The Role of Background Flow Variations in Stratified Flows Over Topography

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Abstract

As the atmosphere and oceans feature density variations with depth, the flow of a density-stratified fluid over topography is central to various geophysical and meteorological applications and has been studied extensively. For reasons of convenience and mathematical tractability, the majority of theoretical treatments of stratified flow over a finite-amplitude obstacle assume idealized background flow conditions, namely constant free-stream velocity and either a homogeneous or two-layer buoyancy-frequency profile. In this work, a numerical model is developed that accounts for general variations in the buoyancy-frequency profile far upstream and the presence of unsteadiness in the free-stream velocity. The model employs a second-order projection method for solving the Euler equations for stratified flow over locally confined topography in a horizontally and vertically unbounded domain — the flow configuration most pertinent to atmospheric applications — combined with absorbing viscous layers at the upper and lateral boundaries of the computational domain. Using this model, a study is first made of the effect of variations in the buoyancy frequency on the generation of mountain gravity waves. Balloon measurements reveal that, apart from a sharp increase (roughly by a factor of 2) at the so-called tropopause, atmospheric buoyancy-frequency profiles often feature appreciable oscillations (typical wavelength 1–2 km). It is found that such short-scale oscillatory variations can have a profound effect on mountain waves owing to a resonance mechanism that comes into play at certain wind speeds depending on the oscillation length scale. A simple linear model assuming small sinusoidal buoyancy-frequency oscillations suggests, and numerical simulations for more realistic flow conditions confirm, that the induced gravity-wave activity under resonant conditions is significantly increased above and upstream of the mountain, causing transient wave breaking (overturning), similarly to resonant flow of finite depth over topography. The effect of temporal variations in the free-stream velocity is then explored for a range of amplitudes and periods typical of those encountered in the field. The simulations reveal that transient disturbances resulting from such variations can be significant, particularly in the nonlinear regime, and
steady states predicted on the assumption of uniform wind may not be attainable.

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Chapter 1

Introduction

In everyday language, the term wave is roughly used to describe any form of a back and forth movement in either space or time. From Newton’s second law, we know that for such a motion to endure, there must be a force persistently attempting to bring the disturbance towards equilibrium. If that restoring force is buoyancy, which stems from gravity, corresponding oscillations are commonly labeled as buoyancy or gravity waves. An elementary example of a gravity wave consists of a fluid particle displaced from its equilibrium position in a pool of hydrostatic fluid, continuously stratified in such a way that its density decreases with height. If the parcel is moved vertically upwards, that is, in a direction opposite to that of the force of gravity, it is in an environment where its weight exceeds the upward oriented force of buoyancy, exerted on it by the ambient fluid. As a result, it is accelerated down, back towards the equilibrium elevation. During its descent, however, in the absence of attenuation effects, it overshoots this altitude and finds itself in the region where now its weight is lower than the buoyancy of its surroundings. Therefore, the particle here feels the upward directed acceleration, which eventually causes it to ascend and once again surpass the point of weight–buoyancy balance. Without a dissipative mechanism, the process continues indefinitely and we designate the parcel’s motion as a gravity wave.

Due to the fact that they are generated by buoyancy, gravity waves may occur within any stratified body of fluid or along a perturbed boundary separating two distinct fluids, such as the air-water interface, for instance. The scientific terminol-
ogy correspondingly distinguishes between internal and surface buoyancy waves. In nature, the largest basins of stratified fluid are oceans and the atmosphere where internal gravity waves range from small-scale shallow-water micro-structures to air-born surges with amplitudes of several hundred meters and velocities in excess of 50 m/s. There is now increasing evidence (Eckermann and Preusse [11]) that atmospheric buoyancy-induced disturbances are responsible for regions of turbulence in the upper troposphere and lower stratosphere, at altitudes 10–15 km, which pose hazard to aviation (Eckermann et al. [10]). They are also accountable for strong surface winds that blow down a mountain along its lee slope. In the ocean, on the other hand, turbulence caused by internal gravity waves in the saline water flow over uneven bathymetry may exert volatile forces on submarines and other autonomous under-sea devices. Dynamics of internal gravity waves also profoundly affects deep-water drilling strategies and airborne laser techniques. Owing to its influence on such a wide range of engineering disciplines, a comprehensive understanding of the internal gravity-wave dynamics can hardly be overemphasized.

In the atmosphere, internal buoyancy waves are predominantly generated when atmospheric air encounters a topographic barrier, a mountain for instance, and consequently is forced to rise. In this study, we are concerned with the structure and behavior of the flow field generated during this natural phenomenon. A property that distinguishes this flow — and all geophysical flows for that matter — from other areas of fluid mechanics is the density stratification of the medium. Hydrostatic density for the U.S. Standard Atmosphere nearly exponentially diminishes with altitude and at the elevation of 70 km it is less than one percent of its see-level value, measured to be 1.2 kg/m³. Such a substantial rate of decay conceals fluctuations in the density profile whose wavelength is much shorter than the roughly 7 km half-life constant, defined as the distance over which the density decreases by a factor of two. These deviations from the exponential downfall can be seen by measuring stratification in terms of the quantity termed buoyancy or Brunt–Väisälä frequency. For incompressible fluids,
considered in this work, the squared buoyancy frequency is defined as

\[
N^*2 = \frac{g}{\bar{\rho}^*} \frac{d\bar{\rho}^*}{dz^*}.
\]  

(1.1)

In this expression, \( g \) is the acceleration of gravity oriented in the negative \( z^* \) direction, while \( \bar{\rho}^* \) and \( d\bar{\rho}^*/dz^* \) denote hydrostatic density and its gradient respectively. It is apparent from (1.1) that a purely exponential variation in \( \bar{\rho}^* \) corresponds a constant value of \( N^* \). Measurements of the atmospheric Brunt–Väisälä frequency, depicted in Fig. 1-1, reveal the presence of nearly periodic fluctuations with the wavelength from one to two kilometers. They are centered at approximately 0.01 Hz throughout the troposphere, that is, up to the height of about 10 km. Above this stratum, there is a transitional region, tropopause, extending from 10 to 12 km above which oscillations are centered at around 0.02 Hz. It is further evident from the figure that amplitudes of these modulations are anywhere from 10 to 60 percent of the mean value of buoyancy frequency in each layer. A qualitatively similar behavior is exhibited by the atmospheric wind speed, in a sense that it is also comprised of two distinct segments, both of which feature 1–2 km variations. In contrast to the buoyancy frequency, however, the mean tropospheric wind speed is roughly parabolic. At this point, it is important to stress that current models of stratified air over topography, largely for the purpose of mathematical tractability and computational simplicity, neglect these short-scale oscillations and assume uniform background flow conditions, namely, constant free-stream velocity and either a single or two-layer Brunt–Väisälä frequency profile. In this study, we present numerical evidence that small departures from the homogeneous background conditions can indeed dramatically affect the behavior of stratified air over an isolated topographic barrier.

We begin the following chapter by describing the physical model of atmosphere that we utilize to study topography induced flows. We subsequently formulate governing equations and boundary conditions associated with this model. The chapter concludes with an introduction of two simplest models for stratified flow over a locally confined topographic barrier, namely, those of Long [23] and Davis [8]. They semi-
analytically describe the steady-state response for a constant and arbitrary buoyancy frequency respectively in the presence of the uniform free-stream velocity. Forecasting the unsteady evolution of internal gravity waves requires a far more rigorous numerical treatment. As a result, third portion of the document is devoted to implementation details of the computational scheme used to solve unabbreviated governing equations. Ultimately, the last two chapters discuss results of our simulations. Particularly, chapter 4 describes effects of small changes in the buoyancy frequency when background flow speed is constant. The treatment of temporal variations in the basic velocity is postponed to chapter 5. In conclusion, we reemphasize that the main conclusion of this study is that small changes in the background flow conditions can profoundly influence the nature of stratified flow over topography.
Chapter 2

Preliminaries

2.1 Physical Model and Governing Equations

Disturbances generated when air is forced to rise over a topographic barrier, a mountain for instance, are observed to possess massive scales. Amplitudes of several hundred meters are not uncommon while horizontal length scales are often of the order of few tens of kilometers. Such immense fronts are typically accompanied by wind velocities on the scale of $10 \text{ m/s}$ and temperature decrease with altitude from about $280 \text{ K}$ on the surface to roughly $200 \text{ K}$ in the mesosphere. These observations in conjunction with buoyancy frequency of the order of $0.01 \text{ Hz}$ may be utilized to identify physical mechanisms that are insignificant in characterizing the behavior of topography induced flow fields. Accordingly, each of the following approximations provides a useful simplification of the fluid mechanics without significantly affecting the character of the motion being studied.

The observed behavior foremost suggests that Mach number, defined as the ratio of the local fluid velocity and the local speed of sound, is much smaller than unity. Specifically, with the speed of sound in air of about $300 \text{ m/s}$, Mach number is in vicinity of $0.03$. Correspondingly, as discussed in detail by Lighthill [21], effects of compressibility may be neglected in modeling atmospheric gravity-wave dynamics.

Moreover, Reynolds number -- the ratio of the horizontal length scale and kinematic viscosity multiplied by the horizontal velocity -- is much larger than unity.
With kinematic viscosity for the U.S. Standard Atmosphere of the order of $10^{-4}$ m$^2$/s, Reynolds number is on the scale of $10^9$. Consequently, as depicted by Baines [1], viscous layers are confined to thin boundary layers and atmosphere may be perceived as an inviscid medium.

As the next simplification, we neglect temperature variations and model air as an isothermal fluid. This assumption is reasonable in view of the fact that between the surface and the 70 km elevation the absolute temperature for the U.S. Standard Atmosphere is within 15% from the constant value of 250 K (Gill [12]). Nevertheless, the influence of temperature variations on the results in this work remains to be explored.

We further remark that Coriolis effects have been excluded from this study and this imposes some restrictions on the applicability of our findings. As pointed out by Baines [1], the assumption of negligible Coriolis acceleration in comparison to its convective counterpart is appropriate for horizontal length scales of few tens of kilometers or less.

Additionally, due to the fact that the nature of topographic barriers if often such that one of the dimensions is much larger than the others, we regard buoyancy-induced disturbances as two-dimensional. It is important to emphasize, however, that this is an assumption valid far away from the barrier’s end. The structure of the flow near the ends is highly three-dimensional.

Flow of an isothermal, incompressible and inviscid fluid in the absence of Coriolis effects is governed by Euler equations. In terms of dimensional quantities, these can be expressed as

$$\rho^* (u_t^* + u^*u_x^* + w^*u_z^*) = -p_x^*, \quad (2.1a)$$

$$\rho^* (w_t^* + u^*w_x^* + w^*w_z^*) = -p_z^* - \rho^*, \quad (2.1b)$$

$$\rho_t^* + u^*\rho_x^* + w^*\rho_z^* = 0, \quad (2.1c)$$

$$u_x^* + w_z^* = 0. \quad (2.1d)$$

Here, $u^*$ and $w^*$ denote components of the velocity vector in the horizontal ($x^*$) and
vertical ($z^*$) direction respectively while $p^*$ depicts pressure. Moreover, $\rho^*$ designates the fluid density and $g$ represents the acceleration of gravity oriented in the negative $z^*$ direction. Quantities $u^*$, $w^*$, $\rho^*$ and $p^*$ are functions of independent variables $x^*$, $z^*$ and time, $t^*$. As the last approximation, we stress that observational evidence of atmospheric topography produced flow fields further suggests that the vertical scale of variations in $w^*$ is much smaller than the scale of vertical hydrostatic density variations. The simplification that applies when this condition is satisfied is the Boussinesq approximation. It is demonstrated below that it amounts to neglecting density changes in Euler equations everywhere except when they give rise to buoyancy, that is, on the right side of (2.1b).

