important ideas whenever needed has made the graduate experience unbelievable. Thank you Terry for providing a great and creative environment to work, and for all the care and attention that I will never forget.

I would like to thank my supervisor, Prof. David Cory, for readily agreeing to guide me on the spin resonance detection using SQUIDs. From David, I learnt to be bold in seeking new directions and push new limits. His enthusiasm for research is highly motivating and inspiring. His ability to understand the intricacies of both theoretical and experimental aspects of quantum information have kept my research relevant to the real world. We started out with the possibility of exploring quantum coherence transfer in a spin-SQUID system but realizing the relevance and importance to connecting to experiments, it was his insight that helped us come up with novel techniques for detecting spin resonance using SQUID. Working for Terry and David has been one of the most enjoyable moments for me, something that I will cherish forever.

I would like to thank Prof. Millie Dresselhaus for her constant encouragement and support ever since the start of grad school. She has been a great source of inspiration to me, both academically and personally. She has always kept her doors open to me for advice and discussions in spite of her busy schedule.

I have benefitted a lot from a close collaboration with Dr. Zac Dutton of Naval Research Laboratory. We met at an APS conference and after just one meeting we became collaborators and friends. Zac’s interest in topics ranging from atom
physics to condensed matter is amazing. His methodical approach to research and also emphasis on the importance to every detail in the EIT work has been extremely valuable. His visits to Cambridge have been extremely fruitful and also thank him for hosting me at Washington D.C. while I was visiting National Institute of Standards and the Naval Research Laboratory.

I would like to thank Dr. Will Oliver of Lincoln Laboratory for collaborating on the EIT project. Many thanks to him for insightful discussion on EIT work and for setting a high standard both in science and writing skills. I would like to thank all the Orlando group members: David Berns, Bill Kaminsky, Janice Lee, Sergio Velanzuela for being wonderful labmates. Having had a chance to share an office with each of them has been wonderful. David has been instrumental in keeping the whole group in good cheer all the time. Janice has taught how to be a strong willed and also to keep going at all times, our tea talks have been wonderful. I also acknowledge great discussions with Sergio and Bill on various topics. I also thank Dr. Dan Nakada and Dr. Jon Habif for being wonderful friends and groupmates while they were at MIT. Also, thanks to Ann Orlando for taking care of the Orlando group just like her family.

I would also like to thank all my groupmates at the Cory group: Ben, Anatoly, Michael, Jamie, Cecilia, Dima, Paola, Sid, Troy, Johnathan, Sekhar, and Karen. They have been wonderful people to know, their enthusiasm for research is quite motivating. In this group, I have always felt like being a part of a family and made research even more enjoyable. Thanks to Ben and Anatoly for always willing to hear any “small” exciting result I have had and for always encouraging me. Thanks to Johnathan for never refusing to help when it came to fixing problems with computers, network etc. and for discussions on spin detection and ESR. Ben and Ceci have been the keys to bringing the whole group together and creating a family like atmosphere. Learning to balance work and play is one of the best lessons I have learnt from them, thank you Ben and Ceci. Thanks to Michael and Jamie for being wonderful people to share the office with and also discussions on various topics. I also acknowledge insightful discussions with Sekhar. Thanks to Paola, Dima, and Sid for being helpful on various occasions. I would also like to thank my friends, Rishi, Ryan, Anuja, Hui, Liang,
Tzu-Ching, and Mridula for making the final year really exciting. They are some people I could rely on any time and for any thing. Many thanks to the members of the badminton club for helping me keep up my favorite sport.

I would like to thank my parents and sister for their encouragement. I owe much of what I have achieved and accomplished to them. After being away from them for over four years has made me realize how important they are to me. It is because of their love and affection that I was able to complete this thesis successfully.

I would also like to acknowledge the support for the thesis work in part by the AFOSR grant No. F49620-01-1-0457 under the Department of Defense University Research Initiative in Nanotechnology (DURINT).
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Chapter 1

Introduction and Overview

Ever since the bold and far-sighted proposal of Feynman on using quantum computers for simulating quantum systems [1, 2], there has been an urge to build quantum information processing systems. Many interesting ideas and proposals based on superconductors, electron and nuclear spin magnetic resonance, quantum dots, ion-traps and photons are being envisaged for building a quantum information processor. While these techniques have been of independent pursuit and there has been extraordinary progress in these areas, it seems that it would be best to build systems that would take advantage of the unique features each of these systems offer.

This thesis is an exploration on superconducting devices, quantum optics, and magnetic resonance. To observe quantum optical effects, one would need systems that behave like atoms. It has been shown that superconductive quantum circuits (SQCs) comprising mesoscopic Josephson junctions can exhibit quantum coherence amongst their macroscopically large degrees of freedom [3]. They feature quantized flux and/or charge states depending on their fabrication parameters, and the resultant quantized energy levels are analogous to the quantized internal levels of an atom [4, 5, 6, 7, 8, 9, 10].

This thesis analyzes quantum optical and magnetic resonance effects on Superconducting Quantum Circuits (SQCs). It extends the SQC-atom analogy to another quantum optical effect associated with atoms: electromagnetically induced transparency [11, 12]. In particular, a quantum optical effect, Electromagnetically In-
duced Transparency (EIT), is proposed in SQC [13]. We propose the demonstration of microwave transparency using a superconductive analog to EIT (denoted S-EIT) in a superconductive circuit exhibiting two metastable states (e.g., a qubit) and a third, shorter-lived state (e.g., the readout state). This technique would be shown to be a sensitive probe of decoherence, besides the prospects of observing other interesting quantum optical effects like AC-stark effect in SQCs [14].

As is well known, SQCs are the most sensitive detectors of magnetic fields [15, 16, 17, 18]. Hence, for increasing the detection sensitivity of magnetic resonance signals it has been proposed to use SQC-based resonance circuits. In particular, the effect of electron spin resonance signal on a Superconducting Quantum Interference Device (SQUID) based non-linear resonant circuit is rigoursly derived. The electron spin resonance signal propagates on the SQUID based circuit as a non-linear behavior in the output voltage of the SQUID that is readily detectable.

1.1 Outline

The thesis is organized as follows.

Chapter 2 introduces the basic concepts of superconducting quantum circuits. Starting with a discussion on Josephson effect and deriving the current and voltage relations, we analyze a simple SQC known as the Superconducting Quantum Interference Device (SQUID) in context of a magnetic sensor. Finally, an “artifical” atom like SQC known as the persistent current qubit is introduced and its Hamiltonian is derived. This would set a stage to discuss quantum optical effects in a superconducting “artificial atom” circuit.

Chapter 3 introduces the concept of Electromagnetically Induced Transparency, first in atoms and its interesting applications for quantum communication. Then an analogy of EIT in atoms is derived in SQCs. The manifestation of EIT in SQCs is discussed, followed by its interesting application as probe of decoherence and its estimation using Bloch equations is presented.

Chapter 4 analyzes EIT in detail. A theoretical model of analyzing the SQC in
the presence of realistic effects like decoherence, tunneling, and multi-level cross talk is analyzed. Analytic expressions for estimation of these effects are presented. Also, simulation results through modeling of the SQC with Bloch equations are presented to show the use of EIT as a tool to distinguish each of these effects.

Chapter 5 analyzes the magnetic resonance effects on a SQUID-based resonant circuit. A theoretical model for analyzing the effect of external magnetic field at rf on a SQUID is presented. The necessary equations to describe the non-linear behavior of the output voltage of the SQUID are derived from the theoretical model. Simulation results of the same are presented. The use of SQUID based resonant circuits for sensitive detection of rf-magnetic field is presented.

Finally, we conclude with a summary of the results presented in the thesis. This also sets a stage for a discussion of the insights obtained from the results for future work.

1.2 Highlights of the Thesis

The main results of the thesis are summarized below.

1. Superconductive quantum circuits (SQCs) comprise quantized energy levels that may be coupled via microwave electromagnetic fields. Described in this way, one may draw a close analogy to atoms with internal (electronic) levels coupled by laser light fields. Here, I present a superconductive analog to electromagnetically induced transparency (S-EIT) that utilizes SQC designs of present day experimental consideration. I discuss how S-EIT can be used to establish macroscopic coherence in such systems and, thereby, utilized as a sensitive probe of decoherence. This work has enabled the integration of quantum optical effects in solid state systems.

2. Another highlight of the thesis is electromagnetically-induced transparency (EIT) in a superconducting quantum circuit (SQC) in the context of various decoherence effects. Here, I extend the results from the work published in 1 by exploring the effects of imperfect dark-state preparation and specific decoherence mechanisms (population loss via tunneling, pure dephasing, and incoherent population exchange).
These effects are some of the important effects that would be crucial to realization of EIT through experiments and also enhance our understanding of the physics of EIT in SQC systems significantly and also in bringing quantum optics and SQCs together. I obtain analytic expressions for the slow loss rate, with coefficients that depend on the particular decoherence mechanisms, thereby providing a means to probe, identify, and quantify various sources of decoherence with EIT.

3. In a quest for sensitive detection of magnetic resonance, we propose novel techniques for efficient detection of electron spin using SQUID based resonant circuits. A theoretical model based on Josephson equations has been proposed to describe the non-linear effects of RF-magnetic fields on a SQUID. The theoretical model is used to obtained simulation results that predict the behavior of the SQUID in RF-magnetic fields.
Chapter 2

Superconducting Quantum Circuits

With the advent of intriguing capabilities of quantum systems for computing, there is a flurry of activity for building a quantum computer [1, 2, 19, 20]. Many interesting proposals based on NMR [21], ion-trap [22], silicon [23], and superconducting quantum circuits [24, 25] have been proposed to build a quantum computer. This thesis focusses on a scalable approach to building quantum information systems - superconducting quantum circuits (SQCs). Classically, these circuits are known to be the most sensitive magnetic detectors (known as the Superconducting Quantum Interference Device, SQUID) [15, 16]. Also these circuits behave like quantum mechanical objects at low-temperatures when well-isolated from the environment and show “artificial” atom-like properties [3, 26]. An example of an SQC that behaves like an atom is a persistent current qubit (pc-qubit). The use of a SQUID and the pc-qubit would be extensively discussed in this thesis in the context of rf-magnetic detection and also for quantum optical effects as an approach to a sensitive probe of decoherence.

2.1 Introduction

The quest for building a quantum information processor (QIP) has led to many quantum systems based on microscopic degrees of freedom like spin (electron or nuclear),
dipole transitions of ions etc. These designs are well thought of, given their isolated nature from the environment. But their isolation from the environment presents a problem - it is difficult to make these systems interact fast enough, without introducing decoherence, for a scalable approach to building a QIP.

Are there macroscopic quantum systems, with macroscopic degrees of freedom that can be easily manipulated and also be scalable? This chapter presents an approach that is based on macroscopic circuit elements, based on either the charge or the flux degree of freedom of a superconducting quantum circuit. Given that these systems are macroscopic in nature, they also present a challenge in isolating them from the environment while retaining their ability of ease of manipulation. Some of the basic features of the SQCs that make them attractive for quantum applications are:

1. Ultra-low dissipative systems - For any system to behave quantum mechanically, it is essential for the system to have ultra-low dissipation. As losing even a quantum of energy gives rise to decoherence, for this reason it is required that any quantum circuit have a low resistance. Hence, superconductors would be an ideal choice for their resistance-less behavior and therefore ideal for quantum circuits.

2. Ultra-low noise systems - Typical thermal fluctuations are in the $kT$ range, which means that the quantum circuit needs to work in a regime that is much less than $kT$ fluctuations and the smallest energy separation between the qubit states be much larger than $kT$ for efficient manipulation in its operating frequency. It turns out that the typical operating range of around 1-20 GHz, which requires that the temperature of the devices should be in the mK range (the temperature equivalent of 1K is about 20 GHz) for most efficient manipulation. This is a requirement that can be readily met with a dilution refrigerator.

3. Low-dissipative and non-linear circuit elements - Quantum computing requires both linear and non-linear circuit elements while being ultra-low dissipative. A well-known electronic element that is both non-linear and non-dissipative at low temperatures is the superconducting tunnel junction (the Josephson junction). As we would see in the sections to follow that the tunneling of Cooper pairs creates a non-linear inductance known as Josephson inductance.
2.2. THE JOSEPHSON EFFECT AND RELATIONS

The above requirements are met by superconducting systems that we are going to study in this chapter and this would set a stage for the work in the thesis. In particular, we study the Superconducting Quantum Interference Device (SQUID) and a persistent current qubit (pc-qubit) that are used as an rf magnetic detector and as artificial atoms.

2.2 The Josephson Effect and Relations

We review the basic aspects of the physics of SQCs. Let us start with the Josephson effect in a Josephson junction [29], which is one of the basic and integral components of superconducting devices. A Josephson junction consists of an insulating layer sandwiched between two superconducting layers.

The electrons in the superconductor that are responsible for resistance-less behavior or superconductivity are all in the same macroscopic state, which can be identified as a complex order parameter $|\psi\rangle$. This macroscopic wavefunction of the Cooper pairs on a superconducting layer can be written as,

$$|\psi\rangle = \sqrt{n_s}e^{i\phi}$$

(2.1)

where, $n_s$ is the density of the superconducting charge carriers. For a thin insulating layer, the electrons pairs can tunnel through the barrier. The wavefunction of the electrons in the left and right electrodes in Fig. 2-1 can be written as $|\psi_1\rangle = \sqrt{n_1}e^{i\phi_1}$ and $|\psi_2\rangle = \sqrt{n_2}e^{i\phi_2}$, respectively. The Hamiltonian of the system is given by,

$$H = H_1 + H_2 + H_t$$

(2.2)

where, $H_1 = E_1|\psi_1\rangle\langle\psi_1|$, $E_1$ is the energy eigenvalue of the state $|\psi_1\rangle$ and $E_2$ is the energy eigenvalue of $|\psi_2\rangle$ and $H_2 = E_2|\psi_2\rangle\langle\psi_2|$, and $H_t = P(|\psi_1\rangle\langle\psi_2| + |\psi_2\rangle\langle\psi_1|)$. $P$ is the tunneling matrix element connecting the two wavefunctions. From the Schrodinger
Figure 2-1: A Superconducting tunnel junction consists of a thin insulating barrier sandwiched between two superconductors. For a sufficiently thin barrier thickness $d$, the macroscopic wavefunctions $|\psi_1\rangle$ and $|\psi_2\rangle$ of the two superconductors interact.

Wave equation, we have,

\[
\begin{align*}
\hbar \frac{d\psi_1}{dt} &= E_1 \psi_1 + P\psi_2 \\
\hbar \frac{d\psi_2}{dt} &= E_2 \psi_2 + P\psi_1
\end{align*}
\]

(2.3)  
(2.4)

If a voltage $V$ is applied across the junction, then $E_1 - E_2 = 2eV$, $e$ is the electron charge. Setting the voltage at the center of the junction to zero, gives, $E_1 = eV$ and $E_2 = -eV$. Then, Eq. 2.3 and 2.4 can be written as,

\[
\begin{align*}
\hbar \frac{d\psi_1}{dt} &= eV \psi_1 + P\psi_2 \\
\hbar \frac{d\psi_2}{dt} &= -eV \psi_2 + P\psi_1
\end{align*}
\]

(2.5)  
(2.6)

From Eq. 2.1, we have,

\[
\hbar \left( \frac{1}{2} \frac{dn_1 e^{i\phi_1}}{dt} \frac{1}{\sqrt{n_1}} + i \sqrt{n_1} e^{i\phi_1} \frac{d\phi_1}{dt} \right) = eV \sqrt{n_1} e^{i\phi_1} + P \sqrt{n_2} e^{i\phi_2}
\]

(2.7)
Equating the real and imaginary parts and solving for \( \frac{dn_1}{dt} \) and \( \frac{d\phi_1}{dt} \), we get,

\[
\frac{d\phi_1}{dt} = -\frac{P}{\hbar} \sqrt{\frac{n_2}{n_1}} \cos(\phi_2 - \phi_1) - \frac{eV}{\hbar} \tag{2.8}
\]

\[
\frac{dn_1}{dt} = \frac{2P}{\hbar} \sqrt{n_2 n_1} \sin(\phi_2 - \phi_1) \tag{2.9}
\]

Similarly, equations for the density \( n_2 \) and the phase \( \phi_2 \) can be written as,

\[
\frac{d\phi_2}{dt} = -\frac{P}{\hbar} \sqrt{\frac{n_2}{n_1}} \cos(\phi_2 - \phi_1) + \frac{eV}{\hbar} \tag{2.10}
\]

\[
\frac{dn_2}{dt} = -\frac{2P}{\hbar} \sqrt{n_2 n_1} \sin(\phi_2 - \phi_1) \tag{2.11}
\]

Hence,

\[
\frac{d\phi_2}{dt} - \frac{d\phi_1}{dt} = \frac{2eV}{\hbar} \tag{2.12}
\]

and

\[
\frac{dn_1}{dt} = \frac{2P}{\hbar} \sqrt{n_2 n_1} \sin(\varphi) \tag{2.13}
\]

The pair current density is given by \( J = \frac{2e}{A_j} \frac{dn_1}{dt} = \frac{(-2e)}{A_j} \frac{dn_2}{dt} \), where \( A_j \) is the area of the junction and \( e \) the electron charge. This gives,

\[
J = J_c \sin \varphi \tag{2.14}
\]

where, \( \frac{(2e)^2 P}{A_j h} \sqrt{n_1 n_2} \) is the critical current density \( J_c \) and \( \varphi = \phi_2 - \phi_1 + \frac{(2e)}{h} \oint A \cdot dl \). Here \( \phi \) is the phase of the wavefunction as in Eq. 2.1 and \( A \) is the magnetic vector potential, and \( \varphi \) is the gauge-invariant phase across the junction. With the above, we finally arrive at the second Josephson relation for voltage that is give as,

\[
V = \frac{\Phi_0}{2\pi} \frac{d\phi}{dt} \tag{2.15}
\]

Equations (2.14) and (2.15) are known as the Josephson relations for current and
voltage for a junction.

Another important characteristic of the Josephson junction, used extensively in the thesis, is the Josephson inductance. This kind of inductance is highly non-linear and makes efficient spin detection possible. Let us understand this phenomena. From the Josephson relation, we have,

\[
\frac{dI}{dt} = I_c \cos \varphi \frac{d\varphi}{dt} \tag{2.16}
\]

\[
= \frac{2\pi V}{\Phi_0} \cos(\varphi)I_c \tag{2.17}
\]

From the above equations, one can define a Josephson inductance, \( L_J \) such that \( V = L_J dI/dt \). Hence, \( L_J = \Phi_0/(2\pi \cos \varphi I_c) \). We see that Josephson inductance is inversely proportional to both the critical current \( I_c \) and to the cosine of the gauge invariant phase across the junction.

With these basic relations, we are ready to understand the basics of classical and quantum circuits made of superconducting systems.

### 2.3 A Superconducting Quantum Interference Device

Let us consider a circuit with two Josephson junctions and a magnetic field threading the two arms of the circuit as shown in the Fig. 2.3. For the gauge invariant phase change \( \Delta \phi \) around the loop to be single-valued, we have,

\[
\Delta \phi = 2\pi n \tag{2.18}
\]

\[
= \sum_i \Delta \varphi_i + \frac{2e}{\hbar} \oint A \cdot dl \tag{2.19}
\]
2.3. A SUPERCONDUCTING QUANTUM INTERFERENCE DEVICE

Figure 2-2: (a) A schematic of the dc-SQUID. The phase across each junction is given as, $\varphi$, while the critical current is $I_c$. The external magnetic field is indicated in terms of the flux $\Phi_{sq}$. (b) The modulation of the SQUID critical current as a function of the applied flux. The period of the critical current is a flux quantum, $\Phi_0$. 
where, $\Delta \varphi_i$ are the phase drops across the $i$th junctions of the loop, and $A$ is the magnetic vector potential. We can rewrite this equation,

$$2\varphi_m + \frac{2e}{\hbar} \oint B \cdot dA = 2\pi n \quad (2.20)$$

$$\varphi_m + \frac{\Phi_{sq}}{\Phi_0} = \pi n \quad (2.21)$$

where, $\varphi_m = (\varphi_2 - \varphi_1)/2$. Equation 2.21 is known as the fluxoid quantization condition. From the Josephson current relation, we have,

$$I_b = I_1 + I_2 \quad (2.22)$$

$$= I_{c1} \sin \varphi_1 + I_{c2} \sin \varphi_2 \quad (2.23)$$

When, the critical currents obey, $I_{c1} = I_{c2} = I_c$, and from the fluxoid quantization, we have,

$$I_b = 2I_c |\cos(\frac{\pi \Phi_{sq}}{\Phi_0})| \quad (2.24)$$

Thus for a two junction devices, like the dc-SQUID, the currents through the arms of the circuit interfere. The result is that the critical current of the dc-SQUID is modulated by the external flux. This is reminiscent of Young’s double slit experiment, and hence, appropriately named the Superconducting Quantum Interference Device (SQUID). It should be noted the word SQUID would implicitly mean the dc-SQUID throughout the thesis.

We can plot the critical current of the dc-SQUID as a function of the external flux. We see that the period of the critical current modulation is the flux quantum or $\Phi_0$. From this, we can clearly see that small changes in the external flux (less than a fraction of $\Phi_0$) leads to a large change in the critical current of the SQUID. A typical size of the SQUID is in the $\mu m^2$ regime, which means that ultra-small changes in magnetic fields (of the order of $\mu G$) would lead to a detectable change in the critical current of the SQUID, hence they are the most sensitive detectors of magnetic field.
ever known [15, 16].

Having understood the function of a simple device like a SQUID and its application for sensitive magnetic detection, we can analyze a circuit, that behaves like a quantum mechanical system, a pc-qubit. Also, the dc-SQUID is used as a magnetometer for the detection of the pc-qubit state, which is described in the next section.

2.4 Artificial Atom: A PC-qubit

At a macroscopic level, electrical circuits are often represented by collective electronic degrees of freedom such as currents and voltages that are treated as classical variables. In quantum circuits, these variables are treated as quantum variables. For example, the charge or voltage on the plates of the capacitors is thought of as a number in classical circuits, but in quantum circuits, the charge (voltage) on the capacitor is represented by a wavefunction [4, 5]. The charge can be a superposition of both positive and negative charge states. Similarly, the current in a loop can be flowing in both the clockwise or anticlockwise states at the same time. These macroscopic effects were proposed in superconducting quantum circuits [3, 28].

As shown in Fig. 2-3(a), the pc-qubit [24, 25] is a superconducting loop interrupted by three Josephson junctions indicated by X. The Hamiltonian of the system can be obtained from the basic Josephson relations [29]. We first determine the Hamiltonian of a single tunnel junction which would be the basis for deriving the Hamiltonian of the qubit. The Josephson junction can be modeled as a non-linear inductor in parallel with a capacitor $C$ (the junctions are regarded as dissipationless). The charging (capacitive) energy is due to the sandwich structure of the junction and is the energy associated with a capacitor $C$,

$$U_c = \frac{1}{2} CV^2 = \frac{Q^2}{2C}$$  \hspace{1cm} (2.25)

The Josephson energy is calculated from $\int_0^t VIdt$. Using the Josephson relation $I =$
Figure 2-3: (a) Schematic of a Persistent Current Qubit inductively coupled to a DC SQUID. Here X indicate a Josephson junction. (b) Two lowest energy levels of the persistent current loop as a function of applied flux $f$ for appropriate design parameters. Classically (dashed lines), we expect a degeneracy point at $f = \frac{1}{2}$. Quantum mechanically (colored lines), the energy levels are separated by a gap $\Delta$. 
I \_c \sin \varphi \text{ and the voltage-frequency relation } \frac{d\varphi}{dt} = \frac{2\pi}{\Phi_0} v, \text{ this yields:}

\[ U_J = E_J(1 - \cos \varphi) \quad (2.26) \]

where \( E_J = \frac{\Phi_0 I}{2\pi} \). The pc-qubit is a loop with three junctions. Hence the total Josephson energy due to the three junctions is given by,

\[ U_J = \sum_i E_{J_i}(1 - \cos \varphi_i) \quad (2.27) \]

where, \( i \) refers to the \( i \)th junction. The capacitive energy of the junctions is given by,

\[ T = \frac{1}{2} \sum_i C_i V_i^2 \quad (2.28) \]

\[ = \frac{1}{2} \sum_i C_i \frac{\Phi_0^2}{2\pi} \varphi_i^2 \quad (2.29) \]

The pc-qubit is designed such that two of the junctions are equal in size, while the third one is scaled by a factor \( \alpha \),

\[ E_{J1} = E_{J2} = E_J, \text{ and } E_{J3} = \alpha E_J \quad (2.30) \]

and

\[ C_1 = C_2 = C, \text{ and } C_3 = \alpha C \quad (2.31) \]

where \( \alpha \) is the ratio of the area of the smaller junction to the bigger junction. From the fluxoid quantization condition, we have,

\[ \varphi_1 - \varphi_2 + \varphi_3 = -2\pi f \quad (2.32) \]

where, \( f = \Phi_{ext}/\Phi_0 \). Hence, the potential energy of the system can be written as,

\[ U_J = E_J(2 + \alpha - 2 \cos \varphi_p \cos \varphi_m - \alpha \cos 2\pi f + 2\varphi_m) \quad (2.33) \]
Figure 2-4: (a) A three-dimensional potential energy plot of the pc-qubit. (b) The double-well potential energy level diagram along the $\varphi_m$ direction.
Hence, the total energy of the system can be written as,

\[ H = \frac{p_p^2}{2M_p} + \frac{p_m^2}{2M_m} + E_J(2 + \alpha - 2 \cos \varphi_p \cos \varphi_m - \alpha \cos 2\pi f + 2\varphi_m) \]  \hspace{1cm} (2.34)

where, the first two terms of the equation are the kinetic energy terms, and \( \varphi_p = (\varphi_1 + \varphi_2)/2 \), and \( \varphi_m = (\varphi_1 - \varphi_2)/2 \), and \( P_p = M_p \dot{\varphi}_p \), and \( P_m = M_m \dot{\varphi}_m \). Here, \( M_p = \frac{\Phi_0^2}{2\pi} \), and \( M_m = \frac{\Phi_0^2}{2\pi} \). Quantizing the Hamiltonian gives flux states in a two-dimensional potential well [24, 25]. It should be noted that the phase and the momentum equivalent defined by \( P \), obey the commutation relation \([\dot{\varphi}, \dot{P}] = i\hbar\).

Where the phase \( \varphi \) and its conjugate variable \( P \) are replaced by the quantum operator equivalent to a quantum Harmonic oscillator:

\[ \varphi \rightarrow \dot{\varphi} \]  \hspace{1cm} (2.35)

\[ P \rightarrow \dot{P} \]  \hspace{1cm} (2.36)

Fig. 2-4 shows the potential energy landscape of the pc-qubit and the double well potential of a pc-qubit along the \( \varphi_m \) direction. For appropriate parameters we can design both two-level and multi-level pc-qubits. Both these kinds of systems would be of interest to this thesis.

## 2.5 Effect of Radiation on the Qubit

From the equation of the Hamiltonian of the system, we can calculate the effect of radiation on the qubit. In the Hamiltonian of the qubit (Eq. 2.33), the only term of interest to calculate the matrix element is the term containing \( f \). This is the parameter that we can control to do spectroscopy on the qubit. To find the matrix element connecting any two transitions consider the case, where we perturb the system with a small time varying magnetic field \( \Delta f(t) \). The term that changes in Eq. 2.33 is \( \cos(2\pi f + 2\varphi_m) \), which due to the perturbation becomes \( \cos(2\pi f + 2\pi \Delta f(t) + 2\varphi_m) \). This term can be rewritten as \( \cos(2\pi f + 2\varphi_m) \cos(2\pi \Delta f(t)) - \sin(2\pi f + 2\varphi_m) \sin(2\pi \Delta f(t)) \). For small \( \Delta f(t) \), the first term \( \cos(2\pi f + 2\varphi_m) \cos(2\pi \Delta f(t)) \)
is a constant and can be pulled into the Hamiltonian. The second term, \( \sin(2\pi f + 2\varphi_m) \sin(2\pi \Delta f(t)) \) can be approximated as, \( \sin(2\pi f + 2\varphi_m)(2\pi \Delta f(t)) \), allowing us to describe \( \Delta f \) as a perturbation on the system.

Hence, the matrix element the one needs to calculate for obtaining the rate of an excitation between two quantum states \( |p\rangle \) and \( |q\rangle \) of the pc-qubit is given as,

\[
(p | \sin(2\pi f + 2\varphi_m) | q) \tag{2.37}
\]

With the discussion from this chapter and equipped with the necessary equations, we will describe quantum optical effects and a highly sensitive approach to rf-detection of magnetic fields using Superconducting Quantum Circuits (SQCs).
Chapter 3

EIT in Superconductive Quantum Circuits

In this chapter I introduce the concept of Electromagnetically Induced Transparency (EIT). Starting with the concept of EIT in atoms, an analogy with superconducting quantum circuits is derived. This is followed by a discussion on how the EIT effect, first observed in atom clouds, can be used to establish macroscopic coherence in SQC systems and, thereby, utilized as a sensitive probe of decoherence. Finally, I am going to highlight the decoherence measurement technique using EIT. This chapter is an extended version of the Superconducting-EIT paper [13].

3.1 Introduction

The human race has always been fascinated in “taming” light, the fastest entity ever known to man. This is evident from the fact that modern communication systems are moving away from electrical to all-optical approaches wherever possible - the fact being that nothing propagates faster than light [30, 31].

Light is made of quantum objects, known as photons [32, 33], which are inherently quantum mechanical in nature. Recently, it has been discovered that information systems based on using quantum mechanical properties of light, for example polarization of photons, can be the most secure form of information processing [34, 35, 36].
This emerging field, of quantum information processing based on the laws of quantum mechanics, has already demonstrated secure communication over optical fiber channels and free space channels using polarized photons as carriers of information [37, 38, 39, 40].

If one starts to use quantum mechanics for communication systems, then evidently one needs to use quantum systems for building them. As we know, quantum systems are extremely fragile, in that any disturbance or measurement process destroys the quantum state of the system. How can we use fragile quantum systems like photons for efficient quantum information processing? How do we process, store, and manipulate photons without disturbing them? These are some of the key sought after questions driving the field of quantum information processing.

In the quest for building a robust quantum optical system, a dramatic breakthrough in optical science in the past decade is the discovery of Electromagnetically Induced Transparency (EIT) [11, 12, 41, 42]. EIT has been the central feature of recent developments in “slowing” and “stopping” light [43, 44, 45, 46, 47, 48, 49, 50, 51]. It has been envisioned that slow and stopped light could be extremely important for optical communications, and particularly for quantum information processing. These techniques are an integral part of long-distance quantum communication systems that combine EIT-based quantum memory and linear optical elements.

Let us understand the physics behind electromagnetically induced transparency. I am going to discuss the importance of certain quantum states known as the dark states, and their consequence on slow and stopped light phenomena in atom clouds. This would provide a good framework for us to develop the idea of EIT in solid state systems like the superconducting quantum circuits and its importance as a sensitive approach to detecting decoherence.

### 3.2 Electromagnetically Induced Transparency

We start with a three-level Λ as shown in Fig. 3-1 and let us understand EIT in such a system. The typical Λ system has two lower states, |1⟩ and |2⟩, that are often Zeeman
3.2. ELECTROMAGNETICALLY INDUCED TRANSPARENCY

Figure 3-1: A three level Λ-atomic system with all the atoms in the ground state $|1\rangle$. The system has two possible transitions, denoted as $\omega_a$ or the probe and $\omega_b$ or the control frequency.

or hyperfine levels of an atom cloud. These two levels are meta-stable because their lifetimes are much larger than the lifetime of the electronically excited state $|3\rangle$. Each of the two meta-stable states $|1\rangle$ and $|2\rangle$ can be resonantly excited to the excited $|3\rangle$ level at the probe and control frequency. Hence, depending on the initial state of the atom system, the atoms can absorb the probe and control frequencies coherently that give rise to Rabi oscillations [52] in $(|1\rangle,|3\rangle)$ and $(|2\rangle,|3\rangle)$ subspace. An interesting question that arises, in this system, is if the absorption of light can be inhibited? This is a non-trivial question, and a solution to the question requires examination of quantum mechanical interference phenomena that can be better understood through dark states.

3.2.1 Dark State Phenomena

It is easy to see that a three level EIT system can be transparent to the control frequency by having all the atoms in the $|1\rangle$ state. As shown in Fig. 3-1, all the atoms are in state $|1\rangle$. This system is transparent to the control frequency as there are no atoms in state $|2\rangle$ to absorb the control frequency. The state $|1\rangle$ is known as the dark state of the system, $|\Psi_{\text{dark}}\rangle$ and is transparent to $\omega_b$. Similarly transferring all the atoms to the state $|2\rangle$ makes the system transparent to the probe frequency,
3. EIT IN SUPERCONDUCTIVE QUANTUM CIRCUITS

![Diagram](image)

Figure 3-2: A example of a dark state that is a superposition of $|1\rangle$ and $|2\rangle$. The probe and control frequency excitation interfere destructively at level $|3\rangle$, thereby inhibiting absorption of radiation. The system continues to be in the superposition state, hence being transparent to both the probe and control frequencies.

hence the dark state of the system is $|\Psi_{dark}\rangle = |2\rangle$. How can this system be made transparent to both the control and probe frequencies at the same time? A solution to this is the heart of quantum inference phenomena that is described below.

### 3.2.2 Quantum Interference Pathways

As is well known in quantum mechanics, the phase of any quantum system plays a crucial role in many interesting effects like Ramsey interferometry [53], Aharonov-Bohm [54] and interference based phenomena [55, 56]. EIT is another interference phenomena, like these effects, where the relative phase of different levels of the quantum system can cause interference in such a manner leading to complete transparency to both the probe and control frequencies. Let us understand this effect mathematically and also intuitively. Consider the state which is a superposition of $|1\rangle$ and $|2\rangle$, known as a dark EIT state, that is transparent to both the probe and control frequencies, as shown Fig. 3-2. The dark EIT state can be be written as,

$$
|\Psi_{dark}\rangle = \frac{\Omega_{23}|1\rangle - \Omega_{13}|2\rangle}{\sqrt{\Omega_{13}^2 + \Omega_{23}^2}}
$$

(3.1)
3.2. ELECTROMAGNETICALLY INDUCED TRANSPARENCY

where, $\Omega_{pq}$ are the Rabi frequencies for a given $|p\rangle \rightarrow |q\rangle$ transition. To understand why this state is transparent to the probe and control frequencies, let us look at the system-light interaction. The system Hamiltonian can be written as (in the $|1\rangle, |2\rangle, |3\rangle$ basis),

$$
\mathcal{H}_{sys} = \hbar \begin{bmatrix}
    \omega_1 & 0 & 0 \\
    0 & \omega_2 & 0 \\
    0 & 0 & \omega_3
\end{bmatrix},
$$

where, $\hbar \omega_p$ is the energy of the $|p\rangle$ state. The interaction Hamiltonian of the light can be written as,

$$
\mathcal{H}_{int} = \begin{bmatrix}
    0 & 0 & \Omega_{13} \cos \omega_{13} t \\
    0 & 0 & \Omega_{23} \cos \omega_{23} t \\
    \Omega_{13} \cos \omega_{13} t & \Omega_{23} \cos \omega_{23} t & 0
\end{bmatrix}.
$$

Here, $\Omega_{pq} \cos \omega_{pq} t$, represents the excitation $|p\rangle \rightarrow |q\rangle$ transition, with amplitude $\Omega_{pq}$ and frequency given by $\omega_{pq} = \omega_q - \omega_p$.

It is useful at this point to go into an interaction frame, with respect to the natural Hamiltonian $\mathcal{H}_{sys}$. The system-light interaction in the interaction frame is given by,

$$
\mathcal{H}_{int} = e^{\frac{i}{\hbar} \mathcal{H}_{sys} t} \mathcal{H}_{pert} e^{-\frac{i}{\hbar} \mathcal{H}_{sys} t}
$$

$$
= \frac{\hbar}{2} \begin{bmatrix}
    0 & 0 & \Omega_{13}(1 + e^{-2i\omega_{13} t}) \\
    0 & 0 & \Omega_{23}(1 + e^{-2i\omega_{23} t}) \\
    \Omega_{13}(1 + e^{-2i\omega_{13} t}) & \Omega_{23}(1 + e^{-2i\omega_{23} t}) & 0
\end{bmatrix}
$$

In a rotating wave approximation (RWA), we can throw away the terms containing frequencies that are twice $\omega_{13}$ and $\omega_{23}$ respectively. This is valid as these fast oscillations average out to zero.

We can write the Hamiltonian in the the rotating wave approximation (RWA)
[52] as,

\[
\mathcal{H}^{(\text{RWA})}_{\text{int}} = \frac{\hbar}{2} \begin{bmatrix}
0 & 0 & \Omega^*_{13} \\
0 & 0 & \Omega^*_{23} \\
\Omega_{13} & \Omega_{23} & 0
\end{bmatrix},
\]

Diagonalizing this Hamiltonian gives us an eigenstate \(\Omega_{23}|1\rangle - \Omega_{13}|2\rangle\) with eigenvalue equal to zero. A zero eigenvalue state means that the action of the Hamiltonian on the dark state, \(|\Psi_{\text{dark}}\rangle\), leaves the quantum state undisturbed, hence transparency to both the probe and control frequencies.

One can also understand this phenomena intuitively as follows - the phase arriving at \(|3\rangle\) due to both the probe and control frequency excitation are opposite in sign. This causes the relative phase between \(|1\rangle\) and \(|2\rangle\) to be opposite in sign. Hence, leading to destructive interference at level \(|3\rangle\). This leaves the system unchanged, thus inhibiting any excitation to \(|3\rangle\) or equivalently transparency to both the probe and control frequencies is achieved at the same time. The two electromagnetic fields (probe and control) then propagate through the atom cloud without absorption.

This is the heart of slow and stopped light, where ultra-slow light propagation due to EIT-based refractive-index modifications in atomic clouds has been observed [43, 44, 45]. I briefly describe the slow and stopped light phenomena, doing a comprehensive review of this phenomena is beyond the scope of the thesis, the reader is pointed to excellent review articles [12, 57].

### 3.3 Slow and Stopped Light

The concept of slow and stopped light arises due to EIT-based modification of the refractive index of atom clouds. It is often mistaken for what is being slowed in context of “slow” light. There are two different propagation speeds of light, one is the phase velocity and the other group velocity. The phase velocity is the speed at which each frequency component of a light pulse travels. This is typically close to the value, \(c = 3 \times 10^8 m/s\), while the group velocity is the speed with which the maximum of a
many frequency component envelope of a light pulse moves.

The phase velocity is related to the refractive index as,

\[ v_{ph} = \frac{c}{n} \]  

(3.7)

where \( n \) is the refractive index of the medium of propagation. The group velocity is associated with the dispersion of the refractive index, that is the variation of \( n(\omega) \) as a function of frequency \( \omega \) as,

\[ v_g = \frac{c}{n + \omega_p \frac{dn}{d\omega_p}} \]  

(3.8)

where, \( p \) is the probe. It is clear from this equation that when \( \frac{dn}{d\omega_p} \) becomes large, the group velocity tends to zero, \( v_g \to 0 \).

In the atomic EIT, it turns out that EIT occurs in a narrow transparency window which also creates a steep variation of \( n(\omega) \) with frequency \([12, 43, 44, 57]\). Hence, the group velocity of light is reduced dramatically. During this process, a fraction of the photons corresponding to the probe signal are converted into a spin-wave via a two-photon (probe and control) process that maps the probe into a superposition of \( |1\rangle \) and \( |2\rangle \) or the dark EIT state. One can dramatically reduce the group velocity to zero, thereby "stopping" light, by turning off the control beam once the pulse with the probe frequency enters the EIT medium. When this happens, the probe is completely mapped onto the long lived states, \( |1\rangle \) and \( |2\rangle \). This stopped light is then stored as long as the decoherence time of the ground states, or until the control beam is turned back on [47, 57]. These remarkable results can have enormous implications for quantum information processing [43, 44]. From the basic understanding of EIT, I introduce the superconducting analog of EIT and its equally exciting implications for solid state systems.
3.4 EIT in Superconducting Quantum Circuits

Superconductive quantum circuits (SQC) comprise quantized energy levels that may be coupled via microwave electromagnetic fields. Described in this way, one may draw a close analogy to atoms with internal (electronic) levels coupled by laser light fields. Here, I present a superconductive analog to electromagnetically induced transparency (S-EIT) that utilizes SQC designs of present day experimental consideration. I discuss how S-EIT can be used to establish macroscopic coherence in such systems and, thereby, utilized as a sensitive probe of decoherence.

3.4.1 SQC-atom analogy

Superconductive quantum circuits (SQC) comprising mesoscopic Josephson junctions can exhibit quantum coherence amongst their macroscopically large degrees of freedom [3]. They feature quantized flux and/or charge states depending on their fabrication parameters, and the resultant quantized energy levels are analogous to the quantized internal levels of an atom. Spectroscopy, Rabi oscillation, and Ramsey interferometry experiments have demonstrated that SQCs behave as "artificial atoms" under carefully controlled conditions [4, 5, 6, 7, 8, 9, 10]. This leads one to the interesting question: Can SQCs exhibit other quantum optical effects associated with atomic systems? This chapter extends the SQC-atom analogy to another quantum optical effect associated with atoms: electromagnetically induced transparency [11, 12]. I proposed the demonstration of microwave transparency using a superconductive analog to EIT (denoted by S-EIT) in a superconductive circuit exhibiting two metastable states (e.g., a qubit) and a third, shorter-lived state (e.g., the readout state). We show that driving coherent microwave transitions between the qubit states and the readout state is a demonstration of S-EIT. Further, I propose a means to use S-EIT to probe sensitively the qubit decoherence rate; the philosophy is similar to proposed EIT-based measurements of phase diffusion in atomic Bose-Einstein condensates [58].
3.4. EIT IN SUPERCONDUCTING QUANTUM CIRCUITS

Figure 3-3: (a) Energy level diagram of a three-level \( \Lambda \) system. EIT can occur in atoms possessing two long-lived states \(|1\rangle, |2\rangle\), each of which is coupled via resonant laser light fields to a radiatively decaying state \(|3\rangle\). (b) Circuit schematic of the PC qubit and its readout SQUID. (c) One-dimensional double-well potential and energy-level diagram for a three-level SQC. Using PC qubit parameters [25, 61, 62], we calculate \( \omega_2 - \omega_1 = (2\pi) 36 \text{ GHz} \) and \( \omega_3 - \omega_2 = (2\pi) 32 \text{ GHz} \). The simulated matrix elements \( \langle p|\sin(2\pi f + 2\phi_m)|q\rangle \) for \((p,q) = (1,2), (2,3), \text{ and } (1,3)\) are, respectively, 0.0704, -0.125, 0.0158.
3.4.2 Description of a Superconducting Quantum Circuit

SQCs also exhibit Λ-like energy level structures [9, 24, 72, 83, 59, 87, 88]. One example is the persistent-current (PC) qubit, a superconductive loop interrupted by two Josephson junctions of equal size and a third junction scaled smaller in area by the factor $\alpha < 1$ (Fig. 3-3b) [24]. Its dynamics are described by the Hamiltonian

$$\mathcal{H}_{PC} = \frac{1}{2} C \left( \frac{\Phi_0}{2\pi} \right)^2 (\varphi_p^2 + (1 + 2\alpha)\varphi_m^2) + E J [2 + \alpha - 2\cos\varphi_p \cos\varphi_m - \alpha \cos(2\pi f + 2\varphi_m)], \quad (3.9)$$

in which $C$ is the capacitance of the larger junctions, $\varphi_{p,m} \equiv (\varphi_1 \pm \varphi_2)/2$, $\varphi_i$ is the gauge-invariant phase across the larger junctions $i = \{1, 2\}$, $E J$ is the Josephson coupling energy, and $f$ is the magnetic flux through the loop in units of the flux quantum $\Phi_0$ [25]. Near $f \approx 1/2$, the qubit potential landscape (second term in Eq. 3.9) assumes a double-well profile. Each well corresponds to a distinct classical state of the electric current, i.e., left or right circulation about the qubit loop, and its net magnetization is discernable using a dc SQUID [25]. As a quantum object, the potential wells exhibit quantized energy levels corresponding to the quantum states of the macroscopic circulating current [60, 62]. These levels may be coupled using microwave radiation [6], and their quantum coherence has been experimentally demonstrated [10].

The three-level Λ system illustrated in Fig. 3-3a is a standard energy level configuration utilized in EIT [11, 12]. It comprises two metastable states $|1\rangle$ and $|2\rangle$, each of which may be coupled to a third excited state $|3\rangle$. In atoms, the metastable states are typically hyperfine or Zeeman levels, while state $|3\rangle$ is an excited electronic state that may spontaneously decay at a relatively fast rate $\Gamma_3$. In an atomic EIT scheme, a resonant “probe” laser couples the $|1\rangle \leftrightarrow |3\rangle$ transition, and a “control” laser couples the $|2\rangle \leftrightarrow |3\rangle$ transition. The transition coupling strengths are characterized by their Rabi frequencies $\Omega_{j3} \equiv -d_{j3} \cdot E_{j3}$ for $j = 1, 2$ respectively, where $d_{j3}$ are the dipole matrix elements and $E_{j3}$ are the slowly varying envelopes of the electric fields.
seen in section 3.2, for particular Rabi frequencies $\Omega_3$, the probe and control fields are effectively decoupled from the atoms by a destructive quantum interference of the two driven transitions. The result is probe and control field transparency [11, 12].

As an analog with atoms, a $\Lambda$ system in the SQC is obtained by tuning the flux bias away from $f = 1/2$. This results in the asymmetric double-well potential illustrated in Fig. 3-3c. The three states in the left well constitute the superconductive analog to the atomic $\Lambda$ system. Using tight-binding models with experimental PC qubit parameters [25, 61, 62] at a flux bias $f = 0.5041$, we estimate the interwell resonant-tunneling ($\Gamma_{\text{rt}}$) rates for states $|1\rangle$, $|2\rangle$, and $|3\rangle$ to be $\Gamma_{1\text{rt}} \approx (1 \text{ ms})^{-1}$, $\Gamma_{2\text{rt}} \approx (1 \mu\text{s})^{-1}$, and $\Gamma_{3\text{rt}} \approx (1 \text{ ns})^{-1}$ respectively. An off-resonant biasing of state $|3\rangle$ decreases its interwell tunneling rate to order $(100 \mu\text{s})^{-1}$ [84]; the off-resonant biasing of states $|1\rangle$ and $|2\rangle$ (as in Fig. 3-3c) will also significantly decrease their interwell tunneling rates. In addition, the intrawell relaxation rate at a similar flux-bias was experimentally determined to be $\Gamma_{3\text{intra}} \approx (25 \mu\text{s})^{-1}$ [61] and, presumably, $\Gamma_{2\text{intra}} < \Gamma_{3\text{intra}}$. Therefore, the "qubit states" $|1\rangle$ and $|2\rangle$ are effectively metastable with respect to the resonantly-biased "readout state" $|3\rangle$. Since $\Gamma_{3\text{rt}} \gg \Gamma_{3\text{intra}}$ [60, 61, 62], a particle reaching state $|3\rangle$ will tunnel quickly to state $|4\rangle$, an event that is detectable by a dc SQUID. Alternatively, for slower detection schemes, one may detune states $|3\rangle$ and $|4\rangle$, and then apply a resonant $\pi$-pulse to transfer the population from state $|3\rangle$ to $|4\rangle$.

### 3.4.3 Superconducting-EIT

Having discussed EIT in atoms, we will now discuss the manifestation of EIT in SQCs through the concept of macroscopic tunneling of flux states as shown in Fig. 3-1 and 3-4. Here, at the flux bias away from $f = 1/2$, the potential energy of the system forms an asymmetric double well. At the appropriate bias point, we can get an energy level configuration as shown in Fig. 3-4. The three levels in the left well form the three level $\Lambda$ system, where $|1\rangle$ and $|2\rangle$ are the meta-stable states of the systems or qubit states and the third level $|3\rangle$ forms the read-out state. Any excitation of population to $|3\rangle$ would lead to the quantum particle to quickly tunnel into the right
Figure 3-4: A three level Λ system in an SQC. The asymmetric potential diagram indicates quantized flux states in the left and right well, which correspond to circulating current in the clockwise and anti-clockwise directions respectively. A transition between two levels of the system can be excited by shining microwave frequencies on resonance with the transition. Any particle reaching level $|3\rangle$ would lead to tunneling to the well on the right hand side, an event detectable by the dc-SQUID.

well, a process that is detectable by the dc-SQUID (surrounding the pc-qubit), as shown in Fig. 3-3 [24]. When the system is in $|1\rangle$, the system can be read out by resonantly exciting the $|1\rangle \rightarrow |3\rangle$ transition. This leads to tunneling of the flux states from the left well to the right well, hence detection through the dc-SQUID. Once the particle tunnels, it is equivalent to making a measurement as flux state gets localized in either the left or right well because of the dc-SQUID measurement process.

Similarly, when the system is the state $|2\rangle$, it can be read-out through the above process by resonant excitation at the $|2\rangle \rightarrow |3\rangle$ transition. But the interesting question would be if such a macroscopic tunneling process can be inhibited in the presence of resonant excitations? This is an analog of making the atom-cloud system transparent to probe and control frequencies. The answer to this question can be easily guessed from the solution to the atom-cloud problem. If we can create a superposition of the qubit states, then a exciting transitions from the qubit states to the read-out state can destructively interfere to inhibit any excitation of the read-out state. Let us understand this inhibition of quantum tunneling through EIT in detail.

Transitions between the quantized levels are driven by resonant microwave-frequency
magnetic fields. Assuming the Rabi frequencies $\Omega_{ij}$ to be much smaller than all level spacings $|\omega_{kl}| \equiv |\omega_k - \omega_l|$, the system-field interaction may be written within the rotating wave approximation (RWA) [63],

$$\mathcal{H}_{\text{int}}^{(\text{RWA})} = \frac{\hbar}{2} \begin{bmatrix} 0 & \Omega_{12}^* & \Omega_{13}^* \\ \Omega_{12} & 0 & \Omega_{23} \\ \Omega_{13} & \Omega_{23} & -i\Gamma_3 \end{bmatrix},$$

in which the decay from state $|3\rangle$ is treated phenomenologically as a non-Hermitian matrix element [63, 64]. For small microwave perturbations of the frustration, $\Delta f$, the associated Rabi frequencies are $\Omega_{pq} = 2\pi \Delta f (\alpha E_3/\hbar) \langle p | \sin (2\pi f + 2\phi_m) | q \rangle$; numerical simulations of the matrix elements are consistent with recent experimental results (see caption to Fig. 3-3c) [60, 61, 62]. A qubit initially in $|1\rangle$ can be prepared in a superposition state $|\Psi\rangle = c_1|1\rangle + c_2|2\rangle$ by temporarily driving the $\Omega_{12}$ field. Applying the $\Omega_{13}$ ($\Omega_{23}$) field then allows the population of state $|1\rangle$ ($|2\rangle$) to be read out through a transition to state $|3\rangle$ followed by a rapid escape to the right well (a readout scheme also utilized by single-junction qubits [9]).