With the air dynamics governed by the Boussinesq form of (2.1), in this study, we are concerned with the structure of flow fields generated when atmospheric wind encounters a mountain and is forced to rise. We investigate this natural phenomenon by considering the flow of an infinitely deep body of air over a locally confined topographic barrier of height $H$ and characteristic length scale $L$. If the typical density and buoyancy frequency of air are $\rho_0$ and $N_0$ respectively while $U_0$ is the free-stream velocity oriented in the positive $x^*$ direction, Euler equations can be made dimensionless and the Boussinesq approximation formally introduced via following arguments.

Due to the fact that the mountain causes the flow to deviate from the hydrostatic conditions present far upstream and downstream, we expect the width of the disturbance, and consequently horizontal coordinate, to scale with $L$. While the only plausible choices for the measure of the horizontal velocity and density are $U_0$ and $\rho_0$ respectively, the proper scale for $z^*$ is less obvious. Since we do not expect the vertical extent of the disturbances to be influenced by $L$, there are two alternatives for the vertical length scale, namely, $U_0/N_0$ and $H$. With the scales for $u^*$ and $x^*$ already established, it turns out that either selection will suffice with the understanding that the choice will influence the scale for $w^*$. We select $U_0/N_0$ as the vertical length scale, which by (2.1d) determines the scale for $w^*$ to be $U_0^2/N_0L$. Lastly, since we do not have observational evidence of the relative importance of pressure gradients and unsteadiness in comparison to the convection, we assume that these terms are of the
same order of magnitude. As a result, we scale time with the convective time scale $L/U_0$ and pressure with $\rho_0 U_0^2$. With these scales, the dimensionless form of Euler equations is

$$\rho (u_t + uu_x + wu_z) = -p_x, \quad \text{(2.2a)}$$

$$\mu^2 \rho (w_t + uw_x + wwu_z) = -p_z - \frac{\rho}{\beta}, \quad \text{(2.2b)}$$

$$\rho_t + u\rho_z + wp_z = 0, \quad \text{(2.2c)}$$

$$u_x + w_z = 0. \quad \text{(2.2d)}$$

In these expressions, $\mu$ is the ratio of the vertical and horizontal length scale, namely,

$$\mu = \frac{U_0}{N_0 L}, \quad \text{(2.3)}$$

and as such it controls the aspect ratio of the topography induced disturbances. In the limit of long waves in comparison to their height, $\mu \to 0$ and the $z$-component of the momentum equation approaches the hydrostatic balance. As it will become apparent from the results presented in section 2.3 and chapters 4 and 5, this limit vastly simplifies governing equations while retaining qualitative properties of the response associated with the finite value of $\mu$.

The second quantity ($\beta$) is termed Boussinesq parameter and is defined as

$$\beta = \frac{U_0 N_0}{g}. \quad \text{(2.4)}$$

It physically represents the ratio of the vertical scale for variations in $w^*$ and the vertical scale for variations in $\bar{\rho}_z^*$, which can be easily seen by using (1.1) to replace $\bar{\rho}_z^*$ in the quotient of $\bar{\rho}_z^* w^*$ and $\bar{\rho}^* w_z^*$ and making the resulting expression dimensionless. For the Boussinesq approximation to be valid, $\beta$ must be much smaller than unity. In the atmosphere, based on the observed properties of the buoyancy induced disturbances, $\beta$ is of the order of 0.01, thus atmosphere behaves as a Boussinesq medium.

The remaining step in the derivation of the governing equations is to make the
Boussinesq approximation. In that regard, we separate density into its hydrostatic and perturbation components. We anticipate the perturbation portion to vanish in the limit of decreasing topography while in the Boussinesq limit ($\beta \to 0$) we expect density to be constant in front of the momentum terms. Both of these requirements can be satisfied by writing density as the sum of hydrostatic and perturbation components. Specifically, we let

$$\rho = \bar{\rho} + \epsilon \beta r$$

(2.5)

where $r$ denotes perturbation density reduced by $\epsilon \beta$ with

$$\epsilon = \frac{N_0 H}{U_0}$$

(2.6)

being, as it will become evident shortly, a measure of nonlinearity. In the limit of either $\beta \to 0$ or $\epsilon \to 0$, in accordance with our expectations, perturbation density approaches zero. Moreover, the dimensionless form of (1.1)

$$\bar{\rho}_z = -\beta \bar{\rho} N^2$$

(2.7)

suggests that when $\beta \to 0$, $\bar{\rho}_z$ approaches zero, which in turn implies that $\bar{\rho}$ is a constant. Since $\bar{\rho}$ is scaled by the characteristic density, we conclude that $\bar{\rho} = 1$. Ultimately, substitution of (2.5) into (2.2) in the limit $\beta \to 0$ yields dimensionless form of governing equations associated with our model of the atmosphere, that is,

$$u_t + uu_x + wu_z = -p'_x,$$

(2.8a)

$$\mu^2 (w_t + uw_x + wu_z) = -p'_x - \epsilon r,$$

(2.8b)

$$\epsilon (r_t + ur_x + wr_z) = w N^2,$$

(2.8c)

$$u_x + w_z = 0.$$

(2.8d)

where $p'$ stands for the perturbation pressure.
Due to the fact that we are considering two-dimensional flow, we remark at this point, that it is convenient to rewrite (2.8) via a stream function, $\Psi$, in terms of which velocity components are defined as

$$u = \Psi_z \quad \text{and} \quad w = -\Psi_x$$

(2.9)

where it is apparent that we scaled the stream function by the quotient $U_0^2/N_0$. The foremost advantage of expressing governing equations in terms of $\Psi$ is that (2.8d) is identically satisfied. The second benefit is that the resulting system of three equations can further be reduced to two equations by subtracting the $x$-derivative of (2.8b) from the $z$-derivative of (2.8a). This eliminates the perturbation pressure and equations (2.8) become

$$\Psi_{zzt} + \mu^2 \Psi_{xzt} + J (\Psi_{zz} + \mu^2 \Psi_{xx}, \Psi) = \epsilon_x,$$

(2.10a)

$$\epsilon_t + J (\epsilon_x, \Psi) = -N^2 \Psi_x$$

(2.10b)

where $J(a, b) = a_x b_z - a_z b_x$.

We conclude the discussion of governing equations by accentuating that in the process of exploring the structure of stratified flows over topography, we will often be concerned with the importance of nonlinear effects. Their significance can be established through the comparison of the solution of (2.10) with the behavior predicted by the linear form of the governing equations. This form can be formally derived by expressing (2.10) in terms of the perturbation stream function defined as

$$\Psi^* = U^* z^* + \psi^*.$$  

(2.11)

It physically denotes the deflection of a streamline from the hydrostatic value it possesses far upstream of the obstacle. Accordingly, we expect that in the limit of the vanishing topography, the perturbation portion of the stream function approaches zero. As a result, we scale $\psi^*$ with $HU_0$, which gives the dimensionless form of (2.11) as

$$\Psi = U z + \epsilon \psi.$$  

(2.12)
In terms of $\psi$, our governing equations become

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) (\psi_{zz} + \mu^2 \psi_{xx}) - r_x = \epsilon J (\psi, \psi_{zz} + \mu^2 \psi_{xx}), \tag{2.13a}
\]

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) r + N^2 \psi_x = \epsilon J(\psi, r). \tag{2.13b}
\]

Nonlinear terms are now confined to the right-hand side of (2.13). In the limit of the small barrier height $\epsilon \to 0$, equations (2.13) become linear in both $\psi$ and $r$.

In order to complete the mathematical formulation of the problem under the consideration, we lastly need to specify boundary conditions. Due to the fact that the topography is presumed to be locally confined in the horizontal direction and that the fluid is considered to be infinitely deep, we require

\[
\psi = 0 \text{ and } r = 0 \text{ at } x = \pm \infty \text{ and } z = +\infty. \tag{2.14}
\]

Along the bottom boundary, on the other hand, the velocity component perpendicular to the barrier is zero, which in turn implies that the scalar product of the velocity and the unit vector orthogonal to the mountain is zero. Algebraically, this can be written in terms of $\psi$ as

\[
-\psi_x = \frac{dh}{dx} (U + \epsilon \psi_x) \text{ at } z = \epsilon h(x) \tag{2.15}
\]

where we have scaled the obstacle profile by its peak height $H$. Equivalently, (2.15) could be obtained by noting that the velocity is always tangential to the barrier’s surface. Since streamlines are defined as lines that are always tangential to the velocity vector, $z = \epsilon h(x)$ is a streamline. Because the value of the stream function is physically meaningless, we can arbitrarily assume it to be zero along $z = \epsilon h(x)$; therefore, the bottom boundary condition can also be written as

\[
\Psi = 0 \text{ at } z = \epsilon h(x). \tag{2.16}
\]
Taking the directional derivative of (2.16) along \( z = \epsilon h \) and rearranging terms yields again (2.15). As it will become apparent in sections 2.2 and 2.3, however, the form (2.16) is more convenient in finding steady-state solutions of (2.10). We lastly mention that throughout this document, we use the topography profile popularly known as the algebraic mountain or *Witch of Agnesi*

\[
h = \frac{1}{1 + x^2}.
\]  

### 2.2 Long's Archetype

The behavior of a stratified fluid elevating over a locally confined obstacle was first studied by Long [23] in 1953. In this pioneering effort, he sought the steady-state solution of the governing equations (2.10) in the presence of the simplest background flow conditions, namely, uniform basic flow and constant Brunt-Väisälä frequency. For this special flow configuration, he found that, under the assumption of no upstream influence (i.e., in the steady state, disturbances do not evolve against the basic flow, and as a result, free-stream conditions are present far upstream of the barrier), nonlinear equations (2.10) remarkably reduce to a single linear equation.

In order to attain this governing equation associated with Long’s archetype, we begin with the steady-state form of (2.10), which for \( \mu = 0 \) is

\[
J(\Psi_{zz}, \Psi) = \epsilon r_x, 
\]  
\[
J(\epsilon r, \Psi) + N^2 \Psi_x = 0. 
\]

We proceed by noting that the second term in (2.18b) for \( N = 1 \) may be rewritten in the Jacobian form as \( J(\Psi, z) \); therefore, this equation becomes

\[
J(\epsilon r - z, \Psi) = 0. 
\]

In the steady-state limit, the Jacobian of an arbitrary property with respect to \( \Psi \)
represents the rate of change of this property along a streamline. Accordingly, (2.19) states that the rate of change of $\varepsilon r - z$ along any streamline is zero. Integrating (2.19) along a streamline suggests that $\varepsilon r - z = C$ where $C$ is a constant whose value depends on the particular streamline along which the integration is performed. At this point, we note that in the absence of the upstream influence, as $x \to -\infty$, $r$ approaches zero while $z$ converges to $\Psi$. This in turn implies that $C = -\Psi$; hence, $\varepsilon r = z - \Psi$. Substitution of this result into (2.18a) gives $J(\Psi_{zz} - z, \Psi) = 0$. Integrating now this expression along a streamline and realizing that far upstream $\Psi_{zz} = 0$ and $z = \Psi$ yields the governing equation for Long’s model

$$\Psi_{zz} + \Psi - z = 0,$$

which is linear in $\Psi$. This result can be further simplified by using (2.12) to express $\Psi$ in terms of $\psi$. For the uniform background flow, based on our scalings, $U = 1$ and (2.20) reduces to the Helmholtz equation

$$\psi_{zz} + \psi = 0.$$  

(2.21)

An inspection of this equality reveals that for every value of $x$, (2.21) is a second order ordinary differential equation in $z$. Therefore, its solution may be written as

$$\psi(x, z) = C_1 \cos(z) + C_2 \sin(z)$$

(2.22)

where $C_1$ and $C_2$ are functions of $x$ and are determined from the top and bottom boundary conditions.