Alternatively, one may achieve S-EIT in a superconductive $\Lambda$ system that is pre-
pared in state $|\Psi\rangle$ by simultaneously applying the microwave fields $\Omega_{13}$ and $\Omega_{23}$ such that

$$\frac{\Omega_{13}}{\Omega_{23}} = -\frac{c_2}{c_1}. \quad (3.11)$$

Under this condition (with $\Omega_{12} = 0$), the state $|\Psi\rangle$ is an eigenstate of $\mathcal{H}_{\text{int}}^{\text{RWA}}$ in Eq. (3.10) with eigenvalue zero; in this dark state, the SQC becomes transparent to the microwave fields. As in conventional EIT, the amplitudes for the two absorptive transitions into $|3\rangle$ have equal and opposite probability amplitudes, leading to a destructive quantum interference and no population loss through the readout state $|3\rangle$.

It should also be noted that S-EIT also provides a means to confirm, without disturbing the system, that one had indeed prepared the qubit in the desired state. Let us say we want to measure a certain superposition state (of $|1\rangle$ and $|2\rangle$). The EIT Rabi intensities, $\Omega_{13}$ and $\Omega_{23}$ can be adjusted as in Eq. 3.11, so that for the desired state that we want to measure there is a zero excitation probability to the read-out state. This prevents the particle from tunneling, thus an indirect interference of the state of the system, thereby preserving its quantum coherence. Hence, this would lead to a non-destructive state measurement of the qubit. One should note that non-destructive measurement does not violate the theory of quantum measurement [57, 63]. EIT only provides a means to measure the dark state non-destructively, while any other superposition of the qubit states is destroyed due to the EIT fields. This is equivalent to measuring the SQC in state $|1\rangle$ non-destructively by turning on the control frequency. Hence, EIT fields only provide a change of basis for non-destructive measurement of the dark state. Similar approaches could be useful for state measurement in other solid state systems like quantum dots.
3.4.4 Probing Decoherence using Electromagnetically Induced Transparency

In a practical SQC, there will be an imperfect preparation as well as a decoherence of the state $|\Psi_{\text{dark}}\rangle$, and this must be measured, characterized, and minimized for quantum information applications. S-EIT is a sensitive probe for this purpose, since deviations in the amplitude and/or relative phase of the complex coefficients $c_i$ from the condition established in Eq. 3.11 result in a small probability $|\langle c_1 \Omega_{13} + c_2 \Omega_{23} \rangle/\Omega|^2$ (where $\Omega \equiv \sqrt{\Omega_{13}^2 + \Omega_{23}^2}$) of the SQC being driven into the readout state $|3\rangle$ on a time scale $\sim \Gamma_3/\Omega^2$. In general, there are two categories of decoherence: *loss* and *dephasing*. *Loss* refers to population losses from the metastable states $|1\rangle$ and $|2\rangle$, and it is present in an SQC due to both intrawell and interwell energy relaxation. *Dephasing* refers to interactions of the SQC with other degrees of freedom in the system that cause the relative phase between $c_1$ and $c_2$ to diffuse. Both types of decoherence act to drive even a perfectly prepared state $|\Psi_{\text{dark}}\rangle$ out of the dark state defined by Eq. 3.11. To characterize this decoherence process, a density matrix approach is ideally suited to fully describe the dynamics of the three level system.

**Bloch Equation: Estimation of Decoherence**

We describe the system with a $3 \times 3$ density matrix, where the diagonal elements $\rho_{ii}$ describe the populations, and $\rho_{ij}, i \neq j$ describe the coherences between levels. In the presence of the EIT fields $\Omega_{13}$ and $\Omega_{23}$ (with $\Omega_{12} = 0$), the Bloch equations govern
the evolution of the density matrix [63]:

\[
\dot{\rho}_{11} = -\Gamma_1 \rho_{11} - \frac{i}{2} \Omega_{13}^* \rho_{31} + \frac{i}{2} \Omega_{13} \rho_{13}, \quad (3.12)
\]

\[
\dot{\rho}_{22} = -\Gamma_2 \rho_{22} - \frac{i}{2} \Omega_{23}^* \rho_{32} + \frac{i}{2} \Omega_{23} \rho_{23}, \quad (3.13)
\]

\[
\dot{\rho}_{33} = -\Gamma_3 \rho_{33} + \frac{i}{2} \Omega_{13}^* \rho_{31} - \frac{i}{2} \Omega_{13} \rho_{13} + \frac{i}{2} \Omega_{23}^* \rho_{32} - \frac{i}{2} \Omega_{23} \rho_{23}, \quad (3.14)
\]

\[
\dot{\rho}_{12} = -\gamma_{12} \rho_{12} - \frac{i}{2} \Omega_{13}^* \rho_{32} + \frac{i}{2} \Omega_{13} \rho_{13}, \quad (3.15)
\]

\[
\dot{\rho}_{13} = -\gamma_{13} \rho_{13} + \frac{i}{2} \Omega_{13}^* (\rho_{11} - \rho_{33}) + \frac{i}{2} \Omega_{23}^* \rho_{21}, \quad (3.16)
\]

\[
\dot{\rho}_{23} = -\gamma_{23} \rho_{23} + \frac{i}{2} \Omega_{23}^* (\rho_{22} - \rho_{33}) + \frac{i}{2} \Omega_{13}^* \rho_{12}, \quad (3.17)
\]

The remaining three elements' equations are determined by $\rho_{ij}^* = \rho_{ji}$. The decoherence rates $\gamma_{ij} = (\Gamma_i + \Gamma_j)/2 + \gamma_{ij}^{(\text{deph})}$ include both loss and dephasing contributions. We concentrate on the regime in which the readout state escape rate $\Gamma_3 = (1 \text{ ns})^{-1} = (2\pi) 130 \text{ MHz}$ dominates all other loss and dephasing rates, thus $\gamma_{13} \approx \gamma_{23} \approx \Gamma_3/2$. Furthermore, we assume that the dephasing rate $\gamma_{12}^{(\text{deph})}$ dominates the metastable state losses $\Gamma_1$ and $\Gamma_2$, thus setting $\Gamma_1 = \Gamma_2 = 0$ and $\gamma_{12} \approx \gamma_{12}^{(\text{deph})}$. Theoretical estimates of the dephasing rates, such as $\gamma_{12}^{(\text{deph})}$, in multi-level systems were recently obtained in Ref. [68].

We illustrate an S-EIT decoherence probe example by applying EIT fields $\Omega_{13} = \Omega_{23} = (2\pi) 150 \text{ MHz}$ to the dark state $|\Psi\rangle = (|1\rangle - |2\rangle)/\sqrt{2}$ and numerically integrating Eqs. (3.12)-(3.17). With $\gamma_{12} = 0$ the system is stationary and no population is driven into $|3\rangle$ ($\rho_{33} = 0$); when we include a dephasing rate $\gamma_{12} = (2\pi) 5 \text{ MHz}$, $\rho_{33}$ is small but nonzero (Fig. 3-6(a)). It exhibits a rapid initial rise with transitory oscillations (see inset Fig. 3-6(a)), reaching its maximum value $\rho_{33}^{(\text{max})}$ within about $T_{ss} = 4 \text{ ns}$. This is followed by a smooth decay with a $1/e$ time of about 80 ns. The solid curve in Fig. 3-6(b) traces the total population $P = \rho_{11} + \rho_{22} + \rho_{33}$ remaining in the system as a function of time. When the excited state maximum $\rho_{33}^{(\text{max})}$ is reached, the total remaining population is $P(T_{ss}) = 0.973$. In contrast, the dashed
3.4. EIT IN SUPERCONDUCTING QUANTUM CIRCUITS

line in Fig. 3-6b illustrates the rapid population loss expected when the same fields are applied to the state $|\Psi\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$ [π out of phase with the dark state in Eq. (4.10)]. In the absence of S-EIT quantum interference, the entire population is lost on a time scale $\sim \Gamma_3/\Omega^2 \approx 4$ ns.

We now use Eqs. (3.12)-(3.17) to show how measuring the slow population loss in S-EIT can be used to extract the decoherence rate $\gamma_{12}$. The elements $\rho_{33}$, $\rho_{13}$, and $\rho_{23}$ in Eqs. (3.14), (3.16), and (3.17) are damped at a rapid rate $\sim \Gamma_3$, allowing their adiabatic elimination [64, 69]; we solve for their quasi-steady state values by setting $\dot{\rho}_{33} = \dot{\rho}_{13} = \dot{\rho}_{23} = 0$. This approximation is accurate once initial transients have passed and the plateau value $\rho_{33}^{(\text{max})}$ has been reached. Using these results in Eq. (3.15) yields an equation for $\dot{\rho}_{12}$ with a strong damping term $\Omega^2/\Gamma_3$, and it too can be solved for its quasi-steady state value. In the limit $\gamma_{12}\Gamma_3/\Omega^2 \ll 1$ we obtain

$$\rho_{12}(t) \approx -\frac{\Omega_3\Omega_{23}}{\Omega^2} \left(1 - \frac{2\gamma_{12}\Gamma_3}{\Omega^2}\right) (\rho_{11}(t) + \rho_{22}(t)).$$

(3.18)

The ratio $2\gamma_{12}\Gamma_3/\Omega^2$ represents the small fractional deviation of $\rho_{12}$ from its dark state value. There is a competition between the "preparation rate" $\Omega^2/\Gamma_3$ (which constantly acts to drive the system into the dark state) and the decoherence rate $\gamma_{12}$ (which attempts to drive it back out).

Plugging our adiabatic solutions for $\rho_{13}$, $\rho_{23}$, and Eq. (3.18) into Eqs. (3.12) and (3.13) reveals that deviations from the dark state cause a loss of the population $P$ at a rate $R = 2\gamma_{12}(\Omega_{13}^2\Omega_{23}^2/\Omega^4)P$. Since we assumed all the population is lost through $|3\rangle$ via the decay term $\rho_{33}\Gamma_3$ in Eq. (3.14), we can equate these two rates. When the maximum $\rho_{33}^{(\text{max})}$ is reached sufficiently fast, the population $P$ is still approximately unity and this yields:

$$\rho_{33}^{(\text{max})} \approx 2\frac{\Omega_3^2\Omega_{23}^2 \gamma_{12}}{\Omega^4 \Gamma_3}.$$

(3.19)

So long as the loss during the initial transient time $t < T_{\text{ns}}$ is negligible, the population will follow a simple exponential decay $P(t) = \exp(-\rho_{33}^{(\text{max})}\Gamma_3 t)$ (as in Fig. 3-6b), and the dephasing rate $\gamma_{12}$ can be easily extracted. The time $T_{\text{ns}}$ is generally the greater
**Figure 3-6:** (a) Population $\rho_{33}$ versus time with EIT fields $\Omega_{13} = \Omega_{23} = (2\pi) 150$ MHz, an initial dark state $\rho_{11} = \rho_{22} = 0.5$ and $\rho_{12} = -0.5$, and with a dephasing rate $\gamma_{12} = (2\pi) 5$ MHz. The inset shows $\rho_{33}$ for early times. $\rho_{33}$ exhibits a rapid rise to a plateau after a short time $T_{ss}$, followed by a much slower decay. (b) The total population $P$ remaining in the system versus time for the same simulation (solid curve). The dashed curve shows the population for the out-of-phase case $\rho_{11} = \rho_{22} = 0.5$ and $\rho_{12} = 0.5$ discussed in the text. (c) The logarithmic plot of (b) up to a decay time of 200 ns.
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Figure 3-7: (a) The maximum plateau value $\rho_{33}^{(\text{max})}$ for different $\gamma_{12}$ (circles). The solid curve shows the prediction (3.19). (b) The remaining population $P = P_{11} + P_{22} + P_{33}$ at the time the plateau is reached $T_{ss}$ for the cases in (a).

of the preparation time $\sim \Gamma_3/\Omega^2$ and the inverse of the decay rate $1/\Gamma_3$; the loss up to $t = T_{ss}$ will be $\sim \rho_{33}^{(\text{max})} \Gamma_3 T_{ss} \sim (2\Omega_3^2 \Omega_{23}^2/\Omega^4) \max(\gamma_{12}/\Gamma_3, \gamma_{12} \Gamma_3/\Omega^2)$, requiring both ratios in the $\max(...) \text{ argument to be small.}$ The first ratio will generally be small for parameters of interest while the second can be made small by choosing an appropriate field strength $\Omega$. When these conditions are satisfied, the method outlined here can be used to measure $\gamma_{12}$. Note also that if one applies the S-EIT fields for a time $T_{S-EIT} \gg T_{ss}$ and observes no loss, then the prepared superposition $|\Psi\rangle$ is unperturbed and one has still obtained an upperbound on the decoherence rate $\gamma_{12}$. We have performed a series of numerical simulations, varying $\gamma_{12}$, to validate the preceding approach. Fig. 3-7a indicates $\rho_{33}^{(\text{max})}$ versus $\gamma_{12}$ and compares the numerical results with the analytic estimate [Eq. (3.19)]. The agreement is good for $\gamma_{12} < (2\pi) 4$ MHz, which corresponds to $2\gamma_{12} \Gamma_3/\Omega^2 < 0.056$. Higher dephasing rates compete more with the preparation rate, making the adiabatic elimination approach used to obtain Eqs. (3.18)–(3.19) less valid. In such cases, one observes a significant loss of $P$ by the time $\rho_{33}^{(\text{max})}$ is reached, as illustrated in Fig. 3-7(b).

In employing the RWA in Eqs. (3.12)–(3.17), we have ignored the field-induced couplings of far off-resonant transitions, which can drive the system out of the dark
state (e.g., in Fig. 3-3c: although detuned by $|\Delta| \approx 4$ GHz, $\Omega_{23}$ weakly drives $|1\rangle \leftrightarrow |2\rangle$). We have performed calculations of the evolution including all such couplings for our parameters and found they induce an additional absorption into $|3\rangle$ at a rate $R_{\text{off-res}} = (2\pi) 19$ kHz. Such transitions will not affect our S-EIT measurement for qubit decoherence rates $\gamma_{12} \gg R_{\text{off-res}}$. The effect is analogous to A.C. Stark shifts and Rayleigh scattering in atoms [52], and it has been shown [64] that the loss rate scales as $R_{\text{off-res}}^{AC} \sim \Gamma_3 (\Omega^2 / \Delta^2)$ for A.C. Stark Shift and $R_{\text{off-res}}^{RS} \sim \Omega^4 / \Gamma_3 \Delta^2$ for Rayleigh scattering, which introduces a maximum allowed field intensity $\Omega^2$ for a given detuning $\Delta$ and decoherence rate $\gamma_{12}$. A detailed analysis of the above effects in SQCs are described in chapter 4.

3.5 Conclusion

We have proposed using the superconductive analog to EIT (S-EIT) to demonstrate macroscopic quantum interference in superconductive quantum circuits. We have shown how S-EIT can be used to measure, with a single pulse of the two S-EIT fields, whether a particular superposition of meta-stable energy levels (a qubit) has been prepared. The technique is distinguishable from previous state measurement schemes [9] in that S-EIT ideally does not disturb the system, preserving its quantum coherence when it has been prepared in the desired state. Furthermore, we have shown how S-EIT can very sensitively probe small qubit errors due to decoherence or imperfect state preparation, and we have obtained analytic expressions for the field strengths required to measure the qubit dephasing rate. A detailed analysis of the theory of EIT in SQCs in the presence of decoherence and also including other realistic effects like tunneling and radiation cross-talk follows in chapter 4.
Chapter 4

Effect of decoherence pathways on s-EIT

This chapter considers the electromagnetically-induced transparency (EIT) in a superconducting quantum circuit (SQC) in context of various decoherence effects. The system is a persistent-current flux qubit biased in a Λ configuration as considered in chapter 2 and Ref. [13], we showed that an ideally-prepared EIT system provides a sensitive means to probe decoherence. Here, we extend this work by exploring the effects of imperfect dark-state preparation and specific decoherence mechanisms (population loss via tunneling, pure dephasing, and incoherent population exchange). These effects are some of the important effects that would be crucial to realization of EIT through experiments and also to enhance our understanding of the physics of EIT in SQC systems significantly. We find analytic expressions for the slow loss rate, with coefficients that depend on the particular decoherence mechanisms, thereby providing a means to probe, identify, and quantify various sources of decoherence with EIT. We go beyond the rotating wave approximation to consider how strong microwave fields can induce additional off-resonant transitions in the SQC, and we show how these effects can be mitigated by compensation of the resulting AC Stark shifts. This chapter is based on Ref.[14].
4.1 Introduction

Superconducting quantum circuits (SQCs) based on Josephson junctions (JJs) exhibit macroscopic quantum-coherent phenomena [3]. These circuits exhibit quantized flux or charge states states, depending on their fabrication parameters. The quantized states are analogous to the quantized internal (hyperfine and Zeeman) levels in an atom, and the SQCs thus behave like "artificial atoms." Spectroscopy [5, 6, 65, 66], Rabi oscillations and Ramsey interferometry [4, 67, 7, 8, 9, 10, 70], cavity quantum electrodynamics [26, 27], and Stückelberg oscillations [55, 56] are examples of quantum-mechanical behavior first realized in atomic systems that have also been recently demonstrated with SQCs.

We recently leveraged the atom-SQC analogy to propose electromagnetically induced transparency (EIT) [11, 12] in superconducting circuits [13], as was discussed in chapter 2. EIT has attracted much attention in atomic systems in the context of slow light [43], quantum memory [47, 73] and nonlinear optics [71]. EIT occurs in so-called "Λ-systems" comprising two meta-stable states, each coupled via resonant electromagnetic fields to a third, excited state. For particular initial states called "dark states," the absorption on both transitions is suppressed due to destructive quantum interference, thus making the atom transparent to the applied fields. Though EIT is often studied in the context of the behavior of a weak 'probe' field in the presence of a stronger 'pump' field, we focus on the case where the two fields have comparable amplitude. In Ref. [13], we analyzed a superconducting persistent-current qubit biased such that it exhibited a Λ-configuration: two meta-stable states (the qubit) and a third, shorter-lived state (the readout state). We showed that EIT provides a non-destructive means to confirm preparation of an arbitrary superposition state of the qubit. Moreover, we showed that the proposed EIT scheme can sensitively probe the qubit decoherence rate using a method analogous to the proposal in Ref. [58] for atomic systems. In addition to our EIT work, several groups have considered the use of "dark states" in SQCs comprising a Λ-configuration to implement adiabatic passage and its application to quantum information processing [59, 72, 83].
4.1. INTRODUCTION

In the present work, we extend and augment our analysis in Ref. [13] with realistic effects which arise in SQCs due to the presence of additional quantized levels (beyond the three-level “Λ-system” model). These effects have qualitatively unique signatures in an EIT experiment, and this work provides a tool for identifying their origin. This allows a more complete understanding of the full level-structure of the SQC system, and it further clarifies the necessary criteria for the experimental observation of EIT. The present work carries the spirit of previous investigations in which additional degrees of freedom (beyond two-level models) were required to explain quantitatively experimental Rabi oscillations in SQCs. Examples of these works include resonant tunneling across the barrier [74], diagonal dipole matrix elements [75], and coupling to additional degrees of freedom outside the SQC, such as micro-resonators [77, 78]. Just as EIT is sensitive to decoherence, it will be similarly sensitive to effects beyond the idealized three-level model.

The effects we investigate arise primarily from differences between SQCs and the atomic systems considered in much of the literature. First, while damping of the excited level is provided naturally by spontaneous emission in atoms, in SQCs, this decay is ‘manufactured’ by resonant biasing across the tunnel barrier followed by fast measurement with a SQUID. This process must be considered in more detail to assure this decay is indeed analogous to spontaneous decay in atoms. Second, the transitions are at microwave rather than optical frequencies, whereas the Rabi frequency coupling rates and dephasing rates tend to be faster than in atomic systems. This means that various couplings in the system can be more comparable to the level spacings and thus the rotating wave approximation (RWA) is often not as valid as in atomic systems. Third, the level structure itself is quite different. In particular, there is typically some degree of dipole-like coupling between all pairs of levels in the system, because selection rules allow all possible transitions [79]. Fourth, in SQCs, there is the possibility of direct resonant tunneling across the barrier, a feature which is absent in atomic systems.

This chapter is organized in the following manner. In Section 4.2, we introduce the proposed system, a persistent-current (PC) qubit [24, 25]. We discuss the conditions
under which the PC qubit exhibits a Λ-configuration amongst its multi-level energy band structure that is conducive for an EIT demonstration. We then present the Hamiltonian and density matrix approaches to analyze the system dynamics. In Section 4.3, we use the Hamiltonian approach to give useful analytic approximations to the full system, which allow us to investigate EIT in a reduced three-level system. We explore effects of population and phase mismatch between the prepared initial state and the desired dark state (as defined by the applied fields), and the effect of detuning the applied fields from their resonances. We also consider the SQUID measurement rate and its effect on the effective decay and frequency of the excited 'read-out' level.

In Section 4.4, we use the density matrix approach to include pure dephasing and incoherent population loss and exchange, generalizing our previous results in Ref. [13]. We explore the effect on EIT in the presence of coherent and incoherent tunneling processes. Generally, one must make the EIT 'preparation rate' (proportional to the microwave field intensities) faster than the decoherence rate in order to observe EIT.

In Section 4.5, we go beyond the rotating-wave approximation (RWA) to examine the important issue of microwave field-induced off-resonant transitions in the spirit of previous work on two-level systems [89]. We conclude that off-resonant transitions cause frequency shifts and losses which depend on the coupling field intensities. Unlike decoherence and detuning, these transitions generally manifest themselves as the field intensities are increased. The high-field frequency shifts are analogous to the AC-Stark shifts observed in atomic systems. We show how the off-resonant effects can be mitigated and, in some cases, eliminated by compensating for the frequency shifts.
4.2 The PC-qubit: A Superconducting Quantum Circuit

4.2.1 The PC-qubit model

The persistent-current (PC) qubit is a superconductive loop interrupted by two Josephson junctions of equal size and a third junction scaled smaller in area by the factor $0.5 < \alpha < 1$ (Fig. 4-1(a)) [24, 25]. Its dynamics are described by the Hamiltonian

$$\mathcal{H}_{pc} = \frac{1}{2} C \left( \frac{\Phi_0}{2\pi} \right)^2 \left( \psi_p^2 + (1 + 2\alpha)\psi_m^2 \right) + E_j \left[ 2 + \alpha - 2 \cos \varphi_p \cos \varphi_m - \alpha \cos(2\pi f + 2\varphi_m) \right], \quad (4.1)$$

in which $C$ is the capacitance of the larger junctions, $\varphi_{p,m} \equiv (\varphi_1 \pm \varphi_2)/2$, $\varphi_i$ is the gauge-invariant phase across the larger junctions $i = \{1, 2\}$, $E_j = I_c\Phi_0/2\pi$ is the Josephson coupling energy, $I_c$ is the critical current of the larger junctions, and $f$ is the magnetic flux through the loop in units of the flux quantum $\Phi_0$ [25].

The qubit potential energy (the second term in $\mathcal{H}_{pc}$) forms a 2D periodic double well potential, a one-dimensional slice through which is shown in Fig. 4-1(b). Each well corresponds to a distinct classical state of the electric current, i.e., left or right circulation through the loop, with a net magnetization that is discernable using a dc SQUID [25]. The relative depth of the two wells can be adjusted by detuning the flux bias to either side of the symmetric point $f = 1/2$. The potential wells exhibit quantized energy levels corresponding to the quantum states of the macroscopic circulating current [60, 61, 62, 90], with the number of levels on each side determined by the depth and frequencies of the wells. In this basis the Hamiltonian can be written...
Figure 4-1: A pc-qubit with a dc SQUID measuring device (a) A pc-qubit, a superconducting loop with two Josephson junctions of equal dimension and the third scaled by a factor \( \alpha \), as shown in the inner loop. The outer loop is a dc-SQUID that is used to measure the magnetic moment of the qubit. (b) A representation of the potential energy of the pc-qubit as a function of \( f \), the magnetic flux in the loop in units of the flux quantum \( \Phi_0 \). The qubit potential can range from an asymmetric double well biased to the right to a symmetric double well and to the an symmetric double well biased to the left for \( f \) ranging from \(< 1/2, = 1/2, \) and \( > 1/2 \) respectively. (c) One-dimensional double-well potential and energy-level diagram for \( f = 0.502 \), in which case we have a three-level system in the left-hand well. States \( |1\rangle \) and \( |2\rangle \) are meta-stable, while \( |3\rangle \) will have significant loss via resonant tunneling to \( |4\rangle \) \((\sigma_{34})\). The right-hand well states undergo fast damping \((\Gamma_4, \Gamma_5, \Gamma_6)\) via the SQUID measurement and intrawell relaxation to lower states. Coupling between the our three-levels is induced by two resonant microwave fields with Rabi frequencies \( \Omega_{13} \) and \( \Omega_{23} \), forming the \( \Lambda \)-system. The qubit parameters we use in calculations are \( \omega_2 - \omega_1 = (2\pi) \) 27.8 GHz and \( \omega_3 - \omega_2 = (2\pi) \) 27 GHz, with matrix elements \( x_{ij} \) for \((i,j) = (1,2), (2,3), \) and \((1,3)\) set to -0.0145, -0.0371, -0.0263, respectively.
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\[ H_{pc} = H_0 + H_{\text{tunnel}}; \]  
\[ H_0 = \hbar \sum_i \omega_i |i\rangle \langle i|, \]  
\[ H_{\text{tunnel}} = \hbar \sum_{i,j \neq i} \sigma_{ij} |i\rangle \langle j|. \]

We note here that two points of view may be taken when discussing the system described by the Hamiltonian in Eq. (4.2). In one picture, the diabatic states (diagonal matrix elements) of the qubit are the uncoupled single-well states of classical circulating current, and these diabatic states are coupled through the tunneling terms (off-diagonal matrix elements). In a second picture, the Hamiltonian is diagonalized, resulting in eigenenergies and eigenstates of the double-well potential. Although the perspectives differ, these two pictures will, of course, lead to identical results; only the interpretation differs. Throughout the paper, we primarily describe the dynamics in terms of diabatic (single-well) states coupled through the tunneling barrier of the double-well potential and driven by harmonic excitation. Exceptions, where they exist, will be clearly noted.

The three-level \( \Lambda \) structure to implement EIT is then provided by the left-hand meta-stable states \( |1\rangle \), \( |2\rangle \) and the fast decaying level \( |3\rangle \) shown in Fig. 4-1(c). Each of these levels are taken to have a finite loss rate \( \Gamma_i^{(t)} \), due to resonant tunneling to a right-well state (at \( \sigma_{ij} \)) followed by relaxation of the right-hand well state \( \Gamma_j \) (which is a sum of the population relaxation to lower levels and damping induced by a fast SQUID measurement of the circulation current of the right-hand well states). In particular, we desire a fast decay rate \( \Gamma_3^{(t)} \), which is achieved by resonantly biasing \( |3\rangle \) and \( |4\rangle \) and a fast SQUID measurement (\( \approx 1-10 \text{ ns} \)), as analyzed in Section 4.3.4. Conversely, we desire states \( |1\rangle \) and \( |2\rangle \) to be long-lived and the tunneling \( \sigma_{25} \) will cause loss and decoherence, which is analyzed in Section 4.4.4 (\( \sigma_{16} \) is negligible by comparison). Rough estimates of the interwell loss rates for resonant-tunneling are \( \Gamma_1^{(t)} \approx (1 \text{ ms})^{-1}, \Gamma_2^{(t)} (1 \mu s)^{-1}, \text{ and } \Gamma_3^{(t)} \approx (1 \text{ ns})^{-1} \) and the off-resonant biasing of states.
|1⟩−|6⟩ and |2⟩−|5⟩ will significantly decrease these rates. In addition, states |2⟩ and |3⟩ can have intrawell relaxation rates $\Gamma_{3-1}$, $\Gamma_{3-2}$, $\Gamma_{2-1}$ (not shown in the diagram). Under similar bias conditions the rate $\Gamma_{3-1} + \Gamma_{3-2} \approx (25 \mu s)^{-1}$ (experimentally measured in [61]) is much slower than $\Gamma_3^{(e)}$. Also note that, $\Gamma_{2-1}$, another source of decoherence of the meta-stable states, will be less than $\Gamma_{3-1} + \Gamma_{3-2}$.

These quantized levels may be coupled using microwave radiation. An applied radiation field $\mu$ can be described in terms of an amplitude, frequency and phase: $\Delta f_\mu = g_\mu \cos(\omega_\mu t + \phi_\mu)$. We find the resulting matrix elements for level transitions (the Rabi frequencies) by treating $\Delta f_\mu$ as a small perturbation in the $\cos(2\pi f + 2\varphi_m)$ term in Eq. (4.1). We write it as $\sin(2\pi f + 2\varphi_m)\sin(2\pi \Delta f_\mu)$, which can be approximated as $\sin(2\pi f + 2\varphi_m)(2\pi \Delta f_\mu)$, leading to a Rabi frequency $\Omega_{ij}^{(\mu)} \equiv (2\pi)g_\mu \alpha E_j x_{ij}/\hbar$, where $x_{ij} = \langle i|\sin(2\pi f + 2\varphi_m)|j\rangle$. The elements $x_{ij}$ we calculate for our proposed parameters are listed in the caption of Fig. 4-1(c). In EIT, we address the SQC with two microwave fields $\Delta f_a = g_a \cos(\omega_a t + \phi_a)$, with $\omega_a \approx \omega_3 - \omega_1$, and $\Delta f_b = g_b \cos(\omega_b t + \phi_b)$, with $\omega_b \approx \omega_3 - \omega_2$. The microwave induced Hamiltonian can then be written as:

$$\mathcal{H}_{\mu\text{-wave}} = \frac{\hbar}{2} \sum_{i,j} \sum_\mu \left( \Omega_{ij}^{(\mu)} e^{-i(\omega_\mu t + \phi_\mu)} + \text{c.c.} \right) |i\rangle \langle j|$$  (4.5)

where $i, j$ runs over the states and $\mu$ runs over the two fields $a, b$. We emphasize that the above approximation is a perturbative approach valid only for small driving amplitudes. In the strongly driven limit, the approximation breaks down, preventing the Rabi frequency from growing without bound [67, 55].

Microwave excitation is used to establish the population of meta-stable states (such as |1⟩ and |2⟩) via photon-assisted tunneling. In this scheme, the population of a meta-stable state is driven via a resonant radiation field into a read-out state (i.e. |3⟩) which quickly tunnels to a measurement state (i.e. |4⟩). This state has opposing current circulation with a unique flux signature that can be measured using a DC-SQUID [25]. In the present scheme, two fields are simultaneously applied (resonant with |1⟩ ↔ |3⟩ and |2⟩ ↔ |3⟩; see Fig. 4-1(c)), and EIT is manifested by a suppression of the photon-assisted tunneling due to quantum interference between
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the two excitation processes.

4.2.2 Evolution model

It is convenient to calculate dynamics from the above Hamiltonian terms in an interaction picture which transforms away the diagonal energies $\mathcal{H}_0$ (4.3). In this frame the total Hamiltonian is then the sum of (4.4) and (4.5):

$$\mathcal{H} = \frac{\hbar}{2} \sum_{i,j} \sum_{\mu} (\Omega_{ij}^{(\mu)} e^{-i(\omega_{ij} t + \phi_{ij})} + c.c.) e^{i(\omega_i - \omega_j) t} |i\rangle \langle j| + \hbar \sum_{i,j \neq i} \sigma_{ij} e^{i(\omega_i - \omega_j) t} |i\rangle \langle j| \quad (4.6)$$

Note that the exponential arguments involve sums of microwave frequencies $\omega_{ij}$ and level splittings $\omega_i - \omega_j$. When these nearly cancel the state is said to be near-resonant and the coupling is strong. However, for most of the terms, this cancelation does not occur, and the term rotates its phase rapidly on the scale of frequencies of interest ($\Omega_{ij}^{(\mu)}$, $\sigma_{ij}$ and $\Gamma_j$). Such terms are neglected in the Rotating Wave Approximation [52].

We can also include incoherent losses from the levels, $\Gamma_i$, by introducing an additional non-Hermitian part of the Hamiltonian $\mathcal{H}_{\text{relax}} = -i\hbar \sum_i (\Gamma_i/2) |i\rangle \langle i|$. This is often done in quantum optics [63, 91] to include non-Hermitian decay of radiatively decaying levels. We then describe the system by a wavefunction $|\Psi\rangle = \sum_i c_i(t) |i\rangle$, with the initial population normalized to unity $\sum_i |c_i(0)|^2 = 1$ (this can decay in time due to the non-Hermitian loss). The evolution of the $|\Psi\rangle$ is then governed from Schrodinger's equation:

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = (\mathcal{H} + \mathcal{H}_{\text{relax}})|\Psi\rangle. \quad (4.7)$$

Besides giving the coherent dynamics, this Schrodinger equation correctly predicts the population relaxation of level $|i\rangle$ at $\Gamma_i$ and also gives the correct dephasings of coherences between $|i\rangle$ and other states at half this rate $\Gamma_i/2$.

When necessary, we use a density matrix approach to include incoherent processes. For example, pure dephasing of a coherence between the two meta-stable states $|1\rangle$
and $|2\rangle$ goes beyond the Hamiltonian approach (4.7). Similarly, incoherent feeding of levels (such as population into $|1\rangle$ from interwell relaxation $\Gamma_{2\rightarrow1}$) goes beyond this description. The density matrix is written $\hat{\rho} = \sum_{ij} \hat{\rho}_{ij} |i\rangle \langle j|$, where the populations in the levels are given by the diagonal terms $\hat{\rho}_{ii}$ and correspond to $|\hat{c}_i|^2$ in the wavefunction description, while the off-diagonal terms $\hat{\rho}_{ij}$ correspond to $\hat{c}_i \hat{c}_j^*$ and describe coherences between levels. The evolution of the density matrix is given by

$$i\hbar \frac{\partial}{\partial t} \hat{\rho} = [\hat{H}, \hat{\rho}] + \mathcal{L}\hat{\rho}.$$  (4.8)

The first term reproduces the part already predicted by the Schrodinger equation (4.7), while the super-operator $\mathcal{L}$, the Lindbladian [63], accounts for other incoherent processes. For pure dephasing of the $|i\rangle \leftrightarrow |j\rangle$ coherence, $\gamma_{ij}$, we introduce a term $\mathcal{L}_{ij,ij} = -\gamma_{ij}$. For a population relaxation from $|j\rangle \rightarrow |i\rangle$, $\Gamma_{j\rightarrow i}$, we introduce $\mathcal{L}_{jj,ii} = +\Gamma_{j\rightarrow i}$. The associated population loss from $|j\rangle$ and decoherences are already included through Eq. 4.7 (via a term $-i\hbar (\Gamma_{j\rightarrow i}/2)|i\rangle \langle i|$).

Throughout the chapter, we consider the model in a number of distinct cases. In each, we include three levels $|1\rangle, |2\rangle, |3\rangle$, coupled by two microwave fields $\Delta f_a, \Delta f_b$ making up our $\Lambda$ system (see Fig. 4-1(c)). We then selectively include additional levels, such as the $|4\rangle, |5\rangle$, and $|e\rangle$ to isolate the contributions of each of them. Numerical results were obtained with a fourth-order Runge-Kutta algorithm [92] solving Eq. (4.8). In it we do not make any RWA assumptions a priori, but instead introduce some cut-off frequency $\omega_{RW_A}$. We examine the phase factors of each term in the evolution and set to zero, the ones with phases rotating faster than $\omega_{RW_A}$.

We compare our numerical results with approximate analytic solutions in many cases. When possible, we use the Schrodinger equation (4.7) to obtain simpler analytic results, though the full density matrix approach is used when dephasing and interwell relaxation are considered (Sections 4.4.1- 4.4.3). In the analytic results we normally make an additional transformation $|\tilde{\Psi}\rangle \rightarrow |\Psi\rangle$, $\hat{H} + \hat{H}_{\text{relax}} \rightarrow \hat{H}$, defined by transformations of each level frequency $\{\tilde{c}_i\} \rightarrow \{c_i\} = \{\tilde{c}_i e^{i\delta_i t}\}$, where the $\delta_i$ are chosen to eliminate time-dependent exponential phase factors in (4.6) (they are usually
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4.3.1 Ideal EIT in a Λ configuration

We first consider the 'ideal' case in which the three levels in the left well (see Fig. 4-1(c)) are well isolated from direct tunneling to other levels, states |1⟩ and |2⟩ are perfectly stable, and |3⟩ quickly decays at some fast rate $\Gamma_3^{(t)}$. This decay is in reality due to resonant coupling of |4⟩ ($\phi_{34}$) and subsequent SQUID measurement $\Gamma_4$, but we will see in Section 4.3.4 how one can derive $\Gamma_3^{(t)}$ in terms of these underlying processes.

We apply fields with nearly resonant frequencies $\omega_a = \omega_3 - \omega_1 + \Delta_{13}$ and $\omega_b = \omega_3 - \omega_2 + \Delta_{23}$ (see Fig. 4-1(c)), where the $\Delta_{13}, \Delta_{23}$ are small detunings. All other couplings are sufficiently detuned to safely eliminate them under the RWA. In this case the transformations to eliminate phase rotating terms are given by $\delta_1 = 0$, $\delta_2 = \Delta_{13} - \Delta_{23}$, $\delta_3 = \Delta_{13}$. The Hamiltonian, written in matrix notation in a basis $\{|1⟩, |2⟩, |3⟩\}$ is

\[
\mathcal{H} = \frac{\hbar}{2} \begin{bmatrix}
0 & 0 & \Omega_{13}^* \\
0 & -2\Delta_2 & \Omega_{23}^* \\
\Omega_{13} & \Omega_{23} & -i\Gamma_3^{(t)} - 2\Delta_{13}
\end{bmatrix}
\]  

(4.9)

where $\Delta_2 \equiv \Delta_{13} - \Delta_{23}$ is the detuning from two-photon resonance. Here we have dropped the $a, b$ labels, $\Omega_{13} \equiv \Omega_{13}^{(a)}$ and $\Omega_{23} \equiv \Omega_{23}^{(b)}$ (see Eq. (4.6)) as there is no ambiguity. The open system loss of |3⟩ due to tunneling $\Gamma_3^{(t)}$ is assumed to dominate...
incoherent population exchange due to intra-well relaxation, allowing a Schrodinger evolution analysis (4.7).

First consider the resonant case $\Delta_{13} = \Delta_{23} = 0$. A qubit initially in the ground state $|1\rangle$ can be prepared in a superposition state $|\Psi_{\text{init}}\rangle = c_1|1\rangle + c_2|2\rangle$ by temporarily driving it with a field resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition. Applying only one field $\Omega_{13}$ ($\Omega_{23}$), then allows the population of a state $|1\rangle$ ($|2\rangle$) to be read out through a transition to state $|3\rangle$ followed by a rapid escape to the right well. In this case, the superposition is destroyed by the absorption of a photon.

However, from (4.7) and (4.9) it follows that if we simultaneously apply both fields and the SQC is in the dark state

$$|\Psi_{\text{dark}}\rangle = \frac{\Omega_{23}}{\Omega} |1\rangle - \frac{\Omega_{13}}{\Omega} |2\rangle,$$

(4.10)

(where $\Omega \equiv \sqrt{|\Omega_{13}|^2 + |\Omega_{23}|^2}$) then (4.7) predicts $\dot{c}_1 = \dot{c}_2 = \dot{c}_3 = 0$. For this particular state, the two absorption processes, $|1\rangle \leftrightarrow |2\rangle$ and $|1\rangle \leftrightarrow |3\rangle$, have equal and opposite probability amplitudes and thus cancel by quantum interference. As a result, no excitation into $|3\rangle$, and thus no tunneling to the right well, will be observed. Note that $|\Psi_{\text{dark}}\rangle$ constrains both the relative intensity and phase of the light fields. Any other (non-zero) values for the relative amplitudes in the two states $|1\rangle, |2\rangle$ will lead to a coupling into $|3\rangle$ and subsequent loss.

An alternative interpretation is obtained by examining the eigensystem of the Hamiltonian (4.9). The dark state $|\Psi_{\text{dark}}\rangle$ has an eigenvalue zero. The other two eigenstates are linear combinations of the excited state $|3\rangle$ and the combination of the stable states orthogonal to $|\Psi_D\rangle$: $|\Psi_A\rangle = (\Omega_{13}|1\rangle + \Omega_{23}|2\rangle)/\Omega$, called the absorbing state (a “bright state”). The system $\{|\Psi_A\rangle, |3\rangle\}$ acts effectively as a two-level system coupled by $\Omega$. The eigenvalues corresponding to the two eigenstates are $(-i\Gamma_3^{(t)} \pm \sqrt{4\Omega^2 - \Gamma_3^{(t)^2}})/4$ and the imaginary parts of these eigenvalues give the loss rates of these states. In the limit $\Omega \gg \Gamma_3^{(t)}$, these rates are both $\Gamma_3^{(t)}/4$ and one observes damped Rabi oscillations. In the limit $\Omega \ll \Gamma_3^{(t)}$, there is an eigenstate $\approx |\Psi_A\rangle$ with a slower damping rate $\Omega^2/2\Gamma_3^{(t)}$. 
Figure 4-2: Suppression of tunneling due to EIT for various ideal wavefunctions (color online) (a) The populations of the states as a function of time in the presence of applied fields $\Omega_{13} = \Omega_{23} = (2\pi) 150$ MHz and the tunneling rate $\Gamma_3^{(t)} = (2\pi) 130$ MHz $= 1/1.2$ ns for the initial state $\rho_{11} = \rho_{22} = 0.5, \rho_{12} = -0.5$ (the dark state). The dotted (red) curve shows $\rho_{11}$, the dashed (blue) $\rho_{22}$ and the thin solid (green) $\rho_{33}$. The total population (sum of the three) is the thick solid (black) curve. There is a slow exponential decay of the population due to the dephasing rate $\gamma_{12} = (2\pi)1$ MHz. (b) The population evolutions (same convention) for the initial state $\rho_{11} = \rho_{22} = 0.5, \rho_{12} = 0.5$ (the absorbing state). (c) The population evolutions for an initial state $|1\rangle$ (which is an equal superposition of the dark and absorbing states).
CHAPTER 4. EFFECT OF DECOHERENCE PATHWAYS ON S-EIT

Figure 4-2(a) shows an example of the lack of tunneling in the presence of applied fields $\Omega_{13} = \Omega_{23}$ for the corresponding dark state $|\Psi_{\text{init}}\rangle = (|1\rangle - |2\rangle)/\sqrt{2}$ (i.e. $\rho_{11} = \rho_{22} = 0.5$, $\rho_{12} = -0.5$). One sees only a barely perceptible population $\rho_{33}$ and a very slow loss of the $\rho_{11}$ and $\rho_{22}$. This is due to a pure dephasing of the state coherence, which we take to be $\gamma_{12} = (2\pi) \times 1$ MHz. The effect of this dephasing is a small exponential loss at a rate we label $R_{\text{EIT}}^{(\gamma_{12})}$, which is discussed and derived in [13] and reviewed in Section 4.4.2. Otherwise the populations remain $\rho_{11} = \rho_{22} \approx 0.5$. EIT thus provides a means to confirm, without disturbing the system, that one had indeed prepared the qubit in a particular desired state of the SQC, preserving its quantum coherence.

By contrast, Fig. 4-2(b) shows the large loss induced when one applies these same fields to the absorbing state, i.e., the state with the same populations but $\pi$ out of phase: $|\Psi_{\text{init}}\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$. In Fig. 4-2(b) we see that there is a large population in the $|3\rangle$ and the entire population has tunneled to the right well within about 10 ns. Note that here we are in the intermediate regime $\Omega \sim \Gamma_{3}^{(t)}$ so we get oscillations with period $\sim \Omega$ strongly damped at $\sim \Gamma_{3}^{(t)}/2$. This is completely analogous to the tunneling which occurs with a single applied field in a two-level scheme.

A general state can be decomposed into dark and absorbing state components. Fig. 4-2(c) shows a case where the initial population is purely in $|1\rangle$ and the same fields are applied. Here the initial state can be written $|\Psi_{\text{init}}\rangle = |1\rangle = (|\Psi_{\text{dark}}\rangle + |\Psi_{A}\rangle)/\sqrt{2}$. Half of the population (the component in the absorbing state) is coupled out over the 10 ns time scale while the dark state component remains. In terms of level populations $\rho_{11}$, $\rho_{22}$, approximately $1/4$ of the population is coherently coupled from $|1\rangle$ to $|2\rangle$.

4.3.2 EIT with imperfect state preparation

One of the useful aspects of EIT is the extremely sensitive manner in which it can measure the amplitude and phase of superpositions in the SQC. When the prepared state has a slightly different phase or population ratio than the state we intend to prepare, EIT could be used to measure these deviations. Such imperfect preparation could arise, for example, due to imperfections in the preparation pulse.
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Figure 4-3: Imperfect state preparation (color online) (a) The time evolution of the population in the left well as a function of initial state of the form $|\Psi_{\text{ini}}\rangle = \sqrt{p_1}|1\rangle - \sqrt{(1 - p_1)}|2\rangle$. The uppermost curve is for the dark state $p_1 = 0.5$. Successively lower curves are for $p_1 = 0.6, 0.7, 0.8, 0.9$ and 1.0. There is a sharp initial decay when there are deviations from the dark state. Inset: The population in the left well at 200 ns versus $p_1$. (b) The population decays out of the left well as a function of the initial phase of the prepared state, $|\Psi_{\text{ini}}\rangle = (|1\rangle - e^{i\theta}|2\rangle)/\sqrt{2}$. $\theta = 0$ (top curve) and the other curves are $\pi/5, 2\pi/5, 3\pi/5, 4\pi/5$, and $\pi$. We see full decay for $\theta = \pi$, the absorbing state $|\Psi_A\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$. Inset: The population in the left well at 200 ns as a function of $\theta$. 
Fig. 4-3(a), shows the populations loss when preparing a state with various initial state population ratios and applying fields $\Omega_{13} = \Omega_{23}$. We again introduce a small dephasing $\gamma_{12} = (2\pi)1 \text{ MHz}$. The inset shows the population at 200 ns, well after the initial transient losses have occurred. This data can be understood using the dark/absorbing basis discussed above. The modulus square of the overlap of the initial state and the dark state $\langle \Psi_{\text{dark}} | \Psi_{\text{init}} \rangle$ gives the population remaining after the fast initial loss of the absorbing component. Postulating that the slower loss (due to dephasing or other effects) is exponential with some rate $R_L$, the population after the fast initial loss is:

$$|\langle \Psi_{\text{init}} | \Psi_{\text{dark}} \rangle|^2 e^{-R_L t}. \quad (4.11)$$

Thus, detecting the fast initial decay of the population can indicate the mismatch between the fields and the prepared state population.

Phase mismatch, or unwanted $z$-rotation, in the qubit preparation shows similar behavior. Figure 4-3(b) shows the population decay from the left well for the state $\left( |1\rangle - e^{i\theta}|2\rangle \right)/\sqrt{2}$. The upper most line is decay due to the perfect state, while the lower lines indicate the decay for varying value of $\theta$. This example indicates that the dark state, is more sensitive to phase mismatch than population mismatch.

### 4.3.3 EIT detuned from resonance

Because EIT is a coherent effect, it only occurs in a narrow range of frequencies near the two-photon resonance. The width of the EIT feature is generally determined by the field intensities, and can be made narrower than the broad resonances of the individual one-photon transitions ($|1\rangle \leftrightarrow |3\rangle$ and $|2\rangle \leftrightarrow |3\rangle$), which are determined by the fast decay rate of $|3\rangle \Gamma_3^{(t)}$.

Figure 4-4(a) shows the results of simulations with the same parameters as Fig. 4-2(a), but with the detuning $\Delta_{23}$ varied. The curves show exponential loss occurring at various rates. We can analyze the results with the Hamiltonian (4.9) and the corresponding Schrodinger equation (4.7). We first adiabatically eliminate [91] the
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Figure 4-4: EIT in the presence of detuning. (color online) (a) Numerical calculation of the total population in time when $\Delta_{13} = 0$ MHz, and at various $\Delta_{23}$ (top to bottom curve) $(2\pi) 0$, $10$, $20$, $30$, $40$ MHz. For $\Delta_{23} = 0$, the decay is due to pure dephasing, while the decay is sharper when $\Delta_{23} \neq 0$. Inset: The population in the left well at 200 ns as a function of the two-photon detuning $\Delta_2 \equiv \Delta_{13} - \Delta_{23}$, with $\Delta_{13} = 0$ (circles). The solid curve shows the prediction (4.13). The diamonds show the case $\Delta_{13} = (2\pi)1$ GHz, varying $\Delta_2$ about the two-photon resonance. (b) The population decay as a function one-photon detuning at two-photon resonance, $\Delta_{13} = \Delta_2 = \Delta$ for (top to bottom) $(2\pi) 3$, $2$, $1$, $0.5$, $0$ GHz. As the one-photon detuning increases, the effective coupling of the fields to the transition reduces, hence reducing the rate of decay. Inset: The population in the left well at 200 ns as a function of the detuning for the dark (diamonds). For comparison we also show the population remaining for the absorbing state $|\Psi_{\text{init}}\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$ at the same detunings (circles).
excited level by setting \( \dot{c}_3 = 0 \) and obtain

\[
c_3 = (\Omega_{13} c_1 + \Omega_{23} c_2) \left( \frac{2\Delta_{13} + i\Gamma_3^{(t)}}{4\Delta_{13}^2 + \Gamma_3^{(t)2}} \right)
\]

(4.12)

This expression is valid for times long compared to the initial transient time \( \text{Min}\{((t))^{-1}, \Delta_{13}^{-1}\} \). Note that for the dark state (4.10) the amplitude \( c_3 \) vanishes. Plugging this expression back into the equations for \( \dot{c}_1, \dot{c}_2 \) then gives a 2 x 2 matrix evolution equation, which can be easily solved by finding for its eigenvalues and eigenvectors. For \( \Delta_2 = 0 \) the eigenvectors are simply the dark states \( |\Psi_D\rangle \) and absorbing \( |\Psi_A\rangle \) states of Section 4.3.1, with eigenvalues \( \lambda_D = 0 \) and \( \lambda_A = -\Omega^2(\Gamma_3^{(t)} + 2i\Delta_{13})/2(4\Delta_{13}^2 + \Gamma_3^{(t)2}) \), respectively. The absorbing component population is damped out at \(-2\text{Re}\{\lambda_A\}\).