The appropriate upper boundary condition is the radiation condition, which requires that all energy is radiated out of the system. Its application to the problem at hand is physically plausible because the obstacle is the source of energy (it causes the formation of gravity waves which are energy carriers) in the infinitely deep pool of fluid. As a result, we expect generated disturbances to endlessly evolve with height. Considering that the energy travels with the group velocity, this boundary condition
ensures that only waves with the vertical component of the group velocity directed upwards reside in the domain of interest. In order to express this condition in terms of $\psi$, following Lilly and Klemp [22], we begin by seeking the dispersion relationship that waves traveling through a medium whose dynamics is guided by (2.21) must obey. In that regard, we emphasize that because $\epsilon$ cancels out of the Helmholtz equation, (2.21) is the governing equation for both the linear and the nonlinear response. Consequently, the dispersion relationship for both the linear and the nonlinear response is the same; nevertheless, this equality is much easier to attain by considering the linear problem. Correspondingly, we note that in the absence on nonlinear effects ($\epsilon \rightarrow 0$) and in the presence of the uniform background flow, equations (2.13) can be combined into a single equation, which for $\mu = 0$ is

$$
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)^2 \psi_{zz} + \psi_{xx} = 0. \tag{2.23}
$$

Assuming plane wave solutions of this expression in the form $\psi = \psi_0 \cos(xk + mz - \omega t)$ where the amplitude ($\psi_0$) is a constant while $k$ and $m$ are the horizontal and vertical wave numbers respectively, yields the dispersion relationship

$$
(\omega - k)^2 = \frac{k^2}{m^2}. \tag{2.24}
$$

The vertical component of the group velocity is obtained by taking the derivative of (2.24) with respect to $m$, which in the steady-state limit ($\omega \rightarrow 0$) is

$$
\frac{\partial \omega}{\partial m} = \frac{k}{m}. \tag{2.25}
$$

This result implies that only those plane waves with the same sign of $k$ and $m$ will possess the upward oriented $z$-component of the group velocity, and as such will be allowed to reside in the system once the steady state is reached.

Before we can impose this boundary condition, we must, however, rewrite (2.22) as the superposition of plane waves. To accomplish this task, we define the Fourier
transform and its inverse with respect to $x$ as

\[
\mathcal{F}\{\psi\} = \int_{-\infty}^{+\infty} \psi e^{-ikx} dx \quad \text{and} \quad \mathcal{F}^{-1}\{\tilde{\psi}\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{\psi} e^{ikx} dk. \tag{2.26}
\]

Converting the sine and cosine in (2.22) into their exponential form and then applying the Fourier transform yields

\[
\psi = \frac{1}{4\pi} \int_{-\infty}^{+\infty} (\tilde{C}_1 - i\tilde{C}_2) e^{i(kx+z)} dk + \frac{1}{4\pi} \int_{-\infty}^{+\infty} (\tilde{C}_1 + i\tilde{C}_2) e^{i(kx-z)} dk. \tag{2.27}
\]

We note that the value of $m$ in the first and the second integral is 1 and $-1$ respectively.

The radiation boundary condition (2.25) requires then the first integral to be zero for $k < 0$ and the second integral to vanish for $k > 0$. This in turn necessitates that $\tilde{C}_2 = i \text{sign}(k) \tilde{C}_1$. Taking the advantage of the fact that $\mathcal{F}\{H[C_1]\} = i \text{sign}(k) \tilde{C}_1$ where $H$ stands for the Hilbert transform, which together with its inverse is defined as

\[
H[C_1] = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{C_1}{x-y} dx \quad \text{and} \quad H^{-1}[\tilde{C}_1] = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\tilde{C}_1}{x-y} dy, \tag{2.28}
\]

the relationship between $C_1$ and $C_2$ may be expressed as

\[
C_2 = H[C_1]. \tag{2.29}
\]

Substitution of (2.29) into (2.22) gives the solution of Long's model as

\[
\psi(x, z) = C_1 \cos(z) + H[C_1] \sin(z) \tag{2.30}
\]

where $C_1$ is determined from the bottom boundary condition

\[
-h = C_1 \cos(\epsilon h) + H[C_1] \sin(\epsilon h) \tag{2.31}
\]

obtained by placing (2.22) into (2.16).

It is apparent from (2.30) and (2.31) that nonlinear effects in Long's model are introduced solely through the bottom boundary condition, which in the linear problem
(\epsilon \to 0) is imposed at \(z = 0\). In the linear regime, (2.31) reduces to \(C_1 = -h\) and the solution of (2.30) is then simply

\[ \psi = -h \cos(z) - \mathcal{H}[h] \sin(z) \]  \hspace{1cm} (2.32)

where the Hilbert transform of the algebraic mountain (2.17) is given by

\[ \mathcal{H}[h] = -\frac{x}{1 + x^2}. \]  \hspace{1cm} (2.33)

For the finite value of \(\epsilon\), on the other hand, (2.30) and (2.31) must be solved numerically. A numerical procedure that accomplishes this task was first constructed by Lilly and Klemp [22] in 1979. The solution using their approach for \(\epsilon = 0.6\) is graphically depicted in Fig. 2-1. It is evident from the figure that the flow field associated with Long’s model takes the form of a columnar disturbance that is slowly modulated in the horizontal direction. From (2.30), we also see that the response recurs in the vertical direction with the period of \(2\pi\). According to simulations of Lilly and Klemp [22], streamlines become vertical, indicating the onset of wave breaking, when \(\epsilon = 0.85\). We conclude the discussion of this model by pointing out that the linear and nonlinear solution closely agree for \(\epsilon < 0.6\).

2.3 Two-Layer Model of Durran [9] and Davis [8]

A more realistic model of the atmospheric gravity-wave dynamics was developed by Durran [9] in 1992. His archetype is analogous to the one adopted by Long except that it accounts for the presence of the tropopause by modeling the Brunt–Väisälä frequency as two layers of the uniform stratification. The tropopause, however, as shown in Fig. 1-1, is not a height of the discontinuous transition in the buoyancy frequency, but rather the 4 km region of the rapid but continuous change in the stratification. To account for this deficit, Davis [8] extended the model of Durran [9] to an arbitrary buoyancy frequency profile. Before discussing the impact of the tropopause on the flow field generated when a stratified fluid encounters a topographic
barrier, we present the derivation of the governing equation associated with the model of Davis [8].

This expression can be easily obtained by rewriting the second term of (2.18b) in the Jacobian form. This gives

\[ J \left( \epsilon r - \int_{z_0}^{z} N^2(z')dz', \Psi \right) = 0. \]  

(2.34)

Physically, this implies that along any streamline, the difference between \( \epsilon r \) and the integral of \( N^2 \), taken from the streamline height far upstream \( (z_0) \) to the altitude of the difference evaluation, is constant. Algebraically, this can also be written as

\[ \epsilon r - \int_{z_0}^{z} N^2(z')dz' = C \]  

(2.35)

where \( C \), just as in the derivation of the Long’s equation, is a constant that depends on the particular streamline. Far upstream of the obstacle, both terms on the left side of this equation are zero; hence, \( C = 0 \). At this point, we remind the reader that based on (2.12), for any value of \( x \), the deflection of a streamline with the elevation \( z_0 \)
in the limit \( x \to -\infty \) is given by \( \epsilon \psi(x, z_0) \). Therefore, the upper limit of the integral in (2.35) for this streamline may be formulated as \( z_0 + \epsilon \psi(x, z_0) = \Psi_0 \). The derivative of (2.35) with respect to \( x \) then gives

\[
\epsilon r_x = N^2(\Psi_0) \frac{\partial \Psi_0}{\partial x}.
\]  

(2.36)

In order to make (2.36) valid for any streamline, we further drop subscripts of \( \Psi \). Substitution of the resulting formula into (2.18a) followed by another integration along a streamline with the integration constant \(-N^2(\Psi)\Psi\) gives the governing equation for the model of Davis [8], that is,

\[
\psi_{zz} + N^2(z + \epsilon \psi) \psi = 0.
\]  

(2.37)

The comparison of (2.21) and (2.37) reveals that the only difference between Long’s equation and that of Davis is the presence of \( N^2(\Psi) \) in (2.37). The fact that this term is a function of the perturbation stream function, however, makes (2.37) nonlinear in \( \psi \), which is in sharp contrast to the model of Long where the nonlinearity entered the problem solely through the bottom boundary condition. The radiation condition (2.29), based on the linear dispersion relationship (2.24), can still be used, nevertheless, if above some height \( z_\infty \) buoyancy frequency is constant at the value \( N_\infty \). In this region then the solution of (2.18) with \( N^2 = N_\infty \) is

\[
\psi_\infty = C_1(x)\cos[N_\infty(z - z_\infty)] + C_2(x)\sin[N_\infty(z - z_\infty)]
\]  

(2.38)

and its \( z \)-derivative is

\[
\psi_\infty = -N_\infty C_1(x)\sin[N_\infty(z - z_\infty)] + N_\infty C_2(x)\cos[N_\infty(z - z_\infty)].
\]  

(2.39)

At \( z = z_\infty \), all sines on the right-hand side of (2.38) and (2.39) are zero. Following Davis [8], using (2.29) to relate the remaining cosine terms gives the radiation
boundary condition for an arbitrary value of $N_c$ as

$$\psi = N_\infty \mathcal{H}[\psi_z].$$  \hspace{1cm} (2.40)

Implementation details of a numerical procedure for solving (2.37) with boundary conditions (2.16) and (2.40) are presented in Appendix A. This computational approach was developed by Davis [8] who originally utilized it to explore effects of the tropopause on the solution of Long. A buoyancy frequency profile that includes the tropopause can be mathematically formulated in several ways. In our simulations, we use

$$N = 1.5 + 0.5 \tanh[c(z - d)].$$ \hspace{1cm} (2.41)

In accordance with the atmospheric stratification, depicted in Fig. 1-1, the Brunt–Väisälä frequency given by (2.41) features two distinct regions with $N$ in the upper stratum being twice as large as its lower layer analogue. The gradient of the transitional region is determined by the parameter $c$ while its height is specified by $d$. The salient feature of the two-layer buoyancy frequency profile is that suitably adjusted tropopause properties $c$ and $d$ may maximize or minimize the steady-state response, in other words, cause a tuning or detuning effect. A flow field is tuned in a sense that the amplitude and steepness of its streamlines is the largest for a prescribed value of $c$. Tuned and detuned patterns for $c=0.6$ and $c=2$ occur when the transition is positioned at $d=25$ and $d=21$ respectively. Maximized streamline amplitudes are nearly twice as large as their minimized counterparts, as illustrated in Fig. 2-2. Streamlines for $d=25$ are also significantly steeper with the streamline that approaches $z=4$ as $x \to -\infty$ overturning above the tip of the topography. This indicates that the wave breaking in the presence of the tropopause, as discovered by both Durran [9] and Davis [8], may take place at the lower value of the nonlinearity parameter than forecasted by Long’s model. Davis [8] has additionally found that the tuning phenomenon takes place regardless of the tropopause steepness although for each $c$ the value of $d$ at which the amplification takes place is different. In agreement with Durran [9], Davis [8] has also established that values of $c$ and $d$ that maximize the
nonlinear response, do not necessarily tune the linear behavior. In fact, circumstances may arise in which the linear theory predicts breaking, but no breaking is present in the nonlinear solution. Therefore, in contrast to Long's model where the limit $\epsilon \to 0$ has featured the agreement between the linear and nonlinear response, for a two-layer profile, the linear solution is not a reliable simulation of the gravity wave dynamics.

Figure 2-2: Response of Davis for $\epsilon = 0.6$ and the buoyancy frequency given by (2.41) with $c = 2$. Tuned behavior occurs when $d = 25$ (solid lines) and detuned when $d = 21$ (dashed lines).
Chapter 3

Numerical Procedure for Unsteady Computations

3.1 Benefits of the Utilized Approach

This segment of the manuscript describes implementation details of the computational model that we utilize in chapters 4 and 5 to simulate the behavior of an infinitely deep stratified fluid rising over an isolated mountain. The model numerically solves governing equations (2.13) with boundary conditions (2.14) and (2.15) using the second-order projection algorithm originally developed by Bell and co-workers [3, 4, 5]. In previous efforts, this scheme has been employed to treat first uniformly and then arbitrarily stratified topographic flows of finite depth by Lamb [19, 20] and Skopović [27] respectively. Here, we apply the procedure to infinitely deep flows of general stratification. In this chapter, we limit our considerations to the constant free stream while in Sec. 5.3 we further extend the approach to account for temporal variations in the incident velocity.

In addition to the fact that it has been successfully used in the past to simulate the gravity-wave dynamics, the second-order projection algorithm also possesses several other properties that make it a suitable choice for solving (2.13). In particular, the scheme is second-order accurate in both space and time. Moreover, it is non-iterative in a sense that it determines nonlinear convective terms at each time step explicitly,
i.e., without iteration. Additionally, the matrix associated with the linear algebra problem \( A \cdot x = b \) that needs to be solved at each time step is symmetric, positive-definite and block-tridiagonal; consequently, it can be effectively treated using, for instance, Cholesky factorization. Ultimately, in the Boussinesq limit, \( A \) does not have to be updated with time. This implies that upon being initially constructed, the matrix can be decomposed and then the value of \( x \) at each time level can be obtained through the matrix-vector multiplication rather than the solution of a linear system.

We begin the discussion of the numerical model with the overview of the second-order projection scheme, presented in sections 3.2 and 3.3. The most challenging aspect of the model construction has been the numerical implementation of boundary conditions on the lateral and the upper boundaries. Appropriate conditions on these boundaries are open boundary conditions, which allow all topography-generated disturbances to leave the domain of interest without reflection. The difficulty arises from the fact that these boundary conditions cannot be analytically posed for the Boussinesq form of Euler equations, as illustrated in detail in Sec. 3.4. As a result, we imitate them by surrounding the inviscid region with layers of varying viscosity whose purpose is to dissipate evolving internal gravity waves before they reflect from the edge of the numerical domain and reenter the region of interest. Details of the numerical implementation of absorbers are discussed in Sec. 3.5. Ultimately, the last section of the chapter describes the testing of the scheme and lists values of the model parameters used to produce simulations in chapters 4 and 5.

### 3.2 Temporal Discretization

Our physical domain is depicted in Fig. 3-1(a). It is discretized with \( I \) cells in the horizontal and \( J \) cells in the vertical direction. The grid is constructed by foremost selecting \( x \) and \( z \) coordinates of cell corners represented by the circles in the figure. Horizontal coordinates of these points are chosen in such a way that their \( x \)-direction distance linearly increases from the middle of the domain to lateral boundaries. In a similar fashion, vertical coordinates of cell corners are picked in such a way that
the $z$-direction distance between grid points linearly increases from the bottom to the top of the domain. We obtain horizontal (vertical) coordinates of cell centers, marked with crosses in Fig. 3-1(a), by averaging $x$ ($z$) values of cell corners. The fluid velocity, density and pressure gradient are defined at cell centers while boundary conditions of these quantities are specified at crosses lying on the domain boundaries — attained as the half-distance between cell corners.

We numerically seek the solution of (2.8) on the rectangular computational domain in Fig. 3-1(b) obtained via one-to-one mapping from the physical one in Fig. 3-1(a). This lattice is quadrilateral; i.e., all cells are unit squares. Accordingly, the space is horizontally bounded by $\xi = 0$ and $\xi = I$ while its vertical boundaries are $\eta = 0$ and $\eta = J$. In our model, the second-order projection algorithm approximates the solution of (2.8) by solving on the computational domain

$$\mathcal{J} \mathbf{U}_t + (\bar{\mathbf{U}} \cdot \nabla_\Xi) \mathbf{U} = -[T]^t \nabla_\Xi p' + \mathcal{J}(b \hat{e}_x - \epsilon \eta \hat{e}_z) + \nu \nabla_\Xi \cdot (\mathcal{J}^{-1}[T][T]^t \nabla_\Xi \mathbf{U}), \quad (3.1a)$$

$$r_t + \mathcal{J}^{-1}(\bar{\mathbf{U}} \cdot \nabla_\Xi) \epsilon r = wN^2, \quad (3.1b)$$

$$\nabla_\Xi \cdot \bar{\mathbf{U}} = 0 \quad (3.1c)$$

with boundary conditions

$$u_t = b \text{ and } w = 0 \quad \text{at} \quad \xi = 0, \quad \xi = I \quad \text{and} \quad \eta = J. \quad (3.2)$$

In these expressions, $\mathbf{U}$ is the velocity vector with the $x$-component $u$ and the $z$-component $\mu^2w$. Moreover, $[T]^t$ is the transpose of the grid transformation matrix

$$[T] = \begin{bmatrix} z_\eta & -x_\eta \\ -z_\xi & x_\xi \end{bmatrix} \quad (3.3)$$

and

$$\mathcal{J} = x_\xi z_\eta - x_\eta z_\xi \quad (3.4)$$

is the Jacobian of the transformation. Lastly, $\nabla_\Xi$ is the gradient with respect to $\xi$.
Figure 3-1: Discretization of the physical and computational domain

and $\eta$ while $\tilde{U}$ is defined as
\[ \mathbf{\tilde{U}} = \begin{pmatrix} \tilde{u} \\ \tilde{w} \end{pmatrix} = [T] \begin{pmatrix} u \\ w \end{pmatrix}. \] (3.5)

Quantities \( \tilde{u} \) and \( \tilde{w} \) are physically velocity components in the \( \xi \) and \( \eta \) directions respectively. Accordingly, along the bottom boundary we require

\[ \tilde{w} = 0 \quad \text{at} \quad \xi = 0. \] (3.6)

The reader should note at this point that in contrast to (2.8a) and (2.8b), (3.1a) contains the body force (b) and viscous terms. The body force is a function of time and it is used to accelerate the flow from rest. The dimensionless kinematic viscosity (\( \nu \)), on the other hand, is zero in the inviscid domain of interest and it varies within viscous layers.

For given values of the fluid velocity and density at the time \( t = n\Delta t \) — designated as \( U^n \) and \( r^n \) respectively where \( n \) is an integer and \( \Delta t \) is the time increment — the second-order projection method obtains \( U^{n+1} \) and \( r^{n+1} \) by foremost explicitly computing nonlinear convective terms in (3.1a) and (3.1b) at the intermediate time level \( n + \frac{1}{2} \). Implementation details of this procedure are presented in [27]. From the knowledge of \( [J^{-1}(\mathbf{\tilde{U}} \cdot \nabla \mathbf{\tilde{r}})\mathbf{\tilde{r}}]^{n+\frac{1}{2}} \), the scheme obtains \( r^{n+1} \) using

\[ \frac{r^{n+1} - r^n}{\Delta t} = -[J^{-1}(\mathbf{\tilde{U}} \cdot \nabla \mathbf{\tilde{r}})\mathbf{\tilde{r}}]^{n+\frac{1}{2}} + N^2 w^{n+\frac{1}{2}} \] (3.7)

where the vertical velocity component at \( t = (n + \frac{1}{2})\Delta t \) is computed via the second-order accurate extrapolation from time levels \( n \) and \( n - 1 \). Specifically, we let

\[ w^{n+\frac{1}{2}} = \frac{3}{2} w^n - \frac{1}{2} w^{n-1}. \] (3.8)

With \( r^{n+1} \) determined, (3.1a) at \( t = (n + \frac{1}{2})\Delta t \) becomes

\[ U_t^{n+\frac{1}{2}} + J^{-1}[T]^t \nabla \cdot \mathbf{p}^{n+\frac{1}{2}} = -J^{-1} \left[ (\mathbf{\tilde{U}} \cdot \nabla \mathbf{\tilde{r}}) \mathbf{\tilde{U}} \right]^{n+\frac{1}{2}} + b^{n+\frac{1}{2}} \hat{e}_x - \epsilon r^{n+\frac{1}{2}} \hat{e}_z \]

\[ + \frac{\nu}{2} J^{-1} \nabla \cdot [J^{-1}[T][T]^t \nabla \cdot (\mathbf{U}^{n+1} + \mathbf{U}^n)] = \mathbf{V}^{n+\frac{1}{2}}. \] (3.9)
At this point, it is important to emphasize that based on (3.1c) divergence free while the second term on the right-hand side of (3.9) is simply the gradient of pressure and as such is gradient free. The second-order projection algorithm takes advantage of these properties and determines $U_t^{n+1}$ by projecting the right-hand side of (3.9) onto the space of divergence free vector fields. In other words, the scheme decomposes $V^{n+\frac{1}{2}}$ into its divergence-free and gradient-free components. The Hodge decomposition theorem, presented in [28], guarantees the uniqueness of the projection for a prescribed set of boundary conditions. Once the value of $U_t^{n+\frac{1}{2}}$ is known, $U_t^{n+1}$ is obtained from

$$\frac{U_t^{n+1} - U_t^n}{\Delta t} = U_t^{n+\frac{1}{2}};$$  \hspace{1cm} (3.10)

and, the pressure gradient at the time level $n + \frac{1}{2}$ is calculated as the difference $V^{n+\frac{1}{2}} - U_t^{n+\frac{1}{2}}$.