In many cases, we are interested in the regime close to the one-photon resonance \( \Delta_{13} \ll \Gamma_3^{(t)} \) for which this reduces to \( 2\Gamma(t)/\Omega(t) \) as in Section 4.3.1.

With a non-zero two-photon detuning \( \Delta_2 \), this process is complicated by the additional evolution term \( \dot{c}_2 = \ldots i\Delta_2 c_2 \), which acts to drive the phase of the SQC out of the dark state and competes with damping of the absorbing component. Solving for the eigensystem in this case we see that, in the limit of small two-photon detuning \( (|\Delta_2| \ll |\text{Re}\{\lambda_A\}|) \), the eigenvalue corresponding to the dark component has a non-zero negative real component, leading to a population loss rate:

\[
-2\text{Re}\{\lambda_D\} \approx \Gamma_2^{(\Delta_2)} = 4 \frac{\Omega_{13}^2|\Omega_{23}|^2 \Delta_{13}^{(t)} \Gamma_3^{(t)}}{\Omega^4}
\]

(4.13)

The prediction \( P = \exp[-(\Gamma_2^{(\Delta_2)} + \Gamma_2^{(\Delta_2)})\ell] \) is plotted in the inset of Fig. 4-4(a) and is seen to agree well with the numerical results (where \( \Gamma_2^{(\Delta_2)} \) was determined by the numerically calculated loss for \( \Delta_{23} = 0 \)). Eq. (4.13) shows how the field strength, via \( \Omega^2 \) in the denominator, determines the frequency width of the EIT feature.

The above analysis indicates that it is only the two-photon detuning \( \Delta_2 \) which affects the relative phase of \( |1\rangle \) and \( |2\rangle \) and therefore affects the dark state. EIT will occur in the presence of a large one-photon detuning \( \Delta_{13} \) and the inset of Figure 4-4(a) shows such a case with \( \Delta_{13} = (2\pi) 1 \text{ GHz} \) and \( \Delta_{23} \) varied about the two-photon resonance. The presence of a transparency peak is still clear. The important
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difference is, because of the large one-photon detuning $\Delta_{13} \gg \Gamma_3^{(t)}$, the damping of the absorbing state $-2\text{Re}\{\lambda_A\}$ is substantially reduced and so both the dephasing $R_L^{(\Delta_2)}$ and detuning $R_L^{(\Delta_3)}$ loss are reduced. The analytic model (4.13) is not valid for large one-photon detunings $\Delta_{13}$ where the strong damping assumption $-2\text{Re}\{\lambda_A\} \gg |\Delta_2|$ does not hold. As a result, one sees non-exponential decay in the large one-photon detuning cases (upper curves of Fig. 4-4(b)).

Figure 4-4(b) shows simulations at two-photon resonance $\Delta_{13} = \Delta_{23}$, varying the one-photon detuning $\Delta_{13}$. The population decay is much slower as the detuning gets larger. For comparison, we also plot the decay for the initial state equal to the absorbing state $|\Psi_{\text{init}}\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$. We note that the analytic model for loss of the dark state (4.13) is invalid for large one-photon detunings, where the absorbing state is not completely damped.

4.3.4 Effective $\Lambda$-system via tunneling and measurement

Thus far we have considered the system to be a three level system with the excited level $|3\rangle$ subject to a fast population decay $\Gamma_3^{(t)}$. Underlying this decay are actually two processes: the fast resonant tunneling to a near degenerate level in the right hand well ($\sigma_{34}$) followed by interwell relaxation and possibly a strong measurement of the population in $|4\rangle$ ($\Gamma_4$); see Fig. 4-1(c). We show here how the picture of a three-level system with a strong damping of $|3\rangle$ (the Hamiltonian (4.9)) is most valid when $\sigma_{34} < \Gamma_4$ but actually has a larger range of validity than one might expect. We derive an expression for $\Gamma_3^{(t)}$ and also see how the tunneling slightly shifts $\omega_3$.

To do this we consider the Schrodinger evolution of the full four-level system Hamiltonian (with the same frame transformation as Eq. (4.9)) and $\delta_{4} = \Delta_{13} + \delta_{34}$, where $\delta_{34} = (\omega_4 - \omega_3)$:
\[
\mathcal{H} = \frac{\hbar}{2} \begin{bmatrix}
0 & 0 & \Omega_{13}^* & 0 \\
0 & -2\Delta_2 & \Omega_{23}^* & 0 \\
\Omega_{13} & \Omega_{23} & -2\Delta_{13}^{(0)} & 2\sigma_{34} \\
0 & 0 & 2\sigma_{34} & -i\Gamma_4 - 2(\Delta_{13}^{(0)} + \delta_{34})
\end{bmatrix},
\]
(4.14)

We have used the notation \(\Delta_{13}^{(0)}\) to distinguish it from \(\Delta_{13}\) which includes the frequency shift of \(\omega_3\) induced by \(|4\rangle\).

To recover our three-level picture, we note that when \(\Gamma_4 \gg \sigma_{34}\) we can adiabatically eliminate level \(|4\rangle\) to obtain \(c_4 = -2c_3\sigma_{34}/[2(\delta_{34} + \Delta_{13}^{(0)}) - i\Gamma_4]\). Plugging this result back into the equation for \(\dot{c}_3\) reveals that our system can be reduced to a three-level system as in (4.9) with \(\Gamma_3^{(t)} = 4|\sigma_{34}|^2\Gamma_4/[\Gamma_4^2 + 4(\delta_{34} + \Delta_{13}^{(0)})^2]\) and \(\Delta_{13} = \Delta_{13}^{(0)} + 4|\sigma_{34}|^2\delta_{34}/[\Gamma_4^2 + 4(\delta_{34} + \Delta_{13}^{(0)})^2]\). Alternatively, when \(\Gamma_4 \ll \sigma_{34}\) we would expect the tunneling to induce a splitting of \(|3\rangle\) and \(|4\rangle\) into two superposition eigenstates (split by \(2\sigma_{34}\)).

We carried out several numerical simulations of the four-level density matrix equations (4.8) for this system, considering first the resonant case (\(\delta_{34} = 0\)). In them we used \(\sigma_{34} = (2\pi)150\) MHz = \((1.2\ ns)^{-1}\), fields \(\Omega_{13} = (2\pi)120\) MHz, \(\Omega_{23} = (2\pi)150\) MHz, the corresponding dark initial state \(\rho_{11} = 0.61, \rho_{22} = 0.39, \rho_{12} = -\sqrt{\rho_{11}\rho_{22}}\) (i.e. full coherence), and a dephasing rate \(\gamma_{12} = (2\pi)2\) MHz. Figure 4-5(a) shows the populations \(\rho_{33}, \rho_{44}\) versus time for cases \(\Gamma_4^{-1} = 1\ ns = ((2\pi)159\ MHz)^{-1}\) (thin, blue curves) and \(25\ ns = ((2\pi)6\ MHz)^{-1}\) (thick, red curves). Fig. 4-5(b) shows the total population remaining versus time. In the fast measurement case, 1 ns, \(\rho_{33}\) and \(\rho_{44}\) are seen to track each other, and we see the exponential decay of the population as in the previous examples. For the slower measurement case, 25 ns, we see \(\rho_{33}\) and \(\rho_{44}\) still track each other, but here there is a large excitation \(\rho_{33}\) (note the different scale), no fast transients in \(\rho_{33}\) to a quasi-steady state value, and non-exponential population decay, all indications of the breakdown of EIT.

To check the validity of the effective three-level model, in Fig. 4-5(c) we compare its predictions for the populations remaining after 100 ns (solid curve) with predictions
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Figure 4-5: Consequences of the measurement state characteristics. (color online) (a) The thinner curves (blue) show populations $p_{44}$ (solid curves) and $p_{33}$ (dashed) as a function of time for a fast read-out $\Gamma_4^{-1} = 1 \text{ ns} < \sigma_{34}^{-1}$ (with scale on the right side). After initial transient period, the two values reach quasi-steady state values, which undergo slow exponential decay. Conversely the thick (red online) curves show a slow measurement case $\Gamma_4^{-1} = 25 \text{ ns} \gg \sigma_{34}^{-1}$ (scale on the left). In this case the populations do not reach a quasi-steady over the time scale plotted. (b) The total population remaining versus time for the same two cases. (c) The population remaining at 100 ns for varying measurement rates (dots) and compared to the prediction of a three-level system with $\Gamma_3^{(t)}$. As the measurement gets slower, this prediction slightly and increasingly underestimates the actual population which should be observed. (d) Numerical results (dots) and three-level model predictions (solid curves) now letting $\delta_{34}$ vary, for the cases $\Gamma_4^{-1} = 1 \text{ ns}$ (black, lower curve) and 10 ns (upper, red curve).
of the full four-level model (dots). The agreement is excellent for the $1/\Gamma_4 \leq 5$ ns, and is still in rough agreement even up to 50 ns. The adiabatic elimination procedure appears to be valid well beyond the expected regime $\Gamma_4^{-1} \ll \Gamma_3^{-1}$. The breakdown of EIT in the 25 ns case is due to $\Gamma_3^{(t)}$ becoming too large (see Section 4.4.2), rather than a breakdown of the effective three-level model.

In Fig. 4-5(d) we show the population remaining for when $\delta_{34}$ is varied (for both $1/\Gamma_4 = 1$ ns (black curves) and 10 ns (red curves)). We have kept the microwave fields on bare resonance $\Delta_{13}^{(0)} = \Delta_{23}^{(0)} = 0$ and accounted for the predicted frequency shifts of $\omega_3$ in the three-level model. As $\delta_{34}$ becomes comparable to $\Gamma_4$ the tunneling rate $\Gamma_3^{(t)}$ goes down as predicted.

In summary, we find that so long as $\Gamma_4 > \sigma_{34}$ the simple three-level system provides an excellent model and even when $\Gamma_4 \sim \sigma_{34}$ or somewhat larger, this model unexpectedly gives very good predictions of the behavior. However, one must be careful of the strong dependence of $\Gamma_3^{(t)}$ on $\Gamma_4$, as this may severely affect the necessary conditions for EIT (which are discussed in Section 4.4.2). Through $\Gamma_4$ the SQUID measurement rate can thus have a large influence on the EIT. When the detuning $\delta_{34}$ becomes comparable to $\Gamma_4$ the tunneling rate is reduced as expected and one must account for the frequency shift of $\omega_3$. For the remainder of the chapter, we will not explicitly include $|4\rangle$ in the calculations, but assume some $\Gamma_3^{(t)} \sim 1$ ns and that the frequency shift is already included in the definitions of $\Delta_{13}, \Delta_{23}$.

### 4.4 Effect of decoherence and qubit tunneling

An outstanding, important issue in the eventual application of SQCs to quantum computing is the identification and suppression of sources of decoherence and unwanted dynamics of the qubit states ($|1\rangle$ and $|2\rangle$). In particular, decoherence from pure dephasing, intrawell relaxation, interwell resonant tunneling [74], and coupling to microscopic degrees of freedom in the junction [76, 77] have all been proposed as potential hurdles in successfully isolating a coherent two-level system for use as a qubit. Use of phase sensitive methods such as EIT could be a fruitful path for ex-
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exploring and differentiating the contributions of these various decoherence processes to qubit dynamics. To learn how EIT is affected by decoherence, we use the density matrix approach here to include pure dephasing, population loss, incoherent population exchange, and resonant coupling to the right well. We find a minimum microwave coupling strength necessary for the observation of EIT in the presence of decoherence and see it contributes a small exponential loss (as seen in the numerical results above). We derive analytic expressions for the loss rates, which are proportional to the decoherence processes but with coefficients which depend on the nature of the process. These results are a generalization of the results for pure dephasing previously published [13].

4.4.1 Density matrix approach

To carry out this analysis, we must go beyond the Schrodinger approach, and introduce the corresponding density matrix evolution (4.8), also referred to as the Bloch equations [63]. We work in the three-level case and transform to the frame defined above Eq. (4.9) and obtain:

\[
\begin{align*}
\dot{\rho}_{11} &= -\Gamma_1^{(t)}\rho_{11} + \Gamma_{2-1}\rho_{22} - \frac{i}{2} \Omega_{13}^* \rho_{31} + \frac{i}{2} \Omega_{13} \rho_{13}, \\
\dot{\rho}_{22} &= -\left(\Gamma_2^{(t)} + \Gamma_{2-1}\right)\rho_{22} - \frac{i}{2} \Omega_{23} \rho_{32}^* + \frac{i}{2} \Omega_{23}^* \rho_{23}, \\
\dot{\rho}_{33} &= -\Gamma_3^{(t)}\rho_{33} + \frac{i}{2} \Omega_{13}^* \rho_{31} - \frac{i}{2} \Omega_{13} \rho_{13} + \frac{i}{2} \Omega_{23}^* \rho_{32} - \frac{i}{2} \Omega_{23} \rho_{23}, \\
\dot{\rho}_{12} &= -\left(\gamma_{12} + \frac{\Gamma_1^{(t)} + \Gamma_2^{(t)} + \Gamma_{2-1}}{2}\right)\rho_{12} - \frac{i}{2} \Omega_{13}^* \rho_{32} + \frac{i}{2} \Omega_{23} \rho_{23}, \\
\dot{\rho}_{13} &= -\frac{\Gamma_3^{(t)} + \Gamma_1^{(t)}}{2}\rho_{13} + \frac{i}{2} \Omega_{13}^* (\rho_{11} - \rho_{33}) + \frac{i}{2} \Omega_{23}^* \rho_{12}, \\
\dot{\rho}_{23} &= -\frac{\Gamma_3^{(t)} + \Gamma_2^{(t)} + \Gamma_{2-1}}{2} \rho_{23} + \frac{i}{2} \Omega_{23}^* (\rho_{22} - \rho_{33}) + \frac{i}{2} \Omega_{13}^* \rho_{21}. 
\end{align*}
\tag{4.15}
\]

For simplicity, we have supposed the resonant case \(\Delta_{13} = \Delta_{23} = 0\) and ignored interwell relaxation of \(|3\rangle\), which is dominated by \(\Gamma_3^{(t)}\). The remaining three elements equations are determined by \(\rho_{ij}^* = \rho_{ji}\). The most important new piece here is the
decoherence rate of $\rho_{12}$: $\gamma_{12} + (\Gamma_1^{(t)} + \Gamma_2^{(t)} + \Gamma_{2-1})/2$. This will decohere the dark state and lead to small losses in the population.

To proceed, we adiabatically eliminate the excited state state coherences $\rho_{13}, \rho_{23}$ as they are strongly damped by $\Gamma_3^{(t)}$. In these equations we can ignore $\Gamma_1^{(t)}, \Gamma_2^{(t)} \ll \Gamma_3^{(t)}$ as well as $\rho_{33} \ll \rho_{11}, \rho_{22}$. We then plug the results back into the remaining equations to obtain:

$$
\dot{\rho}_{11} = -\Gamma_1^{(t)} \rho_{11} + \Gamma_{2-1} \rho_{22} - \frac{|\Omega_{13}|^2}{2\Gamma_3^*} \rho_{11} - \frac{\Omega_{13} \Omega_{23}^*}{2\Gamma_3} \rho_{12}, \\
\dot{\rho}_{22} = -\left(\Gamma_2^{(t)} + \Gamma_{2-1}\right) \rho_{22} - \frac{|\Omega_{23}|^2}{2\Gamma_3^*} \rho_{22} - \frac{\Omega_{13} \Omega_{23}^*}{2\Gamma_3} \rho_{12}, \\
\dot{\rho}_{12} = -\left(\gamma_{12} + \frac{\Gamma_1^{(t)} + \Gamma_2^{(t)} + \Gamma_{2-1}}{2}\right) \rho_{12} - \frac{\Omega^2}{2\Gamma_3^{(t)}} \rho_{12} - \frac{\Omega_{13} \Omega_{23}^*}{2\Gamma_3^{(t)}} \left(\rho_{11} + \rho_{22}\right) \quad (4.16)
$$

We note here a strong damping of the coherence provided by the fields $\Omega^2/2\Gamma_3^{(t)}$. This damping acts to drive the system into the dark state. Adiabatically eliminating $\rho_{12}$ under the assumption that this damping is large yields the result:

$$
\rho_{12} = -\frac{\Omega_{13} \Omega_{23}^*}{\Omega^2 + 2\Gamma_3^{(t)} \left(\gamma_{12} + \frac{\Gamma_1^{(t)} + \Gamma_2^{(t)} + \Gamma_{2-1}}{2}\right)} \left(\rho_{11} + \rho_{22}\right) \quad (4.17)
$$

In the limit that the second term in the denominator vanishes, we get perfect coherence $|\rho_{12}|^2 = \rho_{11} \rho_{22} = |\Omega_{13}|^2 |\Omega_{23}|^2 / \Omega^4$ and plugging (4.17) back into (4.16) yields $\dot{\rho}_{11} = \dot{\rho}_{22} = 0$ (perfect transparency). However, the second term shows how decoherence will drive $\rho_{12}$ out of the dark state, causing excitation $\rho_{33}$. One sees the ratio of the decoherence rate compared with the EIT preparation rate $\Omega^2/\Gamma_3^{(t)}$ determines the degree to which the coherence deviates from the perfect dark state value.

### 4.4.2 Measuring dephasing with EIT

We first show this comes into play for the pure dephasing, which is expected to be the case in many practical implementations and was analyzed previously in [13]. Figure 4-6(a) (solid, red curve) shows the excited state population $\rho_{33}$ when $\gamma_{12} =$
(2π)1 MHz and we apply the fields \( \Omega_{13} = \Omega_{23} = (2\pi)150 \) MHz to the dark state \( |\Psi_{\text{init}}\rangle = |\Psi_D\rangle = (|0\rangle - |2\rangle)/\sqrt{2} \). One sees a small (note the scale in Fig. 4-6(a)) but finite excitation. In particular, there is an initial fast transient behavior to some plateau value (over a time scale determined by \( \text{Min}\{\Omega^2/\Gamma_3^{(t)}, \Gamma_3^{(t)}\} \)), followed by a slow exponential decay. The general behavior of an initial transient rise, Fig. 4-6(a), was seen over a wide parameter regime. This quasi-steady state excitation of \( |3\rangle \) is the origin of the exponential losses at rate \( R_{L}^{(\gamma_{12})} \) in the previous simulations. Fig. 4-6(b) shows the exponential decay for several different dephasing rates \( \gamma_{12} \). The inset of Fig. 4-6(b) plots the populations remaining at 200 ns, which is seen to approach unity as \( \gamma_{12} \to 0 \).

The loss rate can be quantified by considering the evolution of the \( \rho_{11} \) and \( \rho_{22} \) according to (4.16) with (4.17), which gives a 2 × 2 evolution matrix for the populations. Looking at the eigenvalue corresponding to the smallest loss rate, and expanding to first order in \( \gamma_{12} \) gives:

\[
R_{L}^{(\gamma_{12})} = 2\gamma_{12} \frac{[\Omega_{13}]^2[\Omega_{23}]^2}{\Omega^4}
\]

By measuring this decay constant experimentally, one can use (4.18) to extract \( \gamma_{12} \). Note that the two rates are simply related by a constant or order unity, determined by the relative strength of the two microwave field couplings (in the example in Fig. 4-6, the constant is \( 2\Omega_{13}^2\Omega_{23}^2/\Omega^4 = 0.5 \)). A glance at Fig. 4-6(b) reveals how choosing the observation time \( t \sim \gamma_{12}^{-1} \) will give the best sensitivity in the measurement.

The inset of Fig. 4-6(b) shows the prediction (4.18) in comparison with the analytic results and we see good agreement. The adiabatic elimination and the expansion for small \( \gamma_{12} \) require \( 2\gamma_{12}\Gamma_3^{(t)}/\Omega^2 \ll 1 \) and \( \gamma_{12} \ll \Gamma_3^{(t)} \). This ratio is 0.07 for \( \gamma_{12} = (2\pi)5 \) MHz. For higher dephasing rates, the dephasing rate competes with the preparation rate, making the adiabatic elimination approximation in obtaining (4.17) less valid. Such a case is seen in Fig. 4-6(a) (dashed, blue curve) where we plot \( \rho_{33} \) for a case with \( \gamma_{12} = (2\pi)20 \) MHz. The exponential decay occurs with a time scale comparable to the transient time to reach the quasi-steady state plateau. In such
Figure 4-6 continued on next page.
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Figure 4-6: **EIT loss due to pure dephasing** (color online) (a) The population $\rho_{33}$ versus time in the presence of a pure dephasing $\gamma_{12} = (2\pi)1$ MHz (solid, red curve, scale on left) and $\gamma_{12} = (2\pi)20$ MHz (dashed, blue curve, scale on right). In the slow dephasing case $\rho_{33}$ quickly reaches a plateau value $\rho_{33}^{(\text{max})}$ then undergoes a slow exponential decay. For the faster dephasing, the exponential decay time is similar to the time required to reach the maximum value. In each case, the initial state is $\rho_{11} = \rho_{22} = 0.5$ and $\rho_{12} = -0.5$, $\Omega_{13} = \Omega_{23} = (2\pi)150$ MHz and $\Gamma_3^{(\text{c})} = (2\pi)130$ MHz (b) The population decay with varying dephasing rates (top curve to bottom curve) $\gamma_{12} = (2\pi)$ 1, 2, 3, 4, and 5 MHz. Inset: The population in the left well at 200 ns versus $\gamma_{12}$ (dots). The solid curve shows the analytic prediction $\exp(-R_t^{(\gamma_{12})}t)$ based on Eq. 4.18, demonstrating how the population loss can be used as a probe of $\gamma_{12}$. (c) Population remaining at 100 ns versus detuning $\Delta_{13}$ (keeping $\Delta_{23} = 0$) for two different field intensities. In these simulations we used the initial conditions $\rho_{11} = 0.61$, $\rho_{22} = 0.39$, $\rho_{12} = -\sqrt{\rho_{11}\rho_{22}}$, the fields $\Omega_{13} = 0.8\Omega_{23}$, $\Gamma_3^{(\text{c})} = (2\pi)159$ MHz, $\gamma_{12} = (2\pi)1$ MHz. The solid curves are the analytic prediction described in the text for $\Omega_{23} = (2\pi)100$ MHz (blue, upper curve), and 30 MHz (black, lower curve). The dots show the numerical results.
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cases, $\gamma_{12}$ can only be estimated from the tunneling rate by more detailed modeling of the underlying Bloch equations (4.15). We note that the microwave field intensity can be adjusted to control $\Omega^2$ and to bring us into a regime where (4.18) is valid. The breakdown of this inequality occurred in the slow measurement time (25 ns) case plotted in Fig. 4-5(a,b), for which $\Gamma_3^{(t)} \approx 4|\sigma_{34}|^2/\Gamma_4$ became quite large ($\sim$ GHz).

When detuning and dephasing are both present but sufficiently small, the two effects add linearly. In Fig. 4-6(c) we show the population remaining versus detuning $\Delta_{13}$, with $\gamma_{12} = (2\pi)1$ MHz (with $\Delta_{23} = 0$) both in a strong field ($2\gamma_{12}\Gamma(t)/\Omega^2 = 0.016$) and weak field ($2\gamma_{12}\Gamma(t)/\Omega^2 = 0.18$) case. The prediction $P = \exp\left(-\left(R_{\mathcal{L}}^{(\gamma_{12})} + R_{\mathcal{L}}^{(\Delta_3)}\right)t\right)$ holds for the stronger field but overestimates the loss for the weaker field.

We conclude from the above calculations that there are two basic conditions that have be satisfied for a reliable measurement of the decoherence in the system. First, the decoherence rate $\gamma_{12}$ should be much smaller than the loss rate $\Gamma_3^{(t)}$, which holds in systems of interest. Second, we must be able to apply sufficiently strong fields that the preparation rate $\Omega^2/\Gamma_3^{(t)}$ dominates $\gamma_{12}$.

4.4.3 EIT with incoherent population loss and exchange

When the decoherence of $\rho_{12}$ occurs due to population loss and exchange instead of dephasing, the effect on the EIT is much the same, with the $\gamma_{12}$ simply replaced by the total decoherence rate. However, because these processes involve additional changes in the populations, the population loss rate, which we use to diagnose the decoherence rate, will be different.