The difficulty in evaluating the projection, however, arises from the fact that the last term on the right-hand side of (3.9) depends on $U_t^{n+1}$ when $\nu \neq 0$. A possible solution to this problem would be to guess the value of $U_t^{n+1}$ and use Newton-Rhapson scheme to update the guess upon performing the projection. This approach is numerically very costly as the projection is the most expensive segment of the second-order projection algorithm. In order to avoid multiple projections at a single time step, Bell et al [3, 4, 5] approximate $U_t^{n+1}$ in the viscous term of (3.9) by using the pressure gradient from the time level $n - \frac{1}{2}$. Particularly, they solve

$$\frac{U^* - U^n}{\Delta t} + J^{-1}[T]^t \nabla_\Xi p' \cdot n - \frac{1}{2} = -J^{-1}[(\bar{U} \cdot \nabla_\Xi)U]^{n+\frac{1}{2}} + b^{n+\frac{1}{2}} \hat{e}_x - \epsilon r^{n+\frac{1}{2}} \hat{e}_z + \frac{\nu}{2} J^{-1} \nabla_\Xi \cdot \left[ J^{-1}[T]^t \nabla_\Xi (U^* + U^n) \right]$$  \hspace{1cm} (3.11)

for $U^*$, which represents an estimate of $U_t^{n+1}$. As portrayed by Bell and co-workers [3], performing the projection with the estimate of $U_t^{n+1}$ does not alter the accuracy of the scheme; i.e., the second-order projection method still remains second order accurate in both space and time. Once $U^*$ is known, it is substituted for $U_t^{n+1}$ in the viscous term of (3.9) and then the projection is performed. In the next section, we
illustrate implementation details of the projection portion of the algorithm.

### 3.3 Projection

In this section, we are concerned with numerically decomposing an arbitrary two-dimensional vector field \( \mathbf{V} \) into two components \( \mathbf{U}_t \) and \( \nabla p' \) such that

\[
\mathbf{U}_t + \nabla p' = \mathbf{V} \tag{3.12}
\]

and

\[
\nabla \cdot \mathbf{U}_t = 0. \tag{3.13}
\]

More precisely, we computationally seek \( \mathbf{U}_t \) at cell centers in Fig. 3-1(b) given \( \mathbf{V} \) at grid points designated with crosses in the figure and boundary conditions for \( \mathbf{U}_t \) specified at crosses lying on the domain boundaries. In this study, following Bell et al [3, 4, 5] we chose to implement the composition by expressing \( \mathbf{U}_t \) as the linear combination of divergence-free basis vectors \( \mathbf{Y} \). In particular, we write

\[
\mathbf{U}_t = \sum \alpha_i \mathbf{Y}_i + \sum \alpha_j \mathbf{Y}_j \tag{3.14}
\]

where subscripts \( i \) and \( j \) denote basis vectors laying on the domain interior and its boundaries respectively while their weights \( \alpha \) are real constants. At this point, we formally assume that all \( \alpha_j \) are known from boundary conditions for \( \mathbf{U}_t \) and as such can be moved to the right-hand side of (3.12), which now takes form

\[
\sum \alpha_i \mathbf{Y}_i + \nabla p' = \mathbf{V} - \sum \alpha_j \mathbf{Y}_j = \mathbf{V}^B. \tag{3.15}
\]

We proceed by noting that for any scalar function \( \varphi \), the vector field \((\varphi_z, -\varphi_x)\) is divergence free. In terms of the computational domain gradients, \( \mathbf{Y} \) can be expressed as

\[
\mathbf{Y} = \frac{[T]^{-1}}{\mathcal{J}} \begin{pmatrix} \varphi_n \\ -\varphi_\xi \end{pmatrix} \tag{3.16}
\]
where
\[
[T]^{-1} = \begin{bmatrix}
  x_\xi & x_\eta \\
z_\xi & z_\eta
\end{bmatrix}.
\] (3.17)

Moreover, (3.15) can now be written as
\[
\sum \alpha_i \frac{[T]^{-1}}{J} \begin{bmatrix}
  \varphi_\eta \\
  -\varphi_\xi
\end{bmatrix}_i + \frac{[T]^t}{J} \nabla_{\pi} p' = \mathbf{V}^B.
\] (3.18)

Taking the dot product of both sides of this equation with respect to \([T]^{-1}(\varphi_\eta, -\varphi_\xi)_k\) and integrating the resulting expression over the entire domain gives
\[
\iint \mathbf{V}^B \cdot [T]^{-1} \begin{bmatrix}
  \varphi_\eta \\
  -\varphi_\xi
\end{bmatrix} d\xi d\eta = \iint \sum \alpha_i \frac{[T]^{-1}}{J} \begin{bmatrix}
  \varphi_\eta \\
  -\varphi_\xi
\end{bmatrix}_i \cdot [T]^{-1} \begin{bmatrix}
  \varphi_\eta \\
  -\varphi_\xi
\end{bmatrix}_k d\xi d\eta
\]
\[
+ \iint \frac{[T]^t}{J} \nabla_{\pi} p' \cdot [T]^{-1} \begin{bmatrix}
  \varphi_\eta \\
  -\varphi_\xi
\end{bmatrix}_k d\xi d\eta.
\] (3.19)

Applying the divergence theorem to the second term on the right-hand side of this expression and utilizing the fact that \([T]^t J^{-1} \nabla_{\pi} p' \cdot ( ) = \nabla_{\pi} p', [T] J^{-1} ( ) \) yields
\[
\iint \nabla_{\pi} p' \cdot \begin{bmatrix}
  \varphi_\eta \\
  -\varphi_\xi
\end{bmatrix}_k d\xi d\eta = -\iint p' \nabla_{\pi} \cdot \begin{bmatrix}
  \varphi_\eta \\
  -\varphi_\xi
\end{bmatrix}_k d\xi d\eta + \oint p' \begin{bmatrix}
  \varphi_\eta \\
  -\varphi_\xi
\end{bmatrix}_k \cdot \hat{n} ds
\] (3.20)

where \(\hat{n}\) is the outward oriented unit vector normal to the inflow and outflow computational domain boundaries whose unit length is \(ds\). At this instant, it is important to recognize that the first term on the right-hand side of (3.20) is zero because \(\nabla_{\pi} \cdot (\varphi_\eta, -\varphi_\xi) = 0\). The boundary integral is zero as well because in the view of (3.2) the pressure distribution along the inflow and outflow boundaries is the same; hence, the value of the integral along the two boundaries is the same but of the opposite sign. With these results, (3.19) becomes
\[
\iint \sum \alpha_i \frac{[T]^{-1}}{J} \begin{bmatrix}
  \varphi_\eta \\
  -\varphi_\xi
\end{bmatrix}_i \cdot [T]^{-1} \begin{bmatrix}
  \varphi_\eta \\
  -\varphi_\xi
\end{bmatrix}_k d\xi d\eta = \iint \mathbf{V}^B \cdot [T]^{-1} \begin{bmatrix}
  \varphi_\eta \\
  -\varphi_\xi
\end{bmatrix}_k d\xi d\eta.
\] (3.21)
Lastly, in order to complete the formulation, we need to define $\varphi$. Following Bell and co-workers [3, 4, 5], we use

$$\varphi_i = \begin{cases} 2 & \text{at the grid cell corner } i, \\ 0 & \text{elsewhere,} \end{cases} \quad (3.22a)$$

$$\varphi_k = \begin{cases} 2 & \text{at the grid cell corner } k, \\ 0 & \text{elsewhere.} \end{cases} \quad (3.22b)$$

Note that $\varphi$ is therefore defined at grid points denoted by circles in Fig. 3-1(b). Gradients of $\varphi$, on the other hand, are computed at cell centers with the second-order accuracy as demonstrated in detail by Skopovi [27] and Lamb [19] and are non-zero only at crosses surrounding the corner $k$.

The expression (3.21) represents a single equation for each cell corner that is not located on the domain boundaries. In other words, due to the fact that values of $\alpha$ on cell corners along the domain boundaries are known, indices $i$ and $k$ range over all grid points designated with circles in Fig. 3-1(b) that do not lie on the domain boundary. As indicated by Lamb [19] and then Skopovi [27], along the bottom and top impermeable boundaries,

$$\alpha(0, 0) = \alpha(1, 0) = \ldots = \alpha(I, 0), \quad (3.23)$$

$$\alpha(0, J) = \alpha(1, J) = \ldots = \alpha(I, J). \quad (3.24)$$

Along the lateral boundaries, however,

$$\alpha(0, \eta-1) = \alpha(0, \eta) - \frac{1}{2} z_\eta \left(0, \eta - \frac{1}{2}\right) u_t \left(0, \eta - \frac{1}{2}\right), \quad (3.25)$$

$$\alpha(I, \eta-1) = \alpha(I, \eta) - \frac{1}{2} z_\eta \left(I, \eta - \frac{1}{2}\right) u_t \left(I, \eta - \frac{1}{2}\right). \quad (3.26)$$

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3.4 Open Boundary Conditions

We begin the discussion of open boundary conditions by considering the linearized form of one-dimensional advection equation

\[ \zeta_t + c \zeta_x = 0 \]  

where \( \zeta(x,t) \) is to be determined throughout the domain \( 0 \leq x \leq L \). Initially, we presume that \( \zeta_0(x,0) \) is given and that the constant \( c \) is positive. With these assumptions, we are interested in determining how boundary conditions at \( x = 0 \) and \( x = L \) need to be specified in order for the problem under consideration to be well-posed.

It can be easily verified via substitution that \( \zeta(\sigma) \) where \( \sigma = x - ct \) is the solution of (3.27). This implies that, \( \zeta \) is constant along characteristics \( \sigma = x - ct \). The initial condition \( \zeta_0 \) therefore uniquely defines \( \zeta \) in the lower triangular strip in Fig. 3-2. The solution in the remaining portion of the domain is determined by \( \zeta(0,t) \). Consequently, in order to determine the solution of (3.27), in addition to initial conditions, a boundary condition must be imposed at \( x = 0 \); henceforth, no boundary condition is needed at \( x = L \). For \( c < 0 \), on the other hand, solutions of (3.27) are left-propagating waves of the form \( \sigma = x + ct \) and the boundary condition must be imposed at \( x = L \); i.e., specifying the boundary condition at \( x = 0 \) causes the problem to be overdetermined.