Referring back to expression (4.17) and substituting into (4.16), we obtain the evolution matrix for $\rho_{11}, \rho_{22}$. Again finding the eigenvalues to determine the loss rate of the dark state, and expanding to first order in $\Gamma_1^{(t)}, \Gamma_2^{(t)}, \Gamma_{2\rightarrow1}$, respectively, we find:
Figure 4-7: EIT loss with dephasing, open loss, and closed loss. (color online) The population loss after 100 ns for several types of decoherence. The parameters are as in Fig. 4-6(c) but with $\Omega_{23} = (2\pi)150$ MHz. The curves show the analytic predictions (4.18) and (4.19) and the dots show numerical results. The lowest curve is for purely open loss $\Gamma_2^{(t)}$, the middle curve for pure dephasing $\gamma_{12}$, and the top curve shows pure population exchange $\Gamma_{2 \rightarrow 1}$. The horizontal axis shows the corresponding decoherence rate for each case: $\Gamma_2^{(t)}/2$, $\gamma_{12}$, $\Gamma_{2 \rightarrow 1}/2$, respectively.
As in the discussion of the intra-well relaxation of \( |3\rangle \), we can assume that this pathway is negligible (\( \Gamma_{3-1} + \Gamma_{3-2} \ll \Gamma_{3}^{(t)} \)). The present discussion is easily generalized when this cannot be assumed. In this case the population loss rate is the same as before, multiplied by a factor reflecting the proportion of atoms in \( |3\rangle \) which actually tunnel to the right well: \( \Gamma_{3}^{(t)}/(\Gamma_{3-1} + \Gamma_{3-2}) \). This reflects the fact that excitation of \( |3\rangle \) due to decoherence will only be registered as population loss upon tunneling to the right well.

\[ R_{L}^{(\Gamma_{3}^{(t)})} = \Gamma_{1}^{(t)} \frac{\left| \Omega_{23} \right|^{2}}{\Omega^{2}}; \]

\[ R_{L}^{(\Gamma_{2}^{(t)})} = \Gamma_{2}^{(t)} \frac{\left| \Omega_{13} \right|^{2}}{\Omega^{2}}; \]

\[ R_{L}^{(\Gamma_{2-1})} = \frac{\left| \Omega_{13} \right|^{4}}{\Omega^{4}} \] (4.19)

It is interesting to note that the coefficient will depend in different ways on the relative intensities of the two fields depending on the origin of the decoherence. Fig. 4-7 shows the population loss after 100 ns for different kinds of loss, each plotted versus the total decoherence rate of \( \rho_{12} \). The open system loss \( \Gamma_{2}^{(t)} \) is greater than the pure dephasing case because there is a direct population loss on top of the absorption into \( \rho_{33} \) due to decay out of the dark state. The closed system loss (intrawell relaxation \( \Gamma_{2-1} \)) is seen to be smaller than for pure dephasing, however, (4.19) shows that this could be greater or smaller depending on the relative values of \( \Omega_{13} \) and \( \Omega_{23} \).

While we assumed in this discussion that the intra-well relaxation of \( |3\rangle \) was negligible (\( \Gamma_{3-1} + \Gamma_{3-2} \ll \Gamma_{3}^{(t)} \)), the present discussion is easily generalized when this cannot be assumed. In this case the population loss rate is the same as before, multiplied by a factor reflecting the proportion of atoms in \( |3\rangle \) which actually tunnel to the right well: \( \Gamma_{3}^{(t)}/(\Gamma_{3-1} + \Gamma_{3-2}) \). This reflects the fact that excitation of \( |3\rangle \) due to decoherence will only be registered as population loss upon tunneling to the right well.

**4.4.4 Resonant tunneling loss out of left well**

Just as the decay of \( |3\rangle \) is the result of tunneling followed by decay of \( |4\rangle \), interwell tunneling of \( |2\rangle \), which is conceivably a leading order effect in the decoherence, involves tunneling to a near resonant level \( |5\rangle \) (see Fig. 4-1(c)). Here we consider this effect in detail to find the conditions where an effective damping rate \( \Gamma_{2}^{(t)} \) can be used, and also explore conditions where the dynamics are more complicated.

To explore this issue, we have performed numerical simulations for the system
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\{\mid 1\rangle, \mid 2\rangle, \mid 3\rangle, \mid 5\rangle \} where level \mid 2\rangle is detuned from \mid 5\rangle by \delta_{25} \equiv \omega_5 - \omega_2. The Hamiltonian (with the transformation \delta_5 = \Delta_2 + \delta_{25}) is:

\[
\hat{H} = \frac{\hbar}{2} \begin{pmatrix}
0 & 0 & \Omega_{13}^* & 0 \\
0 & -2\Delta_2 & \Omega_{23}^* & 2\sigma_{25} \\
\Omega_{13} & \Omega_{23} & -2\Delta_{13} - i\Gamma_3^{(t)} & 0 \\
0 & 2\sigma_{25} & 0 & -i\Gamma_5 - 2(\Delta_2 + \delta_{25})
\end{pmatrix},
\] (4.20)

Mathematically similar energy level structures have been considered in the context of atomic systems [93].

In the following we take \sigma_{25} = (2\pi)5 \text{ MHz} and set the pure dephasing \gamma_{12} = 0 to isolate the contribution from the presently considered effect. Analogous to Section 4.3.4, when \Gamma_5 \gg \sigma_{25} one can easily reduce the system to an effective-three level system with an additional loss rate \Gamma_2^{(t)} = 4|\sigma_{25}|^2\Gamma_5/(\Gamma_5^2 + 4\delta_{25}^2). The small (black) dots in Fig. 4-8(a) present the population as a function of the detuning \Delta_{13} (keeping \Delta_{23} = 0) after 100 ns of evolution for a case with \Gamma_5^{-1} = 2 \text{ ns} \approx (15\sigma_{25})^{-1} (and \delta_{25} = 0). The results are in good agreement with the prediction one obtains from the loss rate Eq. (4.19) with this predicted tunneling rate \Gamma_2^{(t)} (solid curve). Fig. 4-8(b) shows the computed loss (dots) for resonant fields (\Delta_{13} = 0) compared with the loss expected from the calculated \Gamma_2^{(t)} (solid curve) as a function of \Gamma_5, for two different field strengths. One sees the estimate is good for \Gamma_5^{-1} < 5 \text{ ns}. The inset shows \rho_{55} versus time when \Gamma_5^{-1} = 5 \text{ ns}. One sees it quickly reaches a quasi-steady state plateau, then undergoes an exponential decay.

For larger \Gamma_5^{-1}, Fig. 4-8(b) shows the loss begins to decrease in contrast to the analytic estimate. The red curve in Fig. 4-8(a) shows the population remaining versus detuning \Delta_{13} in a case in for \Gamma_5 = 80 \text{ ns}. Besides the \Gamma_2^{(t)} model incorrectly predicting complete loss of the population after 100 ns, in the numerical results there is the clear appearance of double-peaked structure. This can be understood from the coupling \sigma_{25} giving rise to two eigenstates \(\mid 2\rangle \pm \mid 5\rangle\)/\sqrt{2} split by \(2\sigma_{25}\), each of which gives rise to a distinct EIT resonance. The initial state (no population in \mid 5\rangle) is a superposition
Figure 4-8: Loss due to resonant tunneling to right well. (color online) In the simulations we assume a (resonant) tunneling rate $\sigma_{25} = (2\pi)5$ MHz, with fields $\Omega_{13} = 0.8\Omega_{23}$ and the corresponding dark state $\rho_{11} = 0.61$, $\rho_{22} = 0.39$, $\rho_{12} = -\sqrt{\rho_{11}\rho_{22}}$. (a) The population remaining at 100 ns of fields applied with strength $\Omega_{23} = 50$ MHz, versus the detuning $\Delta_{13}$ (keeping $\Delta_{23} = 0$), but varying the relaxation time $\Gamma^{-1}$. For the small solid dots (black) $\Gamma^{-1} = 2$ ns. The solid curve shows the analytic prediction based on the effective loss rate $\Gamma^{(t)}$ described in the text. The open (blue) dots show a case $\Gamma^{-1} = 15$ ns, in which case a splitting appears in the resonance, contrary to the analytic prediction (dashed curve). The large (red) dots show the case $\Gamma^{-1} = 80$ ns, for which the splitting becomes more pronounced and the loss rate quite small, while the $\Gamma^{(t)}$ model predicts complete loss of the population. (b) The population remaining at 100 ns at the two photon resonance ($\Delta_{13} = \Delta_{23} = 0$) versus $\Gamma^{-1}$. The solid (black) dots show the case $\Omega_{13} = (2\pi)150$ MHz and the open (red) dots show $\Omega_{13} = (2\pi)50$ MHz. They roughly agree with each other and the $\Gamma^{(t)}$ model (solid curve) for $\Gamma^{-1} \ll \sigma_{25} = 32$ ns. However, for larger $\Gamma^{-1}$ the loss becomes slower. The inset shows the population $\rho_{55}$ for the fast (5 ns, solid curve) and slow (80 ns, dashed) relaxation times $\Gamma^{-1}$ (note the different scales). The fast case looks analogous to decoherence (see Fig. 4-6(a)), while oscillations occur in the slow case. (c) The population remaining versus the level detuning $\delta_{25}$ in the case $\Gamma^{-1} = 8$ ns (solid, black dots) and 80 ns (open, red dots). The solid and dashed curves show the $\Gamma^{(t)}$ model predictions.
of these eigenstates and we get oscillations of the population between $|2\rangle$ and $|5\rangle$. The inset in Fig. 4-8(b) (dashed curve) shows these oscillations in $\rho_{55}$ for $\Gamma_5^{-1} = 80$ ns. Because of the weak damping, the oscillations persist and the quasi-steady state is not reached during the time scale plotted. The dotted curve and open dots in Fig. 4-8(b) show an intermediate case $\Gamma_5^{-1} = 15$ ns where the double peak structure is just becoming apparent, and the analytic estimate has begun to break down.

In Fig. 4-8(c) we address the case where the tunneling levels $|2\rangle$ and $|5\rangle$ can be slightly off-resonance ($\delta_{25} \neq 0$). The filled dots show the population remaining for $\Gamma_5^{-1} = 8$ ns versus $\delta_{25}$. The solid curve shows the $\Gamma_2^{(t)}$ model estimate, which correctly accounts for the slower tunneling rate as we move off resonance. The red dots show the same for $\Gamma_5^{-1} = 80$ ns. In this limit, the analytic estimate expression severely overestimates the loss for $\delta_{25} < \Gamma_5$ but as we move off resonance, the coherent tunneling plays less of a role and the effective tunneling decay rate model becomes more accurate.

In summary, tunneling of our lower states will be a source of loss in EIT. The behavior will depend qualitatively on the relative strength of the coherent coupling and the loss rate of the additional quantum level and so can provide us with information about these quantities. In the limit where the loss rate dominates, we see how it reduces to an open system loss of $|2\rangle$ whereas in the other limit we see a qualitative signature (the splitting of the resonance) of coherent coupling to another level.

### 4.5 EIT with radiation cross-talk

To now, we have considered how EIT is affected by decoherence and tunneling to other levels. Another important consideration to include is that all the levels are dipole coupled and so in principle coupled by the microwave fields. The RWA allows us to neglect the majority of the couplings as the dynamics are dominated by couplings which are near resonant. For example, one need not consider the coupling of field $b$ on the $|1\rangle \leftrightarrow |3\rangle$ transition or field $a$ on the $|2\rangle \leftrightarrow |3\rangle$. However, in SQCs, the relative scale of the Rabi frequencies ($\sim 100$ MHz) to the level spacings ($\sim$ GHz) is
somewhat larger than in typical atomic systems. Thus, it is important to know the magnitude and type of effects that these “cross” couplings can have. Here we consider, separately, a case with cross coupling within the three level system, and a case where fields couple to an additional excited level. In general, we find these effects can be characterized analytically in terms of additional loss rates and AC Stark shifts. If one neglected these effects, there are configurations where one may mistakenly attribute a loss rate to a dephasing when in fact it is due to off-resonant field coupling. While we found it was beneficial to turn the microwave coupling strengths $\Omega^2$ up to overcome decoherence and detuning, we will see how this can increase the importance of these cross-talk effects. We will also see how proper understanding of the effects can allow us be mitigate them by properly compensating for the Stark shifts. To isolate these cross-talk effects, we will set other losses and dephasings to zero in the following.

4.5.1 Radiation cross-talk in a three-level system

In the configuration proposed here and in [13], the qubit states $|1\rangle$ and $|2\rangle$ are the first two levels of a slightly anharmonic potential (the left well), while the excited level $|3\rangle$ is the third such level. Therefore, the level spacing $\omega_3 - \omega_2$ is only slightly different than $\omega_2 - \omega_1$ (with the parameters proposed the spacings are $\sim$30 GHz and the difference is 0.7 GHz). As a result, the field $\Omega_{23}$ is only 0.7 GHz detuned from $|1\rangle \leftrightarrow |2\rangle$ (see Fig. 4-9(a)). A rough estimate of this effect was noted in [13]. Here we present an analytic treatment which shows it causes an AC Stark shift which depends on the relative dipole coupling strengths, field intensities, and level spacings. Thus the EIT resonance position can be a function of field amplitudes used, which can be compensated by adjusting the field frequencies.

We consider the Hamiltonian (4.9) but do not invoke the RWA with respect to terms rotating by the mismatch frequency between the $|2\rangle \leftrightarrow |3\rangle$ and $|1\rangle \leftrightarrow |2\rangle$ transitions $\delta \equiv (\omega_2 - \omega_1) - (\omega_3 - \omega_2)$. 
Figure 4-9: **Cross talk in a ladder system.** (color online) (a) Schematic of the dominant cross-talk term: field $b$ (resonant with $|2\rangle \leftrightarrow |3\rangle$) also couples $|1\rangle \leftrightarrow |2\rangle$, detuned by 0.7 GHz. (b) This induces fast oscillations of the ground state populations $\rho_{11}$ and $\rho_{22}$ (and a slow overall drift) as shown here for the initial state $\rho_{11} = 0.61, \rho_{22} = 0.39$ with full coherence and $\Omega_{13} = 0.8 \Omega_{23} = (2\pi)120$ MHz and $\Gamma_3^{(t)} = (2\pi)159$ MHz. (c) Population remaining at 50 ns for $\Omega_{23} = (2\pi)50$ MHz (small, blue dots), 100 MHz (open, green dots), and 150 MHz (large, red dots), with the curves showing the analytic predictions based on the AC Stark shifts described in the text.
\[ \mathcal{H} = \frac{\hbar}{2} \begin{bmatrix} 0 & \beta \Omega_{23} e^{i\delta t} & \Omega_{13} \\ \beta \Omega_{23} e^{-i\delta t} & -2\Delta_2 & \Omega_{23}^* \\ \Omega_{13}^* & \Omega_{23} & -2\Delta_{13} - i\Gamma_3^{(1)} \end{bmatrix}, \]  

where \( \beta \equiv \frac{x_{12}}{x_{23}} \) is the ratio of dipole moments between the additional off-resonant transition and the intended resonant transition for the field \( b \). In the case we are considering \( \beta = -2.55 \) though it should be emphasized that these ratios are strong functions of the parameters and can vary by an order of magnitude.

We performed a numerical propagation of the Optical Bloch Equations (OBEs) for this Hamiltonian for the resonant case \( \Delta_{13} = \Delta_{23} = 0 \) and plot the evolution of \( \rho_{11} \) and \( \rho_{22} \) in Fig. 4-9(b). We see the extra coupling gives rise to a small amplitude, rapid oscillations of both quantities. Considering a toy two-level model with only the off-resonant coupling present predicts population oscillations of period \( (2\pi)/\delta \) and amplitude \( \sim |\beta \Omega_{23}|/2\delta \), in agreement with the numerical results. The small deviations of \( \rho_{11}, \rho_{22} \) from their dark state values gives rise to absorption into \( |3\rangle \) and thus loss. In the toy model, an off-resonant coupling can be accounted for as an AC Stark shift. In particular, \( \omega_1 \) and \( \omega_2 \) are predicted to shift by \( \pm \sim |\beta \Omega_{23}|/4\delta \), respectively. This results in an effective shift of the two-photon detuning \( \Delta_2 \) which can be compensated.

Stated in terms of our exponential loss language, the loss rate \( R_L^{(\Delta_2)} \) (4.13) is still valid but the two-photon detuning \( \Delta_2 \) should be replaced by \( \Delta_2 + \Delta_{AC}^{(12)} \), where

\[ \Delta_{AC}^{(12)} = \frac{|\beta \Omega_{23}|^2}{2\delta} \]  

In Fig. 4-9(c) we plot the population remaining for \( \Delta_{13} \) (keeping \( \Delta_{23} = 0 \)) for three different values of field intensities. The solid curves show the theoretical prediction based on the predicted AC Stark shift. They are in good agreement (the overestimate of loss at the lowest intensities with some detuning is due to the damping of the absorbing that being too weak to efficiently keep the SQC in the dark state).
Importantly, if one adjusts the field frequencies, one can completely avoid loss due to the cross-talk coupling. Strictly speaking, there is a small loss if $|2\rangle$ decays at some small rate $\Gamma_2^{(t)}$, however, this loss is much smaller than the loss already predicted from the associated decoherence (4.19).

### 4.5.2 Off-resonant interaction with an additional excited level

The last important situation we consider is that of coupling to an additional excited level $|e\rangle$, coupled off-resonantly to states $|1\rangle$ and $|2\rangle$ via the two applied fields $a$ and $b$, thus forming a “double-$\Lambda$” system [94], as diagrammed in Fig. 4-10(a). As we will see, the extra coupling gives rise to AC Stark shifts in much the same way as we saw in Section 4.5.1. In addition, because $|e\rangle$ (unlike $|2\rangle$) is quickly decaying, the coupling gives rise to some population loss even when the AC Stark shift is compensated.

Dropping the RWA with respect to terms coupling to level $|e\rangle$ and defining $\delta_{3e} = \omega_e - \omega_3$, the Hamiltonian (using a frame $\delta_e = \Delta_{13} + \delta_{3e}$) is

$$
\mathcal{H} = \frac{\hbar}{2} \begin{pmatrix}
0 & 0 & \Omega_{13}^* & \beta_1^* \Omega_{13}^* \\
0 & -2\Delta_2 & \Omega_{23}^* & \beta_2^* \Omega_{23}^* \\
\Omega_{13} & \Omega_{23} & -2\Delta_{13} - i\Gamma_3^{(t)} & 0 \\
\beta_1 \Omega_{13} & \beta_2 \Omega_{23} & 0 & -2(\Delta_{13} + \delta_{3e}) - i\Gamma_e
\end{pmatrix},
$$

with $\beta_i \equiv x_{14}/x_{13}$. We have assumed some large open loss channel $\Gamma_e$.

In the case where only one of the couplings is present ($\beta_1 = 0$ or $\beta_2 = 0$), the effect is simple to calculate. When $\beta_1 = 0$, one can consider the Schrödinger evolution $\dot{c_e}$ from Eq. (4.23) and adiabatically eliminate $c_e$ to obtain

$$
c_e = -i \frac{\beta_2 \Omega_{23} c_2}{2i\delta_{3e} - \Gamma_e}
$$

where we have assumed $\delta_{3e} \gg \Delta_{13}$. Substituting this back into the equation for $\dot{c}_2$ yields:
Figure 4-10: Coupling to an additional excited level. (color online) (a) Schematic of off-resonant microwave coupling of each of states |1⟩ and |2⟩ to an additional level |e⟩ above the barrier. (b) The population remaining at 100 ns for the same initial state and relative field strengths as in Fig. 4-9. We show the cases $\Omega_{23} = (2\pi)60$ MHz (small solid, blue), 100 MHz (open, green), and 150 MHz (large solid, red). The solid curves show the loss and AC Stark shift predicted in the text (4.26). To isolate and clearly show the effect we have set $\beta = 0$ (from (4.21)) and used $\delta_{3e} = (2\pi)1.5$ GHz, instead of the $(2\pi)10$ GHz we predict for our proposed parameters. We use $x_{14} = 0.0054$ and $x_{24} = -0.0437$ and $\Gamma_e = \Gamma_3^{(t)} = (2\pi)159$ MHz.
revealing that the extra coupling gives rise to a Stark shift and population decay of \( |2 \rangle \). In the large detuning limit \( (\delta_{3e} \gg \Gamma_e) \), the Stark shift is \( |\beta_2 \Omega_{23}|^2 / 4\delta_{3e} \) and we have an effective loss rate from \( |2 \rangle \), \( \Gamma_{2}^{(e)} = |\beta_2 \Omega_{23}|^2 \Gamma_e / 4\delta_{3e}^2 \). Analogous results occur when \( \beta_2 = 0 \), leading to a Stark shift of the two-photon resonance of opposite sign. The population loss rates will in turn contribute to the decoherence and cause exponential loss from the EIT as discussed in Section 4.4.

When both are present, the shifts and loss rates are not simply the sum of the two separate contributions, due to interferences between them. A case where summing the two contributions clearly does not work is \( \beta_1 = \beta_2 \) (equal dipole ratios). In this case (and only this case) the dark state \( c_2/c_1 = -\Omega_{13}/\Omega_{23} \) is also completely decoupled from \( |4 \rangle \). Thus no population loss or Stark shift is induced in this case. To obtain an expression in the general case, we follow the following procedure. We adiabatically eliminate \( c_3 \) and \( c_e \) from the Schrodinger equation from Eq. (4.23), obtain the 2 x 2 evolution matrix for \( c_1 \) and \( c_2 \), then solve for the eigenvalues of this matrix. One of the values has a large imaginary part (which reduced to \( \Omega^2/\Gamma_3^{(t)} \) in the limit \( \beta_1 = \beta_2 = 0 \) and corresponds to the absorbing state. The other has a small imaginary part which vanishes when \( \beta_1 = \beta_2 = 0 \) and corresponds to the dark state. We investigated the imaginary part of this eigenvalue in the limit that \( \Omega^2/\Gamma_3^{(t)} \gg \Delta_{13}, \Delta_{23} \) and \( \beta_1 \Gamma_e/\delta_{3e} \ll 1 \) and then minimized this expression with respect to the two-photon detuning \( \Delta_2 \) (when \( \Delta_{23} = 0 \)) to obtain the Stark shift. The loss rate and shift of the resonance obtained were:

\[
\begin{align*}
R_L^{(e)} &= \frac{\Omega^2 |\Omega_{23}|^2}{\Omega_1^2} \frac{\Omega^2 \Gamma_e}{4\delta_{3e}^2 + \Gamma_e^2} |\beta_1 - \beta_2|^2; \\
\Delta_{AC}^{(e)} &= -\frac{\delta_{3e}}{4\delta_{3e}^2 + \Gamma_e^2} (\beta_1 - \beta_2)(\beta_1 |\Omega_{13}|^2 + \beta_2 |\Omega_{23}|^2).
\end{align*}
\]

These expressions reduce to the simpler cases above \( (\beta_1 = 0 \) or \( \beta_2 = 0 \) and also
disappear when $\beta_1 = \beta_2$. Interestingly, the relative sign of the dipole moments plays an important role. For example if $\beta_1 = -\beta_2$ the loss rate is actually twice what one would expect from the sum of the individual couplings. In this case, the dark state $\Psi_D$ for $|3\rangle$ is actually the absorbing state for $|4\rangle$.

In Fig. 4-10(b) we plot the population remaining versus detuning $\Delta_{13}$ for three different field intensities in the system we have been considering, but now considering the coupling to the additional level $|e\rangle$. The dots show numerical propagation of the OBEs corresponding to Eq. (4.23). Note that to isolate the effect studied at present, we have ignored the cross-talk considered in Section 4.5.1 (by setting $x_{12} = 0$) and set the pure dephasing $\gamma_{12} = 0$. The solid curves then show the analytic estimate based on the Stark shifts and loss rates (4.26). One sees excellent agreement.

It should be noted that this analysis should be able to account for the effect of multiple excited levels $e_j$ by simply summing their contributions $\sum_j R^{(e_j)}_L$ and $\sum_j \Delta^{(e_j)}_{AC}$. Because of the large frequency differences between each successive level, coherent interference between contributions from different $|e_j\rangle$ will not occur.

### 4.6 Conclusion

We have described in detail a proposal for demonstrating a quantum optical effect, EIT, in a SQC. In this context, EIT will manifest itself as the suppression of photon-induced tunneling from stable states $|1\rangle, |2\rangle$ through some read-out state $|3\rangle$, due to quantum mechanical interference for two paths of excitation. This provides a method of unambiguously demonstrating phase coherence in these systems. We have provided a thorough and mostly analytic treatment of EIT in the presence of complicating effects due to decoherence and multiple levels in SQCs, which will be important in guiding experimental implementation and observation of EIT and other quantum interference effects.

We analyzed in detail first the basic considerations of EIT such as imperfect dark state preparation, and one- and two-photon detuning and determined the expected experimental signatures. Under appropriate conditions, we obtained an expression
for the total population as a function of time Eq. (4.11), which describes a fast loss of the absorbing component, followed by a small exponential loss of the system. Over shorter times, EIT thus provides a method to confirm the successful preparation (and coherence) of the particular dark state defined by the microwave fields applied. For longer times, the observed loss rate will be a function of both the detuning from two-photon resonance as well as decoherence effects. We also discussed the important issue of how the measurement state |4⟩ plays a role in the decay of the read-out state |3⟩ and saw how the biasing condition of these levels and the SQUID measurement rate can have a large effect on the parameters of the effective three-level system.

We then discussed in detail how decoherence due to dephasing of the qubit coherence, incoherent population loss or exchange, and tunneling of levels through the barrier affects the loss rate. Measuring these loss rates can then be a powerful tool which sensitively probes these various processes. We obtained the coefficients for the loss rates, which depend differently on the field strengths, depending on the underlying decoherence processes. For the case of primarily coherent resonant tunneling, we found that the EIT will exhibit a qualitatively different double-peaked structure. Probing these effects with EIT can aid in understanding and minimizing decoherence and give information about the full multi-level structure of the SQC. A potentially interesting future investigation is to learn the signature from coupling to other quantum degrees of freedom, such as the microresonators postulated in [77, 78].

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Finally we have found that the microwave fields themselves can cause additional
loss rates and AC Stark shifts of the EIT resonance which must be accounted for when one uses stronger field strengths. Importantly, we found that, these effects can become more pronounced with $\Omega^2$, meaning there will be some intermediate field strength which balances these considerations with the decoherence and detuning effects. Also, we showed how these effects could be mitigated by proper compensation of the Stark shifts.
Chapter 5

A SQUID Based RF-Magnetic Field Detector

In this chapter, I propose the first direct measurement of transverse spin magnetization using a SQUID based resonant circuit. Starting with a brief review of the main results of chapter 2, on SQUIDs, I derive a theoretical model based on the Josephson equations to describe the non-linear effects of an RF-magnetic field on a SQUID. This could lead to an enhanced sensitivity for spin detection using SQUIDs and can have significant implications for NMR/ESR applications.