We these results in mind, we attempt to formulate the outflow lateral boundary condition for the two-dimensional flow of an inviscid, incompressible Boussinesq fluid. In that regard, we begin by foremost assuming that amplitudes of disturbances reaching the right boundary of the physical domain are small enough so that nonlinear terms in (2.13) can be neglected. With \( \epsilon \to 0 \) and \( U = 1 \), the hydrostatic \( (\mu \to 0) \) form of (2.13) becomes

\[ \psi_{zzt} + \psi_{xxx} = r_x, \]  

\[ r_t + r_x = -N^2 \psi_x. \]  

At this point, we define the Fourier transform of \( \psi \) and its inverse \( (\tilde{\psi}) \) with respect
Figure 3-2: Characteristics of the linear one-dimensional advection equation

to $z$ as

$$\mathcal{F}\{\psi\} = \int_{-\infty}^{+\infty} \psi e^{-imz} dz \quad \text{and} \quad \mathcal{F}^{-1}\{\tilde{\psi}\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{\psi} e^{imz} dm.$$ (3.29)

Applying (3.29) to (3.28) with $N=1$ gives

$$-m^2 \tilde{\psi}_t - m^2 \tilde{\psi}_x = \tilde{r}_x, \quad \text{(3.30a)}$$

$$\tilde{r}_t + \tilde{r}_x = -\tilde{\psi}_x \quad \text{(3.30b)}$$

where $\tilde{r}$ is the Fourier transform (3.29) of $r$ with respect to $z$. In the matrix-vector form, (3.30) may be expressed as

$$\begin{pmatrix} m^2 \tilde{\psi} \\ \tilde{r} \end{pmatrix}_t + \begin{pmatrix} 1 & 1 \\ m^{-2} & 1 \end{pmatrix} \begin{pmatrix} m^2 \tilde{\psi} \\ \tilde{r} \end{pmatrix}_x = 0. \quad (3.31)$$

We decouple this system of equations by letting $\varphi = m^2 \tilde{\psi} - m\tilde{r}$ and $\zeta = m^2 \tilde{\psi} + m\tilde{r}$.
Substituting these expressions into (3.31) yields
\[
\begin{pmatrix}
\varphi \\
\zeta
\end{pmatrix}_t + \begin{pmatrix}
1 - \frac{1}{m} & 0 \\
0 & 1 + \frac{1}{m}
\end{pmatrix}
\begin{pmatrix}
\varphi \\
\zeta
\end{pmatrix}_x = 0.
\] (3.32)

The solution of each of the two equations embodied in (3.32) is a right- or left-propagating wave depending on the value of \( m \). The equation for \( \varphi \), supports right-propagating waves for all values of \( m \) except \( m \in [0, 1] \). In a similar fashion, the equation for \( \zeta \) supports right-propagating waves for all values of \( m \) except \( m \in [-1, 0] \). The boundary condition at the right domain boundary is constructed by noting that the topography is the only source of gravity-wave disturbances. As a result, \( \varphi \) must be set to zero for \( m \in [0, 1] \) and \( \zeta = 0 \) for \( m \in [-1, 0] \). This leads to the outflow boundary condition
\[
\ddot{\psi}_t + \left( 1 + \frac{1}{|m|} \right) \ddot{\psi}_x = 0 \quad \text{when} \quad -1 \leq m < 0 \quad \text{and} \quad 0 < m \leq 1,
\] (3.33)
which is not uniformly valid for all \( m \).

### 3.5 Viscous Layers

Due to the fact that open boundary conditions for the Euler equations cannot be analytically posed, in our numerical model, we imitate them by surrounding the inviscid region of interest with three layers of varying viscosity, namely, two lateral layers and the upper absorber. They are schematically represented by gray areas in Fig. 3-3. In the upper layer, we use the viscosity profile
\[
\nu = 0.1 \mu^{-1} \sin \left( \frac{z - z_0 \pi}{Z - z_0} \right),
\] (3.34)
so \( \nu \) increases gradually from zero at the lower edge \( (z = z_0) \) to \( \nu = 0.1 \mu^{-1} \) at the upper edge \( (z = Z) \) of the viscous layer. The upper edge coincides with the top boundary of the computational domain. In side layers, to ensure numerical stability at the inflow
and outflow boundaries, the viscosity is first increased in a way analogous to (3.34), then is held constant at \(0.1\mu^{-1}\) for \(M\) number of grid points and finally is brought back to zero again sinusoidally over the span of \(M\) points.

A typical contour plot of the viscosity profile is presented in Fig. 3-4 from which it is evident that nearly two thirds of the computational domain are occupied by absorbing layers. In order to accommodate the bottom boundary of the computational domain where we enforce the inviscid boundary condition (i.e. zero velocity perpendicular to the topography), viscosity in lateral absorbing layers is linearly increased in the vertical direction from zero at the topography to the layer value over 20 grid points, as indicated by curving contours in the bottom left and right corners of Fig. 3-4.

With this viscosity profile, we solve (3.11) for \(U^*\) on gray cells. Boundary conditions associated with this problem are specified on crosses along lateral boundaries and the top boundary as well as grid points marked with \(\otimes\). Boundary conditions at
Figure 3-4: Contour plot of the viscosity profile for a typical simulation. Viscosity is gradually increased from the inviscid region (white) to the maximum value $\nu = 0.1 \mu^{-1}$ (black).

$\xi = 0$, $\xi = l$ and $\eta = J$ are given by (3.2) while those on $\otimes$ are given by

$$U^* = U^n + \Delta t \left\{ \frac{B}{2} [\mathcal{J}^{-1} \nabla \cdot [\mathcal{J}^{-1} [T][T]' \nabla \times U^n] + b^{n+\frac{1}{2}} \hat{e}_x - \varepsilon r^{n+\frac{1}{2}} \hat{e}_z ight\} - \mathcal{J}^{-1} [T][T]' \nabla \times d^{n-\frac{1}{2}} - \mathcal{J}^{-1} [(\tilde{U} \cdot \nabla \times U)]^{n+\frac{1}{2}}. \right\}$$

(3.35)

### 3.6 Simulation Parameters and Model Testing

In all unsteady simulations presented in this document, we use two values of $\mu$, namely, 0.05 and 1/6. We set $M = 50$ for results with $\mu = 1/6$ and $M = 25$ for runs with $\mu = 0.05$. Moreover, the grid is linearly stretched along both the horizontal and the vertical such that the height of the bottom cells is 15 times smaller than the height of the cells at the upper boundary, while the width of the middle cells is
15 times smaller than the width of the left- and right-boundary cells for simulations where \( \mu = 1/6 \), and 10 times smaller for runs where \( \mu = 0.05 \).

For computations associated with \( \mu = 1/6 \), we use two different sizes of the physical domain, which we designate as large and small. The width of the large domain is \( x \in [-30, 30] \) and its upper boundary is positioned at \( z = 60 \). The domain is discretized with 1200 grid points in the horizontal and 245 points in the vertical direction. In runs corresponding to the large size, lateral viscous layers are placed at \( x = \pm 18 \) while the onset of the upper absorber is \( z = 40 \). Unless otherwise indicated, simulations with \( \mu = 1/6 \) are conducted on the large physical space.

The width of the small domain, on the other hand, is \( x \in [-22, 22] \) and its upper boundary is at \( z = 50 \). This space is discretized with 860 grid points in the horizontal and 205 points in the vertical direction. In all computations associated with this domain size, lateral absorbers are placed at \( x = \pm 10 \) while the upper damper begins at \( z = 28 \).

For \( \mu = 0.05 \), we use the physical domain bounded by \( x \in [-15, 15] \) and \( z \in [\varepsilon h, 60] \). This computational lattice contains 800 grid cells in the horizontal and 245 cells in the vertical direction while viscous layers start at \( x = \pm 9 \) and \( z = 40 \).

Lastly, in regard to the temporal discretization, upon nearly impulsively accelerating the flow from rest via forcing function

\[
b = \begin{cases} 
\frac{1}{0.05} \left[ 1 - \cos \left( \frac{2\pi}{0.05} t \right) \right] & \text{for } t \leq 0.05\mu, \\
0 & \text{for } t > 0.05\mu,
\end{cases} \tag{3.36}
\]

we use the time step \( \Delta t = 0.015\mu \) for runs with \( \mu = 1/6 \) and \( \Delta t = 0.02\mu \) with \( \mu = 0.05 \).

The numerical method of solution was tested in various ways. First we verified that the nearly hydrostatic unsteady response for the uniformly stratified flow over topography approaches, and in the steady state closely resembles, Long’s solution. This comparison is presented in Fig. 3-5, which contrasts the behavior predicted by our model for \( \varepsilon = 0.6 \) and \( \mu = 0.05 \) to that of Long at two different times, namely, \( t = 50 \) and \( t = 95 \). We emphasize that this was a rigorous test of the performance of the upper viscous layer as in the absence of oscillations in the stratification profile.
disturbances are not trapped, so all energy is radiated upwards.

Figure 3-5: Comparison of the behavior predicted by the unsteady model when $\mu = 1/6$ (solid lines) to the solution of Long (dashed lines) for $\epsilon = 0.6$ (a) $t = 50$; (b) $t = 95$. 
Secondly, as a test of lateral absorbing layers, we compared the response on the large domain where viscous layers are at $x = \pm 18$ against that obtained on the small domain where layers are positioned at $x = \pm 10$. Typical results are shown in Fig. 3-6 from which it is apparent that layers are effectively absorbing waves exiting the inviscid region without impacting the flow field within the region of interest. We remark that this comparison is made at $t = 150$, which corresponds to the largest time at which we present our unsteady results. Lastly, we verified that our numerical results do not vary appreciably when the spatial resolution is changed.

Figure 3-6: Comparison of the resonant response for $F = 1$, $\mu = 1/6$, $\phi = 0$ and $t = 150$ when absorbing layers are positioned at $x = \pm 10$ (dashed lines) with the response obtained for the same set of parameters when the inviscid region extends to $x = \pm 18$ (solid lines).
Chapter 4

Variations in Buoyancy Frequency

Following attempts to account for the presence of the tropopause, discussed in Chapter 2, more recent studies of stratified air over topography have concentrated on modeling oscillations with the wavelength of one to two kilometers, which, as illustrated in Fig. 1-1 are present throughout the atmospheric buoyancy frequency and the background velocity. The first insight into the significance of these variations was provided by Phillips [25] in 1968 who studied the collision of two plane gravity waves in an infinitely deep pool of the uniformly stratified fluid. In particular, he considered a nonlinear interaction of two wave-trains with the same horizontal but opposite vertical wave numbers in the basic flow that is a superposition of a homogeneous stream and small sinusoidal vertical oscillations of the wavenumber twice the internal-wave vertical wavenumber. He remarkably demonstrated that for this special flow configuration, the energy during the interaction propagates in the horizontal direction. This horizontal energy propagation is the unanticipated feature of his investigation as it suggests that the energy in infinitely deep stratified flows may be trapped in the vertical direction, prompting the waveguide-like behavior. He also established that the same effect takes place for the uniform free-stream velocity and sinusoidal variations in the buoyancy frequency.

Motivated by findings of Phillips [25], Prasad and Akylas [26] (hereafter referred to as PA) explored implications of the wave-trapping mechanism on stratified flows over an isolated locally confined obstacle. Their work represents the first concrete evidence
of the importance of the trapping mechanism in defining the character of these flows. Correspondingly, we next provide an overview of their theory in the ground based coordinate system, which is in contrast to their original work formulated in the frame of reference traveling with the basic flow.

4.1 Theory of Prasad and Akylas [26]

In their study, Prasad and Akylas [26], as in Long [23] and Davis [8], considered the flow of infinitely deep fluid over a locally confined topographic barrier in the presence of a homogeneous background flow. In contrast to these previous works, however, their Brunt–Väisälä frequency consists of a mean profile with oscillations superposed on it. Denoting by $\lambda$ the typical length scale of the oscillations, this introduces an additional parameter which we take to be

$$F = \frac{U_0 \pi}{\lambda N_0}. \quad (4.1)$$

In Prasad and Akylas [26], the parameter $F$ plays the role of a Froude number as it defines the critical-flow regime in which resonance occurs, in analogy with resonant flow over topography in a channel of finite depth (Grimshaw and Smyth [14], Grimshaw and Yi [13]).