5.1 Introduction

A wide variety of techniques have been developed to detect spin magnetization, ranging from simple coil based resonant circuits to more exotic optical and magnetic-force resonance microscopy based methods. The latter technique has been extended to single spin detection[95, 96] for specific samples. Single spin detection requires intersystem connection between singlet and triplet states or long $T_1$ relaxation times. Here, we propose a general approach to high sensitivity magnetic detection using SQUIDs.

As seen in chapter 2, superconducting devices are the most sensitive detectors of magnetic flux, for example, the SQUID can detect flux changes of the order of $10^{-5}\Phi_0$. 

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This level of sensitivity has enabled efficient detection of flux qubit, where the sensitivity requirement is quite high [24].

But these techniques involve measurement of static magnetic fields. In a series of experiments, McDermott et al. [17, 18] were able to detect magnetic resonance signals of nuclei at low magnetic fields using SQUIDs. Using this technique, nuclear spin detection at 100 μ-T fields, with the corresponding resonance frequency in the kHz range, have been detected. The vast majority of magnetic resonance work are carried out at high fields to enable direct observation of the chemical or g-shifts. Here, we introduce a high frequency SQUID based NMR detection. We show that non-linear properties of the SQUID can be used to efficiently detect ESR or NMR signals. Here, electron spins are used as example of spin resonance to motivate experiments and also to enable initial high frequency/low field study.

### 5.2 RF-magnetic field generated by ESR

Here we are concerned with detection bulk spin magnetization from an ensemble of electrons placed in a static external magnetic field, \( B_0 = B_z \hat{z} \). In the dilute spin case (where there is zero electron-electron dipole interaction) in the absence of an hyperfine interaction with nuclear spins, and also in the absence of spin-orbit interaction, the time dependence of the bulk magnetization is well described by the Bloch equations [52],

\[
\frac{dM}{dt} = g_\beta_e (M \times B) + \frac{1}{T_1} (M_0 - M_z) - \frac{1}{T_2} (M_x + M_y)
\]

where, \( \mu_e = -g_\beta_e S \), and \( M_0 = \frac{1}{V} \sum_{1,N} \mu_i \). Here, \( g \) is the g-factor of the electron, \( \beta_e = \frac{e\hbar}{2m_e} \) is the Bohr magneton, and \( S \) is the spin of the electron. The precession frequency of the electron is given by \( \omega_e = \frac{\mu_e B_z}{\hbar} \).

In analogy to inductive detection schemes of magnetic resonance, there are two convenient approaches to coupling the transverse (\( M_x \) or \( M_y \)) spin magnetization to a SQUID resonant circuit: the spin system can be excited and the transient magneti-
zation can be coupled; a steady state solution of the continuous coupling of a drive field.

Hence, a realistic signal (an ESR signal) driving the SQUID would be a decaying magnetization due longitudinal and transverse relaxation. Here, we are going to study the behavior of a SQUID based resonant circuit driven by an oscillating magnetic field with and without relaxation of the electron spin. Let us start with an ideal SQUID that is driven by an non-decaying rf-magnetic field.

5.3 Theoretical Model

A SQUID is a superconducting loop interrupted by two Josephson junctions [15, 16]. A single junction consists of a metal-insulator-metal layer that is characterized by $\varphi$ - the gauge invariant phase of the electron wave function. The operation of a SQUID is based on two effects, the Josephson tunneling and fluxoid quantization (as described in chapter 2). We start with the basic Josephson relations. For a junction driven by a current $i$, we have,

$$i = I_c \sin \varphi \quad (5.2)$$

$$V = \frac{\Phi_0}{2\pi} \frac{d\varphi}{dt} \quad (5.3)$$

where, $\varphi$ is the phase across the junction, $I_c$ is the critical current of the junction, and $\Phi_0$ the superconducting flux quantum. One can define an effective inductance of a single junction, also known as the Josephson inductance (derived in chapter 2), as $L_J = \Phi_0 / (2\pi \cos(\varphi) I_c)$. The dependence of the Josephson inductance on the current and phase is non-linear, hence giving rise to non-linear properties of the SQUID. It is the non-linear behavior of the SQUID used in resonance that we are proposing as a novel and sensitive rf-magnetic field detector.

Let us start by analyzing a simple resonant circuit consisting of an ideal SQUID (that has zero junction capacitance and resistance). Understanding this simple circuit will help us analyze more complicated circuits based on SQUIDs. As shown in Fig.
Figure 5-1: A SQUID based resonant circuit with ideal junctions in parallel with a capacitor. The Josephson inductance and the capacitor form the LC-equivalent resonant circuit. The external flux $\Phi_{\text{ext}}$ drives the SQUID circuit.

5.1, a Josephson inductance and the external capacitor form the resonant circuit. For the circuit in Fig. 5-1, we have from the fluxoid quantization condition,

$$\varphi_1 - \varphi_2 + 2\pi \frac{\Phi_{\text{ext}}}{\Phi_0} = 0 \quad (5.4)$$

where, $\varphi_{1,2}$ are the phases across junctions 1 and 2, respectively. For the branch currents in the SQUID, with a symmetric configuration, we have,

$$i_1 = I_c \sin \varphi_1 \quad (5.5)$$
$$i_2 = I_c \sin \varphi_2 \quad (5.6)$$

hence,

$$i = 2I_c \sin \varphi_p \cos \varphi_m \quad (5.7)$$
where, \( \varphi_{p,m} = (\varphi_1 \pm \varphi_2)/2 \), we use the \( \varphi_{p,m} \) basis for the sake of convenience. From Kirchoff current law. For figure 5-1, we have,

\[
-C \frac{dV_{out}}{dt} = i
\]

(5.8)

From the gauge-invariant phase across the junctions, the voltage of the SQUID : for the loop consisting of junction 1 and external capacitor C, can be written as,

\[
V_{out} = \frac{d\varphi_1 \Phi_0}{dt} + \frac{d}{dt} \oint A \cdot dl
\]

(5.9)

\[
= \frac{d\varphi_1 \Phi_0}{dt} + \frac{d}{dt} \int B \cdot da
\]

(5.10)

\[
= \frac{d\varphi_1 \Phi_0}{dt} + \frac{d\Phi_{ext}}{dt}
\]

(5.11)

where, \( A \) is the magnetic vector potential, \( B \) is the magnetic field, \( a \) the area of the SQUID, and \( \Phi_{ext} \) is the external flux through the SQUID. Similarly the voltage of the SQUID for the loop with junction 2 and external capacitor C can be written as,

\[
V_{out} = \frac{d\varphi_2 \Phi_0}{dt}
\]

(5.12)

It should be noted that there is no time dependent external flux term for the loop involving junction 2. This reflects that there is no flux enclosed by the loop consisting of junction 2 and external capacitor. Since the voltages across both the loops are the same, they have to be equal. Hence, the voltage of the SQUID can be conveniently written as the average of the two loop voltages as,

\[
V_{out} = \left( \frac{d\varphi_1 \Phi_0}{dt} + \frac{1}{2} \frac{d\Phi_{ext}}{dt} \right)
\]

(5.13)

We can write the current through the capacitor as,

\[
-C \left( \frac{d^2 \varphi_1 \Phi_0}{dt^2} + \frac{1}{2} \frac{d^2 \Phi_{ext}}{dt^2} \right) = 2I_c \sin \varphi_p \cos \varphi_m
\]

(5.14)
From the fluxoid quantization condition (Eq. 5.12), we have

\[-C\left(\frac{d^2\Phi_p}{dt^2} + \frac{1}{2}\frac{d^2\Phi_{\text{ext}}}{dt^2}\right) = 2I_c \sin \varphi_p \cos \left(\frac{\pi \Phi_{\text{ext}}}{\Phi_0}\right)\] (5.15)

This equation is a non-linear differential equation which can be solved numerically. The output voltage can be estimated from Eq. 5.13, by solving for \(\varphi_p\) in Eq. 5.15. This basic analysis of the SQUID circuit shows that the SQUID is highly non-linear and this non-linearity can be used to build sensitive rf-magnetic detector. This is a basic model that would be used to build more realistic models, because the junction has its characteristic resistance and capacitance that need to be included. We include the Resistively Capacitively Shunted Junction (RCSJ) model of the SQUID and derive the relations for SQUID frequency response.

### 5.3.1 SQUID equations including RCSJ Model

We have worked with an ideal junction with no resistance or capacitance. In reality, there are quasi-particles generated due to the breaking of Cooper pairs, that give rise to a normal conduction channel through the junction. Also, the junction has its inherent capacitance due to the insulating layer. These two effects give rise to the resistance and capacitance of the junction, hence the resistively and capacitively shunted junction, also known as the RCSJ model [15].

The resonant circuit including the RCSJ model of the junction is shown in Fig. 5-2. The junction is characterized by a normal junction resistance \(R_n\), and a capacitance \(c_j\). From Fig. 5-2, the branch currents can be written as,

\[i_{1,2} = I_c \sin \varphi_{1,2} + \frac{d\varphi_{1,2}}{dt} \frac{\Phi_0}{2\pi R_n} + \frac{d^2\varphi_{1,2}}{dt^2} \frac{\Phi_0}{2\pi c_j}\] (5.16)

The total current through the SQUID is given by, \(i = i_1 + i_2\),

\[i = 2I_c \sin \varphi_p \cos \varphi_m + \frac{d\varphi_p}{dt} \frac{\Phi_0}{\pi R_n} + \frac{d^2\varphi_p}{dt^2} \frac{2\Phi_0}{2\pi c_j}\] (5.17)
Figure 5-2: A SQUID based resonant circuit including the RCSJ model of a junction. The SQUID with the external resistor and capacitor form the resonant circuit. The SQUID is the non-linear inductive element of the circuit due to the Josephson inductance.
The current through the capacitor is given by,

\[ i_c = -C \left( \frac{d^2 \varphi_p \Phi_0}{dt^2} + \frac{1}{2} \frac{d^2 \Phi_{\text{ext}}}{dt^2} \right) \]  \hspace{1cm} (5.18)

and the current through the resistor,

\[ i_R = - \left( \frac{d\varphi_p \Phi_0}{dt} + \frac{1}{2} \frac{d\Phi_{\text{ext}}}{dt} \right) \frac{1}{R} \]  \hspace{1cm} (5.19)

Applying Kirchoff’s current law, we have \( i_R + i_c = i \),

\[ -C \left( \frac{d^2 \varphi_p \Phi_0}{dt^2} + \frac{1}{2} \frac{d^2 \Phi_{\text{ext}}}{dt^2} \right) = 2I_c \sin \varphi_p \cos(-\frac{\Phi_{\text{ext}}}{\Phi_0}) + \frac{d\varphi_p \Phi_0}{dt} \frac{1}{R} \]

\[ + \frac{d^2 \varphi_p}{dt^2} \frac{1}{2} \frac{\Phi_0}{2\pi} + \left( \frac{d\varphi_p \Phi_0}{dt} + \frac{1}{2} \frac{d\Phi_{\text{ext}}}{dt} \right) \frac{1}{R} \]  \hspace{1cm} (5.20)

This differential equation can be solved for \( \varphi_p \), which on substitution in Eq. 5.13 gives the output voltage of the SQUID. Fig 5-3, shows the time domain plot of the external drive, \( \Phi_{\text{ext}} = 10^{-5} \Phi_0 \), at the resonant frequency of the SQUID. The steady state output in the time domain, and the Fourier transform of its power spectrum are shown in the middle and lower plots. As we see from figure 5-3, the dominant frequency term is the resonant frequency. This indicates the SQUID behaves like a linear inductive element at low drive amplitudes. At these amplitudes we can derive a linear approximation of the SQUID equations.

### 5.3.2 Linear Approximation

Although the SQUID circuit in Fig. 5-2 is a non-linear system, we expect the resonance behavior to remain under a linear approximation and this is likely to be the most pronounced effect for low drive amplitudes. To do so, we need to find an appropriate regime where the equation of the circuit can be solved analytically. To better understand the behavior of the SQUID circuit, one can analytically estimate the steady solution in the linear regime, where the amplitude of the drive is small. For extremely small fields \( \Phi_{\text{ext}} \ll \Phi_0 \), one can linearize Eq. 5.20, by approximating
Figure 5-3: A typical drive (upper) of the SQUID and the steady-state output voltage (middle) and its power spectrum (bottom). The amplitude, \( A = 10^{-4} \Phi_0 \), of the flux driving the SQUID is in the linear regime.
\[ \cos(-\pi \frac{\Phi_{\text{ext}}}{\Phi_0}) \approx 1 \text{ and } \sin \varphi_p \approx \varphi_p. \] Eq. 5.20 can be written as,

\[ (C + 2c_j)(\frac{d^2 \varphi_p}{dt^2}) \frac{\Phi_0}{2\pi} + (\frac{2}{R_n} + \frac{1}{R}) \frac{d \varphi_p}{dt} \frac{\Phi_0}{2\pi} + 2I_c \varphi_p = -1/2(C \frac{d^2 \Phi_{\text{ext}}}{dt^2} + \frac{1}{R} \frac{d \Phi_{\text{ext}}}{dt}) \] \hspace{1cm} (5.21)

This equation can be solved analytically and a comparison with the simulation can then be made. Let us consider the following differential equation,

\[ a \frac{d^2 q}{dt^2} + b \frac{dq}{dt} + cq = f_1 \cos(\omega t) + f_2 \sin(\omega t) \] \hspace{1cm} (5.22)

This equation represents an LCR resonant circuit driven by a source \( f_1 \cos(\omega t) + f_2 \sin(\omega t) \). The equation that was derived from linearization of the circuit in figure 5-1 is also of the same form. This equation can be solved exactly. The transient part of the solution to Eq. 5.22 is given by,

\[ q_t(t) = h e^{-(b/2a)t} \sin \left( \frac{\sqrt{4ac - b^2}}{2a} t + \theta \right) \] \hspace{1cm} (5.23)

where, \( h \) is a constant and phase \( \theta \), and the steady state solution can be written as,

\[ q_s(t) = A_1 \cos(\omega t) + A_2 \sin(\omega t) \] \hspace{1cm} (5.24)

where, \( A_1 \) and \( A_2 \) are the two constants that would depend on the drive and also on the coefficients of the Eq. 5.22. Thus, substituting the steady state solution into Eq. 5.22, we find the coefficients \( A_1 \) and \( A_2 \) as,

\[ A_1 = \frac{f_1 p_1 - f_2 p_2}{p_1^2 + p_2^2}, \quad A_2 = \frac{f_1 p_2 + f_2 p_1}{p_1^2 + p_2^2} \] \hspace{1cm} (5.25)

where, \( p_1 = c - a\omega^2 \) and \( p_2 = b\omega \). The solution of the above equation can be compared with the simulation result in regimes where the circuit behaves linearly. We find that the appropriate regime for linear approximation to be valid is the range when the amplitude of the drive is \( \leq 10^{-4} \Phi_0 \). The circuit looks like a linear resonant
5.4. A SQUID BASED NON-LINEAR AMPLIFIER

circuit at low drive amplitudes. A comparison of the simulation result is made with the linear theory, as shown in Fig. 5-4. A close agreement with linear theory indicates that the circuit is driven in the linear regime. This indicates that the SQUID can be approximated as a linear inductor at low drive amplitudes \( A \leq 10^{-4} \Phi_0 \). The parameters used for modeling the circuit are, critical current \( I_c = 10^{-6} \text{A} \), junction resistance \( R_n = 500 \Omega \), output resistance \( R = 50 \Omega \), capacitance of the junction \( c_j = 10^{-15} \text{F} \), external capacitance \( C = 6.6 \times 10^{-10} \text{F} \), area of the squid = \( 100 \mu m^2 \). These parameters give a resonance frequency, \( f = 500 \text{MHz} \) and a \( Q \) of about 83. The magnetic field through the SQUID required to produce a flux amplitude of \( 10^{-4} \Phi_0 \approx 0.2 \text{mG} \). These parameters are consistently used throughout the thesis.

5.4 A SQUID based non-linear amplifier

From the Eq. 5.20, we clearly saw that the behavior of the SQUID is highly non-linear. In order to observe these non-linear effects, the drive amplitude has to be in the appropriate regime. Fig 5-5, shows the non-linear effect of the SQUID as the amplitude of the drive is increased to \( 10^{-2} \Phi_0 \). This is clearly seen from the presence of the higher order harmonics of the drive frequency, though they are orders of magnitude smaller than the resonant frequency terms in the power spectrum. Fig. 5-6, shows the amplitude of SQUID voltage with varying drive amplitudes. It is clear that the non-linear effects become much more important when the drive amplitude is about \( 10^{-3} \Phi_0 \). This is characterized by the bending of the resonance peak to the left side, thereby pulling the resonant frequency away from the designed resonance frequency, 500 MHz. The peak bends to the left due to the fact that the drive is about \( \Phi_{dc} = 0 \), hence any variation in the flux leads to reduced critical current or equivalently increasing the effective Josephson inductance. An increase in the inductance causes a reduction in the resonant frequency, and hence the peak bends to the left.

From these results we notice a distinct feature of the SQUID resonant circuit. It is highly non-linear in its behavior at drive amplitudes \( A \geq 10^{-3} \Phi_0 \). This feature can be
Figure 5-4: The simulation result from linear theory (solid line) and numerical solution (dotted line) of the SQUID equation (Eq. 5.20). The amplitude of the drive is for (a.) $A = 10^{-4}\phi_0$ and (b.) $A = 10^{-3}\phi_0$. There is close agreement between the numerical result and linear theory when the drive amplitude is in the linear regime (a.), beyond which the circuit becomes nonlinear (b.).
Figure 5-5: A typical drive (upper) of the SQUID and the steady-state output voltage (middle) and its power spectrum (bottom). This is an example to show non-linear effects when the amplitude, $A = 0.01$, of the drive is larger than that for the linear drive.
Figure 5-6: The amplitude of the output voltage as a function of frequency of the drive. The amplitude A labels each trace of the drive. A is incremented from $5 \times 10^{-4} \Phi_0$ to $10^{-2} \Phi_0$ in units of $5 \times 10^{-4} \Phi_0$ from one trace to another (We will use the compact notation of $A = 10^{-4} \Phi_0 (0 : 5 : 100)$ to indicate this. The non-linearity of the SQUID gives rise to a resonant peak bend to the left.

Figure 5-7: The simulation result of linear theory. The resonance peak remains at the designed resonance (500 MHz), while the amplitude of the resonant response increases linearly with the amplitude A of the drive. Here the A values label each trace and A ranges from $5 \times 10^{-4} \Phi_0$ to $10^{-2} \Phi_0$ in units of $5 \times 10^{-4} \Phi_0$. 
used as a non-linear amplifier due to the sharp rise in the peak output voltage at the loaded resonance frequency. One advantage of the non-linearity is that it provides a means to increase the sensitivity for spin detection. From the Fig 5-6, we see that the resonant frequency of the SQUID changes by about 5 MHz for a change of $2 \times 10^{-3} \Phi_0$ of the drive. The number of spins required to produce a flux of $10^{-4} \Phi_0$ is about $10^6$. The dimensions of the SQUID are assumed to be $10 \mu m \times 10 \mu m$ and for a static $B$ field of 200 G gives rise to a resonance frequency of 500 MHz. If one can detect frequency changes of about 0.25 Hz, then single spin detection would be possible. This would be an unprecedented success of SQCs if the $Q$ of the circuit can be made sufficiently high, that it would allow $\approx 1$ Hz resolution.

Figure 5-7, shows the amplitude of the SQUID voltage with varying drive amplitudes for a linearized model of the SQUID. It is quite clear that the resonant peak remains fixed while the amplitude of the peak increases linearly with the drive.

### 5.5 Effect of loop inductance on SQUID equations

For realistic circuits, there is always a loop inductance. The effectiveness of the non-linearity of the SQUID, is determined by the ratio of the loop inductance to Josephson inductance. This ratio would characterize the non-linear behavior of the circuit: the higher the ratio, the more linear the behavior. For efficient spin detection, lower ratios are preferable. The effect of the loop inductance is to produce shielding current that opposes the external flux, thereby reducing the effectiveness of the flux generated by the spins. This reduction in the flux due to the spins would lead to the reduction in the effective drive amplitude, hence reducing the non-linearity of the SQUID. To understand this behavior we proceed as follows.

The fluxoid quantization condition in the presence of loop inductance can be written as,

$$\varphi_1 - \varphi_2 = -2\pi (\frac{\Phi_{ext}}{\Phi_0} + \frac{L I_{circ}}{\Phi_0})$$

(5.26)

$$I_{circ} = -\frac{\Phi_0}{\pi L} (\varphi_m + \frac{\pi \Phi_{ext}}{\Phi_0})$$

(5.27)
Figure 5-8: A SQUID based resonant circuit with loop inductance, indicated as L.

Here, the loop inductance is given by $L$ and the circulating current by $I_{circ}$. The output voltage in Fig. 5-8 can be written as,

$$V_{out} = \frac{d\varphi_p}{dt} \frac{\Phi_0}{2\pi} + \frac{1}{2} \frac{d\Phi_{ext}}{dt} + \frac{dL}{dt} \frac{I_{circ}}{2}$$ \hspace{1cm} (5.28)

$$= \frac{d\varphi_p}{dt} \frac{\Phi_0}{2\pi} + \frac{d\varphi_m}{dt} \frac{\Phi_0}{\pi}$$ \hspace{1cm} (5.29)

From Kirchoff current law for Fig. 5-8, we have,

$$-C \left( \frac{d^2 \varphi_p}{dt^2} \frac{\Phi_0}{2\pi} + \frac{d^2 \varphi_m}{dt^2} \frac{\Phi_0}{\pi} \right) = 2I_c \sin \varphi_p \cos \varphi_m + \frac{d\varphi_p}{dt} \frac{\Phi_0}{\pi R_n} + \frac{d^2 \varphi_p}{dt^2} \frac{2C_j \Phi_0}{2\pi} + \left( \frac{d\varphi_p}{dt} \frac{\Phi_0}{2\pi} + \frac{d\varphi_m}{dt} \frac{\Phi_0}{\pi} \right) \frac{1}{R}$$ \hspace{1cm} (5.30)

In the above equation, we have both $\varphi_p$ and $\varphi_m$ as unknowns. Hence, we need another simultaneous equation that together with Eq. 5.30, can be solved for the variables $\varphi_p$ and $\varphi_m$. The second equation can be obtained from the circulating current equation.
5.5. EFFECT OF LOOP INDUCTANCE ON SQUID EQUATIONS

$I_{circ}$. The circulating current can be written as the difference in the branch currents of the SQUID as,

\[
I_{circ} = \frac{i_1 - i_2}{2} \tag{5.31}
\]

\[
= (I_c \sin \varphi_m \cos \varphi_p + \frac{d\varphi_m}{dt} \frac{\Phi_0}{2\pi R_n} + \frac{d^2\varphi_m}{dt^2} \frac{\Phi_0}{2\pi c_j}) \tag{5.32}
\]

The fluxoid quantization condition can be written,

\[
\frac{\Phi_0}{\pi L} (\varphi_m + \frac{\pi \Phi_{ext}}{\Phi_0}) = -(I_c \sin \varphi_m \cos \varphi_p + \frac{d\varphi_m}{dt} \frac{\Phi_0}{2\pi R_n} + \frac{d^2\varphi_m}{dt^2} \frac{\Phi_0}{2\pi c_j}) \tag{5.33}
\]

This is the second equation that together with Eq. 5.30 can be solved for $\varphi_p$, to find the output voltage of the device,

\[
V_{out} = \frac{d\varphi_p}{dt} \frac{\Phi_0}{2\pi} + \frac{d\varphi_m}{dt} \frac{\Phi_0}{\pi} \tag{5.34}
\]

The solutions to $\varphi_p$ and $\varphi_m$ are a little more complicated when compared to the case when there is no loop inductance since two simultaneous non-linear differential equations need to be solved. The typical drive and its output voltage of the SQUID are shown in Fig 5-9, in the upper and middle plots. The lower plot shows the power spectrum of the output voltage. As seen, for small drives and for $\beta = L/L_J = 1/1.6$, the non-linearity is small due to the shielding current reducing the effective flux amplitude. Hence, the most dominant frequency term is at the drive frequency, just like the case when the loop inductance is zero.

The non-linear effect of the SQUID with loop inductance is shown in Fig. 5-10 and 5-11. We see that the non-linear effect is smaller than for the case when the loop inductance is zero. It should be noted that the simulations for frequency response of the SQUID were done while increasing the frequency sweep from low to high. In general, a SQUID shows hysteretic frequency behavior depending on the fabrication parameters. Hence, the bifurcation point, the frequency at which the SQUID response slope switches, for the forward and backward sweep is different - the
Figure 5-9: A typical drive (upper) of the SQUID resonant circuit including loop inductance and its output voltage (middle) and power spectrum (bottom). The amplitude, $A = 10^{-4} \Phi_0$ of the drive is such that the circuit is in the linear regime.
hysteresis is higher for $\beta = L/L_J \geq 1$. For efficient spin detection it is preferable to have as low hysteresis as possible.

### 5.5.1 Effect of Loop Inductance - An intuitive approach

The resonance frequency of the circuit is characterized as a function of the loop inductance. From Fig. 5-10, we find that the resonance frequency remains fixed at 500 MHz, even with increasing inductance of the loop. This is rather puzzling. This effect can be better understood by first looking at the circuit in the extreme limit of $L \gg L_J$.