Specifically, the theory of Prasad and Akylas [26] applies to the nearly hydrostatic response ($\mu \ll 1$) in the case that small sinusoidal oscillations of wavelength $\lambda$ are superposed on a uniform mean buoyancy profile:

$$N^2(y) = 1 + \mu^2 \{q_1 \sin(2Fy) + q_2 \cos(2Fy)\}. \quad (4.2)$$

For this choice of $N$, the background flow is nearly uniformly stratified and, according to Long’s model, the steady-state response, to leading order in $\mu$, is a columnar sinusoidal disturbance with wavelength equal to $2\pi$ in the vertical direction; the weak background-flow variations in (4.2), small deviations from the hydrostatic limit, and unsteady-evolution effects merely modulate this columnar disturbance. Accordingly,
the perturbation streamfunction $\psi$, in terms of which $u = 1 + \psi_y, v = -\psi_x$, is posed as

$$\psi = \{A(x, Y, T)e^{iy} + cc\} + O(\mu^2), \quad (4.3)$$

where $cc$ denotes the complex conjugate, and the asymptotic theory furnishes an evolution equation for the complex envelope $A$ of the columnar disturbance as a function of the streamwise coordinate $x$, the 'stretched' vertical coordinate $Y = \mu^2y$ and the 'slow' time $T = \mu^2t$.

From Phillips [25], resonance is expected to occur when the wavelength of the background-flow variations is about one half of the response wavelength, so $F \approx 1$. Setting then

$$F = 1 + \sigma \mu^2, \quad (4.4)$$

$\sigma = O(1)$ being a resonance detuning parameter, it follows that the linear ($\epsilon \ll 1$) hydrostatic response is governed by the evolution equation

$$A_T - iA_xY - \frac{1}{4}e^{2i\sigma Y}QA_x^* = 0, \quad (4.5)$$

where $A^*$ is the complex conjugate of $A$ and $Q = \sqrt{2} - iq_1$; the last term of this equation accounts for the coupling of the induced columnar disturbance with the mean-flow variations. In addition, from (2.16) and (4.3), the linearized boundary condition on the topography is

$$a = -\frac{\epsilon}{2} h(x) \quad (Y = 0), \quad (4.6)$$

where $a$ is the real part of $A$.

The critical Froude number at which resonance occurs can be found by solving equation (4.5) subject to (4.6) for the steady-state linear hydrostatic response. To this end, we write

$$A = \hat{A}e^{i\sigma Y} \quad (4.7)$$

so that, upon substitution into (4.5), $\hat{A}$ satisfies an evolution equation with constant coefficients. The steady version of this equation corresponds to the following equation
system

\[ \sigma \hat{a} + b_Y - \frac{1}{4} q_2 \hat{a} + \frac{1}{4} q_1 \hat{b} = 0, \quad (4.8a) \]

\[ \sigma \hat{b} - \hat{a}_Y + \frac{1}{4} q_1 \hat{a} + \frac{1}{4} q_2 \hat{b} = 0 \quad (4.8b) \]

in terms of the real and imaginary parts of \( \hat{A} = \hat{a} + i \hat{b} \). Moreover, \( \hat{a} \) must satisfy the boundary condition (4.6). The solution is found to be

\[ \hat{a} = -\frac{\epsilon}{2} h(x) e^{-sY}, \quad \hat{b} = \frac{\epsilon}{2} h(x) \frac{q_1 + 4s}{q_2 + 4\sigma} e^{-sY}, \quad (4.9) \]

where

\[ s^2 = -\sigma^2 + \frac{1}{16} \left( q_1^2 + q_2^2 \right). \quad (4.10) \]

From (4.9) and (4.10), it is clear that the steady-state response becomes singular, and hence resonance occurs, when

\[ \sigma = -\frac{1}{4} q_2, \quad q_1 > 0. \quad (4.11) \]

Combined with (4.4), this in turn determines the critical Froude number \( F = F_{\text{crit}} \),

\[ F_{\text{crit}} = 1 - \frac{1}{4} q_2 \mu^2, \quad q_1 > 0 \quad (4.12) \]

at which the flow is resonant according to the linear hydrostatic theory.

The PA theory, in spite of its striking conclusions, is valid for a rather idealized set of background flow conditions. Particularly, it assumes infinitely many wavelengths of sinusoidal variations in the buoyancy frequency whose amplitude is presumed to be small in comparison to the mean value about which they oscillate. Based on Fig. 1-1, atmospheric Brunt-Väisälä frequency, however, features a limited number of finite amplitude modulations, which are not purely periodic. Accordingly, in the following section, we utilize the unsteady computational model, described in chapter 3, to verify postulates of the PA theory under more realistic environmental conditions. Our goal is to demonstrate the existence of the resonance and provide the numerical evidence.
of the flow sensitivity to changes in properties of the oscillations.

### 4.2 Uniform Mean Brunt–Väisälä Frequency

In all our computations, we have used the algebraic topography profile \((2.17)\) and have taken \(\epsilon = 0.6\). For uniformly stratified \((N=1)\) hydrostatic flow over the topography \((2.17)\), as mentioned in Sec. 2.2, according to Long’s model, nonlinear effects come into play roughly above this value of \(\epsilon\) which, however, is still well below the critical value of \(\epsilon = 0.85\) required for overturning (wave breaking).

Our first choice of flow conditions mimics those considered in Prasad and Akylas [26], namely the upstream buoyancy-frequency profile is given by \((4.2)\) with \(\mu^2 q_1 = 0.25 \cos \phi, \mu^2 q_2 = 0.25 \sin \phi (-\pi < \phi < \pi)\). The sinusoidal oscillations are terminated at \(z = 12\pi\) and above this height the buoyancy frequency is held constant at \(N = 1\); the numerical stratification profile thus comprises 12F oscillation periods, in contrast to the asymptotic theory that assumes infinitely many oscillations. According to \((4.12)\), for this choice of \(\mu^2 q_1\) and \(\mu^2 q_2\), resonance is possible when \(-\pi/2 < \phi < \pi/2\); for any \(\phi\) in this range, the critical Froude number then is given by

\[
F_{\text{crit}} = 1 - \frac{\sin \phi}{16}.
\]

As a first example, we take \(\phi = 0\) and \(F = 1\), corresponding to exact resonant conditions according to \((4.13)\). Fig. 4-1(a) shows streamline patterns for this set of parameters and \(\mu = 1/6\) at \(t = 50\) and \(t = 150\), while Fig. 4-1(b) contrasts the response at \(t = 150\) against the solution we would have obtained at the same time in the absence of buoyancy-frequency oscillations far upstream; by \(t = 150\), the response for constant \(N\) has essentially reached the steady state predicted by Long’s model. It is evident that the variations of the Brunt–Väisälä frequency alter the nature of the flow dramatically: the gravity-wave activity is markedly increased over the topography, the streamline peak amplitudes associated with the resonant response at \(t = 150\) being nearly twice as large as those of uniformly stratified flow. In the resonant
response, moreover, we see the emergence of upstream-wave propagation along with a disturbance of opposite sign forming on the downstream side of the topography. This behavior, which is characteristic of resonant flow in a channel of finite depth (Grimshaw and Smyth [14], Grimshaw and Yi [13]), was also found in Prasad and Akylas [26] and must be attributed to the trapping effect caused by the interaction of the induced disturbance with the background buoyancy-frequency oscillations. In the asymptotic theory, the response was tracked in terms of the scaled time $T = \mu^2 t$, corresponding to significantly larger values of $t$ than those in Fig. 4-1; at such later times, typically, wave breaking occurs and the upstream disturbance evolves into solitary waves or bores.

Fig. 4-2 shows two snapshots of the response for $\mu = 0.05$, $F = 1$ and $\phi = \pi/3$ instead of $\phi = 0$. From (4.12), this change of the phase $\phi$ lowers the value of the critical Froude number to $F_{\text{crit}} = 0.957$ so $F = 1$ is supercritical under the present flow conditions. As a result, the response in Fig. 4-2 is quite different from that shown in Fig. 4-1(a): the streamline peak amplitudes are generally smaller, there is no upstream-wave propagation, and apparently steady state is approached as the transients are swept downstream. Fig. 4-2 also shows, for comparison, the hydrostatic steady-state response corresponding to the present flow conditions, as obtained following the numerical procedure of Davis [8]; this confirms that the unsteady response indeed reaches steady state. On the other hand, by lowering the Froude number to the critical value of $F_{\text{crit}} = 0.957$ given by (4.12), the response again becomes resonant (see Fig. 4-3). (We remark that the smaller value of $\mu = 0.05$, which is closer to the hydrostatic limit, rather than $\mu = 1/6$, was used in Fig. 4-2, only to facilitate comparison against the hydrostatic steady state; the difference between the resonant and non-resonant behavior is controlled by the value of $F$ and is not affected by this choice of $\mu$.)

The results of fully numerical simulations reported above support the conclusions reached by the asymptotic theory of Prasad and Akylas [26]. Superposing sinusoidal variations on a uniform mean buoyancy-frequency profile indeed causes trapping of gravity wave disturbances in the vertical direction, and the flow behaves as if it were
Figure 4-1: Resonant response for $F = 1$, $\phi = 0$ and $\mu = 1/6$ (a) evolution in time: dashed line $t = 50$, solid line $t = 150$; (b) comparison between the resonant response (solid line) and the solution for uniform buoyancy frequency profile (dashed line) at $t = 150$. 
Figure 4-2: Non-resonant response for $F = 1$, $\phi = \pi/3$ and $\mu = 0.05$ at $t = 50$ (dotted line) and $t = 100$ (dashed line). The corresponding hydrostatic steady-state solution (solid line) is also shown for comparison.

Figure 4-3: Time evolution of resonant response for $F = 0.957$, $\phi = \pi/3$ and $\mu = 1/6$: dashed line $t = 50$, solid line $t = 150$. 
in a channel of finite depth. The response turns out to be very sensitive to small changes in the background-flow conditions and, near the critical Froude number, features increased gravity-wave activity and upstream influence. In the following section, we explore the role of this resonance mechanism when the buoyancy profile takes into account the tropopause.

### 4.3 Effects of Tropopause

We shall model the transition from the troposphere to the stratosphere by the dimensionless mean buoyancy-frequency profile

\[
\langle N \rangle = 1.5 + 0.5 \tanh 2(y - d), \quad (4.14)
\]

where \( d \) is a parameter that controls the tropopause height. Consistent with Fig. 1-1(a), the profile (4.14) is such that \( \langle N \rangle \) varies smoothly from 1 in the troposphere \( (y \ll d) \) to 2 in the stratosphere \( (y \gg d) \). Referring to Fig. 1-1, a typical value of the mean buoyancy frequency in the troposphere is \( N_0 = 0.01 \text{sec}^{-1} \) and the wind speed is in the range 5–30 m/sec, so the characteristic length scale \( U_0/N_0 \sim 0.5 - 3 \text{ km} \). Generally, the atmospheric tropopause height varies from 10 to 14 km which, in terms of the dimensionless parameter \( d \), would then correspond to \( d \sim 5 - 25 \).

The steady-state hydrostatic response for a two-layer buoyancy-frequency profile with a finite jump at the interface of the two layers was studied by Durran [9] and later Davis [8] considered the same problem for a continuously varying buoyancy frequency as in (4.14). They both find specific tropopause heights for which the hydrostatic steady-state response exhibits enhanced or reduced gravity-wave activity. For \( \epsilon = 0.6 \), for example, such ‘tuned’ and ‘detuned’ gravity-wave flow fields occur when the tropopause is placed in the vicinity of \( d = 21 \) and \( d = 25 \), respectively. In fact, consistent with Durran [9] and Davis [8], we find multiple ‘tuned’ and ‘detuned’ responses that recur periodically with changing \( d \), the lowest value of \( d \) that corresponds to a ‘tuned’ response being in the neighborhood of \( d = 6 \).
For two-layer buoyancy-frequency profiles with sharp interface, Durran [9] and Davis [8] also noted that their numerical procedures failed to converge to a hydrostatic steady-state solution at certain tropopause heights. While Durran [9] suggested that this anomaly could indicate a nonlinear-resonance phenomenon, Davis [8] attributed the lack of convergence to numerical difficulties.