In this limit, the circuit consists of a capacitor and resistor in parallel with a superconducting ring of inductance $L$. The equivalent circuit is shown in figure 5-11. From the flux quantization condition, the total flux through the loop has to be quantized. Hence,
Figure 5-11: The simulation result of the resonant response of the SQUID with loop inductance (\( L = 0.1 \text{ nH}, 0.5 \text{ nH}, 5 \text{ nH}, \text{ and } 50 \text{ nH} \)) and \( L_J = 1.6 \times 10^{-10} \text{H} \).

Figure 5-12: Equivalent circuit of the SQUID resonant circuit when the loop inductance, \( L \), is much bigger than the Josephson inductance, \( L \gg L_J \).
\[ \Phi_{\text{ext}} + LI_{\text{circ}} = n\Phi_0 \]  

(5.35)

As the SQUID is driven about \( \Phi_{dc} = 0 \), we have

\[ \Phi_{\text{ext}} = -LI_{\text{circ}} \]  

(5.36)

Hence, currents through the two arms of the SQUID would be such that \( i_1 = -i_2 \). This results in the total current in the loop, \( i = i_1 + i_2 = 0 \). There is no current flowing external to the ring except for the circulating current, hence there is zero voltage across the SQUID. This behavior is seen from the amplitude of the resonance peak becoming smaller with increase in the loop inductance.

To understand the results for the case when \( L > L_J \) requires additional correction terms to predict the amplitude of resonance peak as a function of loop inductance. The equation for the output voltage can be written as,

\[ V_{\text{out}} = \frac{d\Phi_2}{dt} \frac{\Phi_0}{2\pi} + \frac{L}{2} \frac{di_2}{dt}. \]

(5.37)

Similarly, taking the loop consisting of junction 1 and the capacitor, we have,

\[ V_{\text{out}} = \frac{d\Phi_1}{dt} \frac{\Phi_0}{2\pi} + \frac{L}{2} \frac{di_1}{dt} + \frac{d\Phi_{\text{ext}}}{dt} \]

(5.38)

From the Josephson inductance of each of the junctions, Eq. 5.37, 5.38 can be written as,

\[ V_{\text{out}} = L_J \frac{di_2}{dt} + \frac{L}{2} \frac{di_2}{dt}. \]

(5.39)

and

\[ V_{\text{out}} = L_J \frac{di_1}{dt} + \frac{L}{2} \frac{d}{dt} i_1 + \frac{d\Phi_{\text{ext}}}{dt} \]

(5.40)

From the definition of the circulating current, \( I_{\text{circ}} = (i_1 - i_2)/2 \), we find from Eq.
5.39 and 5.40 that

\[ 2L_J \frac{dI_{circ}}{dt} + \frac{L}{2} \frac{d}{dt} I_{circ} + \frac{d\Phi_{ext}}{dt} = 0. \quad (5.41) \]

which gives,

\[ I_{circ} = \frac{-\Phi_{ext}}{2L_J + L}. \quad (5.42) \]

Hence, we have,

\[ I_{circ} = \frac{\Phi_{ext}}{L} (1 - \frac{2L_J}{L}). \quad (5.43) \]

From this it is clear that, when \( L \gg 1 \), Eq. 5.43 gives the flux quantization condition, \( I_{circ} = -\frac{\Phi_{ext}}{L} \). Hence, the circulating current would adjust itself to completely shield the external flux, hence a zero SQUID voltage. Eq. 5.43 gives the correction term to the circulating current. Hence, for a fixed drive, the first order correction term to \( \varphi_m \) from the fluxoid quantization condition is given by,

\[ \Delta \varphi_m = L \Delta I_{circ} \quad (5.44) \]

\[ \Delta \varphi_m = \Phi_{ext} \frac{2L_J}{L} \quad (5.45) \]

From Eq. 5.45, we find the correction to the term involving phase \( \varphi_m \) in Eq. 5.30. From this, we see that the amplitude of the terms involving \( \varphi_m \) are proportional to \( \Phi_{ext} \frac{2L_J}{L} \). Hence the voltage of the SQUID goes as \( \frac{\Phi_{ext}}{L} \). This behavior is clearly seen in Fig. 5-12, where the voltage is proportional to \( 1/L \).

As discussed earlier when the loop inductance is large, the circuit behaves linearly. One can see this linear behavior when the loop inductance is larger than the Josephson junction. The resonance of the circuit is fixed at 500 MHz, and is analyzed as a function of changing amplitude of the drive. The circuit behaves linearly with increasing amplitude of the drive, which is clear in Fig. 5-13.
Figure 5-13: An example to show that a large loop inductance makes the SQUID circuit look like a linear circuit. The drive amplitude is, $A = 10^{-4} \Phi_0$ (0:5:55), the loop inductance, $L = 100 \times 10^{-9} \text{H}$ and the Josephson inductance $L_J = 1.6 \times 10^{-10} \text{H}$. From the equivalent circuit in Fig. 5-12, we find that the SQUID behaves linearly. As seen in the figure, the resonant response of the SQUID increases linearly with the amplitude, $A$, of the drive.
5.5.2 Effect of Josephson Inductance

When the Josephson inductance dominates, \( L_J \gg L \), we find that the equations of the SQUID circuit are similar to the case when the loop inductance is zero. From the solution of the linearized model in Eq. 5.21, we find that at resonance the coefficients of the steady-state terms are,

\[
A_1 = -f_2/p_2, \quad A_2 = f_1/p_2
\]  

(5.46)

where, \( f_1 \) and \( f_2 \) are the amplitudes of the drive, and \( p_2 = b\omega \), where \( b \) is the coefficient of the damping term involving \( d\varphi_p/dt \) in Eq. 5.30. The coefficient, \( f_1 \) is proportional to \( \omega^2 \) and \( f_2 \) is proportional to \( \omega \). Hence, \( A_1 \) is proportional to the \( \omega \) and \( A_2 \) is a constant. From figure 5-14, we clearly see the \( \omega \) dependence of the peak voltage of the SQUID. This result indicates that the lower the ratio \( \beta \), the higher the peak voltage - hence, increasing the sensitivity of the circuit. Hence, it is desirable to have the Josephson inductance as large as possible when compared to the loop inductance for
efficient detection of spins.

5.5.3 **Effect of constant flux bias - $\Phi_{dc}$**

From chapter 2, the effect of external flux on the critical current of the SQUID was obtained. The critical current, $I_c$ is modulated by the flux penetrating the SQUID, which is given as,

$$I_c = I_0 \cos\left(\frac{-\pi \Phi_{dc}}{\Phi_0}\right)$$

(5.47)

where, $I_c$ is the critical current, $I_0$ is the critical current at zero flux, $\Phi_{dc}$ is the external flux through the SQUID. This leads to the critical current modulated by the external flux. A modulation of the critical current leads to the modulation of the Josephson inductance, $L_J = \Phi_0/(2\pi \cos(\varphi) I_c)$. Hence, the resonant frequency of the SQUID is modulated (see Eq. 5.20 and 5.30). From the Fig. 5-15 we see this behavior with the resonant frequency being reduced with increasing flux up to $\Phi_0/2$. This is because the resonant frequency is proportional to the critical current; hence the resonance frequency is reduced with increasing $\Phi_{dc}$ (up to $\Phi_0/2$). The amplitude dependence of the peak voltage is analogous to the effect of varying the Josephson inductance (section 5.4.2), which increases linearly with the frequency.

5.5.4 **Effect of $T_2$**

From section 5.2.1, the effect of $T_2$ is an exponential decay of the transverse magnetization. From the linear approximation solution in Eq. 5.21, we find that the effect of the $T_2$ leads to an exponential decay in the peak output voltage.

At the resonance frequency, we find from Eq. 5.34, that $A_1 = -f_2/p_2$, and $A_2 = f_1/p_2$ where $f_1 = e^{-\left(f_2/p_2\right)} \cos(\omega t)$ and $f_2 = e^{-\left(\frac{1}{T_2}\right)} \sin(\omega t)$. Hence, the peak output voltage of the SQUID resonant circuit exponential decreases with decreasing $T_2$. This behavior is seen in the Fig. 5-15.
Figure 5-15: The effect of constant external flux $\Phi_{dc}$ is the modulation of the critical current of the SQUID, hence the Josephson inductance increases with increasing flux. Correspondingly, this effect decreases the resonant frequency of the SQUID.
Figure 5-16: The effect of increasing $T_2$ leads an exponential increasing the resonant response of the SQUID.
5.6 Q enhancement using impedance transform

We find that the squid circuit with a load, looks like an LRC circuit with a Josephson inductance being the L, effective resistance that is parallel combination of the junction resistance $R_n$ and the external resistor given by $R$, and effective capacitance given by a parallel combination of the junction capacitances and the external capacitor.

The Q-factor of the circuit determines the peak or resonant response. For the circuit with load resistor $R$, we find,

$$Q = \omega_0 R_{\text{eff}} C_{\text{eff}}$$  \hfill (5.48)

$$R_{\text{eff}} = R || R_n/2$$  \hfill (5.49)

$$C_{\text{eff}} \approx C$$  \hfill (5.50)

For $R_n = 500\,\Omega$, $R = 50\,\Omega$, and $C = 6.6 \times 10^{-10}\,F$, we find $Q = 83.3$.

The normal channel of the junction, has a resistance that is a function of the temperature of the device. At low temperature, at 15 mK, the resistance of the junction can be up to 1 MΩ. If one can increase the effective circuit resistance, then the Q of the given circuit can be enhanced by orders of magnitude. A technique for enhancing Q is to match the load resistor through an impedance transforming circuit. For the resonant circuit consisting of the SQUID and the external capacitor (see Fig. 5-17), an impedance matching transformer (an L-circuit) is used to match with the 50 Ω load.

The output impedance of this circuit is given as,

$$Z(\omega) = j\omega L + (j\omega C_0 + 1/R)^{-1}$$  \hfill (5.51)

We can solve the above equation, so that the real part of $Z(\omega)$ is equal to a 50 Ω load,

$$z_o = \frac{2R_n}{1 + R^2\omega^2 C_0}$$  \hfill (5.52)
Figure 5-17: An example a high-Q impedance matching circuit.

When $R^2 \omega^2 c_0 \gg 1$, then one can write the capacitance required as,

$$c_0^2 = \frac{4}{\omega^2 z_0 R_n}$$  \hspace{1cm} (5.53)

where, $z_0$ is the load resistance. From this equation we get, $c_0 = 0.1$ pF and $L = 400$ n H. We find that Q increases from 83 to $8.3 \times 10^5$. Hence, increasing the Q of the circuit increases the sensitivity of the SQUID. If the Q can be significantly enhanced, it would be possible to detect tens of spins and even single spin. This is a significantly increases the detection efficiency by orders of magnitude with SQUIDs. The SQUID detected NMR by McDermott et al. [17, 18] used about $10^{15}$ spins, where the SQUID is operated in the linear regime.

### 5.7 Signal to Noise Ratio

The main source of noise we consider is the Johnson noise of the normal conduction channel created by quasi-particles. The resistance of the normal channel is given as
$R_n$ of the RCSJ model. The root mean square voltage of Johnson noise is given by,

$$V_{rms}^2 = 4k_BT\Delta f R$$  \hspace{1cm} (5.54)

where, $k_B$ is the Boltzman constant, $T$ is the temperature, $R$ is the resistance, and $\Delta f$ is the bandwidth of the circuit. For a resistance of 1 MΩ, at 15 mK, with an assumed bandwidth of 5 MHz results in a noise voltage of 1 pV. Comparing this with the output of the SQUID for small drive amplitude ($10^{-4}\Phi_0$), we get a signal to noise of about 100.

### 5.8 An example with a fabricated SQUID

Here, we use an already fabricated SQUID to simulate its non-linear effects. We use the numbers from the SQUID parameters to estimate the non-linear effects on the resonant circuit. The parameters of the SQUID are, critical current, $I_c = 2.3\mu A$, loop inductance, $L = 0.06nH$, external capacitance of the resonant circuit, $c = 1.5 nF$, capacitance of junction, $c_j = 5 pF$. As we see from Fig. 5-18, the resonance frequency is 500 MHz, and, the peak amplitude is a little lower than the simulations in Fig. 5-10, the loop inductance assumed for simulation (Fig. 5-10) was $L = 0.1 nH$. Hence, the ratio $L_j/L$ is approximately 1.1, the shielding current is higher that generates a higher shielding flux, for the fabricated SQUID, thereby reducing the output voltage of the SQUID. The results from Fig. 5-18 indicate that the already fabricated SQUID can be used as an efficient non-linear spin detector.

### 5.9 Conclusion

We have proposed novel techniques for the efficient detection of electron spin resonance based on a SQUID resonant circuit. We derived a theoretical model based on Josephson equations to describe the non-linear effects due to an rf-magnetic field on a SQUID. Starting with a simple circuit, ignoring the loop inductance and the RCSJ model, we formulated a simple theory of the effect of the rf-magnetic field on
Figure 5-18: The simulation result of the resonant response of the SQUID with loop inductance ($L = 0.06 \text{ nH}$), $L_J = 0.07 \text{ nH}$.

an ideal SQUID. This simple analysis helped us get insight into the behavior of the SQUID. Based on this simple model, a full model including the loop inductance and the RCSJ model, was used to derive the SQUID equations. The non-linear behavior led us to propose a sensitive approach to spin detection. This could lead to significant advances for spin detection using SQUIDs.
Chapter 6

Conclusions

6.1 Summary of results

We have proposed two new applications of SQCs: (1.) The theory of EIT in SQCs and its use as a probe of decoherence and (2.) A SQUID based sensitive rf-magnetic field detector.

We proposed using the superconductive analog to EIT (S-EIT) to demonstrate macroscopic quantum interference in superconductive quantum circuits. We have shown how S-EIT can be used to measure, with a single pulse of the two S-EIT fields, whether a particular superposition of meta-stable energy levels (a qubit) has been prepared. The technique is distinguishable from previous state measurement schemes [9] in that S-EIT ideally does not disturb the system, preserving its quantum coherence when it has been prepared in the desired state. Furthermore, we have shown how S-EIT can sensitively probe decoherence, and we have obtained analytic expressions for the field strengths required to measure the SQC dephasing rate.

Then we showed a detailed analysis of the theory of EIT in SQCs in the presence of decoherence and also including other realistic effects like tunneling and radiation cross-talk. In the context of SQCs, we saw that EIT manifested itself as the suppression of photon-induced tunneling from stable states $|1\rangle, |2\rangle$ through a read-out state $|3\rangle$, due to quantum mechanical interference for two paths of excitation. This provides a method of unambiguously demonstrating phase coherence in these systems.
We have provided a thorough and mostly analytic treatment of EIT in the presence of complicating effects due to decoherence and multiple levels in SQCs, which would be important in guiding experimental implementation and observation of EIT and other quantum interference effects. We analyzed in detail first the basic considerations of EIT such as imperfect dark state preparation, and one- and two-photon detuning and determined the expected experimental signatures. We then discussed in detail how decoherence due to dephasing of the qubit coherence, incoherent population loss or exchange, and tunneling of levels through the barrier affects the loss rate. We obtained the coefficients for the loss rates, which depend differently on the field strengths, depending on the underlying decoherence processes. Probing these effects with EIT can aid in understanding and minimizing decoherence and can give information about the full multi-level structure of the SQC. Finally, in the EIT work, we have found that the microwave fields themselves can cause both additional loss rates and AC Stark shifts of the EIT resonance, which must be accounted for when one uses stronger field strengths.

In the second part, we introduced novel techniques for the efficient detection of electron spin resonance based on a SQUID resonant circuit. We derived a theoretical model based on Josephson equations to describe the non-linear effects due to an rf-magnetic field on a SQUID. Starting with a simple circuit, ignoring loop inductance and the RCSJ model, we formulated a simple theory of the effect of the rf-magnetic field on an ideal SQUID. This simple analysis helped us get an insight into the behavior of the SQUID. Based on this simple model, a full model including the loop inductance and the RCSJ model, was used to exactly derive the SQUID equations. The non-linear behavior led us to propose a sensitive approach to spin detection. This could lead to new sensitive techniques for spin detection using SQUIDs.

6.2 Scope for future work

From the proposal on EIT based decoherence detection, we learnt that EIT leads to a unique signature for each of the effects like dephasing, detuning, cross talk, tunneling
etc. A potentially interesting future investigation is to learn the signature from interaction with other quantum degrees of freedom, such as the microresonators postulated in [77, 78] through EIT. Another possibility is to demonstrate other quantum optical effects as discussed in this thesis, one of the examples would be the AC-stark effect. This would be important to EIT as well as to quantum computation as cross-talk is unavoidable due to the presence of more than two levels for a typical SQC based quantum system. An interesting area of interest is to analyze the effect of noise on EIT based decoherence detection. A major issue for SQCs is noise. The $T_2$ measurement techniques based on spin-echo are prone to noise. EIT might be promising in terms of noise immunity and would be interesting for future research.

In second part of the thesis, we proposed a SQUID based sensitive rf-magnetic detector. We proposed to sensitively detect electron spin resonance using inherent non-linearity of the SQUID. Though SQUID and ESR by themselves are quantum mechanical, we analyzed the classical effect of the ESR signal on SQUID resonant circuits. An important extension of this analysis that would be interesting to do is to fully assess the potential of the device to study the quantum mechanically coupled systems of the spins and the SQUID. This would be also interesting in the context of quantum memory based on spins and SQUIDs, where quantum coherence can be transferred from a SQUID to a spin system and vice versa. This would enable preserving the SQUID quantum coherence in the electron spin system where the coherence times are orders of magnitude larger than SQUIDs. This can have enormous implication for quantum memory applications.

The two topics proposed in this thesis have the potential of opening up a wide range of interesting work on both the theoretical and experimental front in the area of quantum information processing.
Appendix A

Optical Bloch Equations

% Code for solving Optical Bloch equations
function [t,x] = bloch()

    N = 10000;               % Number of points in the time domain
    theta=0;                % Relative phase between 1 and 2
    p1=0.5;                 % Starting population of level 1
    p2=1-p1;                % Starting population of level 2

% Parameters
    omegal3=2*pi*150*1e6;  % Rabi Frequency of 1-> 3 transition
    omega23=2*pi*150*1e6;  % Rabi Frequency of 2-> 3 transition
    gammal2=2*pi*1*1e6;    % Dephasing rate of 1,2 pair level
    gamma3=2*pi*130*1e6;   % Tunneling rate of level 3
    gamma13=gamma3/2;      % Dephasing of 1,3 due to tunneling from 3
    gamma23=gamma3/2;      % Dephasing of 2,3 due to tunneling from 3
    deltap=2*pi*5000*0*1e6; % Detuning of probe
    deltaq=2*pi*5000*1e6;  % Detuning of control
    tmax=1e-6;              % Max time

% ODE 45 is used as the diff. eq. solver %%%%%%
[t,x] = ode45(@odebloch,linspace(O,tmax,N),
APPENDIX A. OPTICAL BLOCH EQUATIONS

\[
\begin{bmatrix}
  p_1 & p_2 & 0 & -\sqrt{p_1}(\cos(\theta)+i\sin(\theta)) \\
  -\sqrt{p_2}(\cos(\theta)+i\sin(\theta)) & 0 & 0
\end{bmatrix}
\]

function drhodt = odebloch(t,x)
    drhodt = zeros(6,1); % initialize a column vector for drho
    drhodt(1) = - i/2*omega13*(conj(x(5)) - x(5)); % rho11
    drhodt(2) = - i/2*omega23*(conj(x(6)) - x(6)); % rho22
    drhodt(3) = - gamma3 * x(3) + i/2*omega13*(conj(x(5))
                - x(5)) + i/2*omega23*(conj(x(6)) - x(6)); % rho33
    drhodt(4) = - (gamma2 + i*(deltaq - deltap)) *x(4)
                - i/2* omega23*conj(x(6))+ i/2* omega23* x(5); % rho21
    drhodt(5) = - (gamma3/2 + i * deltap) *x(5) + i/2*
                omega13*(x(1) - x(3))+ i/2* omega23* x(4); % rho31
    drhodt(6) = - (gamma3/2 + i * deltaq) *x(6)+ i/2* omega23*
                (x(2) - x(3)) + i/2* omega13* conj(x(4)); % rho23
end

lp=x(:,1) + x(:,2)+x(:,3); % Total population (diagonal elements or rho)
T=1:size(lp);
figure;
plot(T/N*1,lp,'color',[0 0 0], 'linewidth',[2]);
Appendix B

Zero loop inductance SQUID equations

% Code for the case when loop inductance is zero.
function [t,x] = hillnewr(omega)
N = 5000; % Number of points in the time and Fourier plots
omega=omega*10^-8*2*pi; % Frequency of external drive
B = 1*10^-6; % Critical current
omegamax = 10*10^-8*2*pi; % Sampling Frequency
phi0= 2*10^-15; % Superconducting flux quantum
cj= 50*10^-15; % Junction capacitance
tmax=N/omegamax; % Max. time evolution
c=6.3664*10^-10; % External capacitance
Rn=500; % Junction resistance
R=50; % External resistance
A=10^-4; % Amplitude of drive in units of \Phi_0
p= 0; % Constant flux bias

tt=tmax/N

% ODE45 is used as the non-linear diff. eqn. solver
[t,x] = ode45(@HillODE,linspace(0,tmax,N),[0 0 0 0])

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function didt = HillODE(t,x)
    didt = zeros(2,1);  
    didt(1) = -(2*B*sin(x(2))*cos(-pi*A*cos(omega*t))
                + x(1)*phi0/pi/Rn + (x(1)*phi0/2/pi
                - 1/2*A*phi0*omega*sin(omega*t)) * 1/R ...
                - A*c*omega*omega*cos(omega*t)*phi0/2)*2*pi/(c+2*cj)/phi0;
    didt(2) = x(1);
end

Vout = x(:,1)*phi0/(2*pi) 
      + 1/2*A*sin(Omega*t)*Omega*phi0;  
% Output voltage
Vinput = (p + A*cos(omega*t))*phi0;  
% Input voltage
T=0:(size(t)-1);

figure;subplot(3,1,1);
plot(T*tt, Vinput,'b');
xlabel('time');
ylabel('\Omega_{ext}');

subplot(3,1,2);
plot(T*tt, Vout,'b');
xlabel('time');
ylabel(' V_{out}');

Y = fft(Vout);  
% Fourier transform of output voltage
f = fftshift(Y.*conj(Y));  
% Power spectrum
subplot(3,1,3);
plot((T-N/2)/tmax,log(f));
xlabel('frequency');
ylabel('Power ');
end
Appendix C

SQUID equations with loop inductance

% Code for the case with loop inductance.
function [t,x] = hillnewr(omega)

N = 5000; % Number of points in the time and Fourier plots
omega=omega*10^-8*2*pi;
B = 1*10^(-6); % Critical current
omegamax = 10*10^-8*2*pi; % Sampling Frequency
phi0= 2*10^(-15); % Superconducting flux quantum
cj= 50*10^(-15); % Junction capacitance
tmax=N/omegamax; % Max. time evolution
c=6.3664*10^-10; % External capacitance
Rn=500; % Junction resistance
R=50; % External resistance
L=10^-10; % Loop inductance
A=10^-4; % Amplitude of drive in units of \Phi_0
p= 0; % Constant flux bias

%%% ODE45 is used as the non-linear
APPENDIX C. SQUID EQUATIONS WITH LOOP INDUCTANCE

% diff. eqn. solver

% [t,x] = ode45(@HillODE,[0:tt:tmax],[0 0])

[t,x] = ode45(@HillODE,linspace(0,tmax,N),[0 0 0 0])

function didt = HillODE(t,x)
    didt = zeros(4,1); % initialize a column vector for dx
    didt(1) = -(2*B*sin(x(2))*cos(x(4))/phiO + x(1)/pi/Rn ... 
               + (x(1)/2/pi + 2 * x(3)/2/pi)*1/R ... 
               - ((B*sin(x(4))*cos(x(2))/phiO + x(3)/2/pi/Rn ... 
                   + (x(4) + (p + A*cos(omega*t))*pi)/pi/L))*2*pi/cj 
               *1/2*c/2/pi)*2*pi/(c+2*cj);
    didt(2) = x(1);
    didt(3) = -((B*sin(x(4))*cos(x(2))/phiO + x(3)/2/pi/Rn ... 
                 + (x(4) + (p + A*cos(omega*t))*pi)/pi/L))*2*pi/cj ;
    didt(4) = x(3);
end

Vout = x(:,1)*phi0/(2*pi) + 1/2*A*sin(Omega*t) 
      *Omega*phi0; % Output voltage

Vinput = (p + A*cos(omega*t))*phi0; % Input voltage

T=0:(size(t)-1);
figure;subplot(3,1,1);
plot(T*tt, Vinput,'b');
xlabel('time');
ylabel('\Omega_{ext}');

subplot(3,1,2);
plot(T*tt, Vout,'b');
xlabel('time');
ylabel(' V_{out}');

Y = fft(Vout); % Fourier transform of output voltage
f = fftshift(Y.*conj(Y)); % Power spectrum

subplot(3,1,3);
plot((T-N/2)/tmax,log(f));
xlabel('frequency');
ylabel('Power ');

end
Appendix D

Publication List

D.1 Doctoral Work


3. A SQUID based RF-magnetic field detector.
K.V.R.M. Murali, D.G. Cory, and T.P. Orlando. (to be submitted)

D.2 M.S. Work

D.3 B.S. Work


(Selected by Virtual Journal of Quantum Information, September 2002).

Bibliography