For the purpose of validating and clarifying the steady-state results of Durran [9] and Davis [8], we have carried out unsteady numerical simulations using the buoyancy-frequency profile (4.14) close to the hydrostatic limit \( (\mu = 0.05) \). We find that the unsteady response generally approaches the steady state predicted by the previous studies; a typical example is shown in Fig. 4-4 for \( d = 3 \). We also investigated the nature of the response for buoyancy profiles with sharp transition at the tropopause for the values of \( d \) at which Durran [9] and Davis [8] experienced difficulties converging to a hydrostatic steady state. Near these special values of \( d \) (\( d = 4.5, 11, 17.3, 23.5, \ldots \)), it turns out that the steady-state response approaches the uniform stream far upstream and downstream much more slowly than Long’s solution, and a sufficiently large computational domain is required along \( x \) in order to achieve convergence (see Fig. 4-5). Also our numerical simulations indicate that the corresponding unsteady response tends to steady state, albeit relatively slowly, and there is no sign of a nonlinear resonance.

Next, we superpose on the mean buoyancy profile (4.14) sinusoidal oscillations similar to those used in (4.2) for the uniform mean profile:

\[
N^2(y) = \left\{ 1.5 + 0.5 \tanh [2(y - d)] \right\}^2 + \mu^2 \left\{ q_1 \sin(2Fy) + q_2 \cos(2Fy) \right\}
\]  
(4.15)

with \( \mu^2q_1 = 0.25 \cos\phi, \mu^2q_2 = 0.25 \sin\phi \) as before. Fig. 4-6 shows the unsteady response for \( \mu = \frac{1}{6}, \phi = 0 \) and \( F = 1 \) at \( t = 50 \) and \( t = 150 \) when the tropopause is placed at \( d = 21 \). Note that, for uniform mean buoyancy frequency equal to 1, this choice of flow parameters would correspond to resonant conditions in the troposphere \( (y \ll d) \) according to (4.13), but for the buoyancy-frequency profile (4.15), which includes the tropopause, strictly the resonance equation (4.13) is no longer valid. Nevertheless,
Figure 4-4: Comparison of the response for tropopause height $d = 3$ and $\mu = 0.05$ at $t = 50$ (dashed line) with the corresponding hydrostatic steady-state solution (solid line). At this time, steady state has essentially been reached below $y = 3$.

Figure 4-5: Contrast between Long’s solution (solid line) and the steady-state hydrostatic response for $d = 23.25$ computed using a domain of width $x = \pm 50$ (dotted line) and $x = \pm 100$ (dashed line).
upon comparing Fig. 4-6 with Fig. 4-1, the salient features of resonant response identified earlier — increased wave amplitudes over the topography and upstream-wave formation — are still present, suggesting that the resonance mechanism persists when the tropopause is taken into account; even when trapping occurs in the troposphere only, resonance is still possible. Finally, Fig. 4-7 shows the response for the same flow parameters as in Fig. 4-6 but with $\phi = \pi/3$. As expected, this response is not resonant and approaches steady state as the non-resonant case shown in Fig. 4-2.

Figure 4-6: Time evolution of resonant response for $F = 1$, $\phi = 0$ and $\mu = 1/6$ in the presence of a tropopause at $d = 2\ell$, dashed line $t=50$, solid line $t=150$. 
Figure 4-7: Response at $t=150$ for the same flow parameters as in Fig. 4-6 but with $\phi=\pi/3$. 
Chapter 5

Temporal Modulations in Free-Stream Velocity

5.1 A Survey of Prior Studies

The vast majority of previously conducted studies on the subject of stratified flows over topography, primarily for the matter of mathematical manageability and numerical expense, assumes a constant background velocity. However, as Fig. 1-1 indicates, a typical atmospheric free stream profoundly varies with height. In addition to exhibiting spatial variations, basic flows also regularly fluctuate with time. In this study, we are interested in the effect of these temporal modulations, especially those with periods ranging from the order of minutes to the order of hours.

On the theoretical side, one of the first comprehensive efforts to account for aforementioned time variations is the work of Bell [6] who in 1975 derived the linear steady-in-the-mean solution for the horizontally incident and sinusoidally oscillating background velocity. He established that the equation governing his model is non-linear with respect to the basic flow. As a result, buoyancy waves in his model were produced not only at the fundamental frequency, but also at all of its harmonics. In the further development, Bannon and Zehnder [2] constructed a model that is more appropriate for atmospheric conditions by adding a steady component to the sinusoidally varying free stream of Bell [6]. In spite of the fact that it is restricted
to a hydrostatic mountain, their steady-in-the-mean solution encompassed Coriolis acceleration and provided perhaps first insight into the importance of temporal modulations in the atmospheric context by indicating that the instantaneous mountain drag may be larger than that exerted by a steady wind. We lastly cite the work of Hines [15] who allowed the steady portion of the background velocity adapted by Bannon and Zehnder [2] to vary with height, but slowly so that WKB approximation can be used to analyze the resulting flow field. Although his analysis is likely the first substantial effort to consider simultaneously shear and unsteadiness in a basic flow, it is limited to a monochromatic mountain profile.

Among computational studies, we point out the attempt of Lott and Teitelbaum [24] who investigated the formation of unsteady linear gravity waves in a free stream that starts from zero and returns to zero after a finite time. Their analysis is limited to a monochromatic topography, but it includes effects of shear. We also acknowledge the work of Eckermann and co-workers [7] who, for the purpose of forecasting mountain waves in the field, recently devised a hybrid methodology. This approach combines linear theory to estimate the near-field response close to the topography with a ray-tracing technique that accounts for gradual variations of the buoyancy frequency and wind speed in the far field.

These earlier attempts to account for temporal changes in the background velocity, nevertheless, possess several limitations. Primarily, they are linear in nature and they neglect short-scale buoyancy frequency variations. In chapters 2 and 4, these mechanisms have been found to play a substantial role in shaping the character of the response with a homogeneous free stream. At this point, therefore, a natural question to ask is how does the flow behave in the absence of these constraints. In attempt to answer this question, we first derive a linear hydrostatic theory for sinusoidally oscillating background velocity, which is valid for any Brunt–Väisälä frequency. The theory and details of its numerical implementation are discussed in the following section. Furthermore, as explained in Sec. 5.3, we extend our nonlinear computational model to handle a time-varying background flow. Equipped with these two tools, in Sec. 5.4, we gain an insight into the significance of the transience and nonlinearity by
examining the wave response due to a gradually increasing background flow from zero to a constant value in the presence of uniform stratification. Following this effort, we turn our attention to 1-2 km modulations in the buoyancy frequency. We explore their importance by considering the transient response to a basic velocity that is the superposition of a uniform stream and harmonic oscillations. Specifically, in Sec. 5.5 we concentrate on frequencies that are of the order of the atmospheric Brunt–Väisälä frequency while Sec. 5.6 analyzes the behavior at periods that are of the order of hours.

5.2 Linear Theory for Monochromatic Variations

We commence the derivation of the linear steady-in-the-mean solution that accommodates any stratification by taking the Fourier transform (2.26) of our governing equations (2.13). In the limit $\epsilon \to 0$, this yields

$$\left( \frac{\partial}{\partial t} + ikU \right)^2 \left( -k^2 \mu^2 \tilde{\psi} + \tilde{\psi}_{zz} \right) - N^2 k^2 \tilde{\psi} = 0. \tag{5.1}$$

We remind the reader that the use of (2.26) is appropriate only if the topography is locally confined so that $\psi \to 0$ as $x \to \pm \infty$. Following Bell [6], (5.1) can be mathematically simplified by introducing the horizontal coordinate ($\xi$) traveling with the time-varying free stream. In terms of $x$ and $U$, $\xi$ is defined as

$$\xi = x - \int_0^t U(\tau) d\tau. \tag{5.2}$$

The relationship between $\tilde{\psi}$ and the Fourier transform with respect to $\xi$, denoted as $\hat{\psi}$, is then

$$\hat{\psi} = \tilde{\psi} e^{ik \int_0^t U(\tau) d\tau}. \tag{5.3}$$

Substituting (5.3) into (5.1) gives the linearized governing equation (2.13) in terms of $\hat{\psi}$ as

$$\hat{\psi}_{zztt} - k^2 \mu^2 \hat{\psi}_{tt} - N^2 k^2 \hat{\psi} = 0. \tag{5.4}$$
We construct the bottom boundary condition by placing the Fourier transform (2.26) of (2.15) into (5.3) to obtain

$$\hat{\psi} = -i\tilde{h} \frac{\partial}{\partial t} \left( e^{ik \int_0^T U(\tau) du} \right) \quad \text{at} \quad z = 0. \tag{5.5}$$

Here, $\tilde{h}$ designates the Fourier transform of the ridge. Following Bannon and Zehnder [2], at this point, we restrict our considerations to the background velocity of the form

$$U = 1 + \Delta \cos(\omega_0 t) \tag{5.6}$$

where both $\Delta$ and $\omega_0$ are constants. With this basic flow, the time derivative in (5.5) may be eliminated by introducing the Bessel functions of the first kind ($J_n$) defined by

$$\exp \left[ \frac{y}{2} \left( q - \frac{1}{q} \right) \right] = \sum_{n=-\infty}^{+\infty} q^n J_n(y) \tag{5.7}$$

where $n$ is an integer. For $\ln(q) = i\phi$, it follows that

$$e^{iy \sin \phi} = \sum_{n=-\infty}^{+\infty} e^{i\phi n} J_n(y). \tag{5.8}$$

Substitution of (5.6) and (5.8) into (5.5) with $y = k\Delta / \omega_0$ and $\phi = \omega_0 t$ formulates the bottom boundary condition as

$$\hat{\psi} = -\tilde{h} \sum_{n=-\infty}^{+\infty} \frac{k + \omega_0 n}{k} J_n \left( \frac{k\Delta}{\omega_0} \right) e^{i(k+\omega_0 n)t} \quad \text{at} \quad z = 0. \tag{5.9}$$

Based on (5.9), we seek the solution of (5.4) in the form

$$\hat{\psi} = \tilde{h} \sum_{n=-\infty}^{+\infty} W J_n \left( \frac{k\Delta}{\omega_0} \right) e^{i(k+\omega_0 n)t} \tag{5.10}$$

where $W$ is the function dependent on $z$, $k$ and $n$. Substitution of (5.10) into (5.4), reveals that $W$ must satisfy the second order ordinary differential equation with
respect to $z$

$$W_{zz} + k^2 \left[ \frac{N^2}{(k + \omega_0 n)^2} - \mu^2 \right] W = 0. \tag{5.11}$$

Before discussing a varying stratification, we consider the case of constant Brunt-Väisälä frequency. In this scenario, the solution of (5.11) can be determined analytically and it takes form

$$W = C_1 e^{imz} + C_2 e^{-imz} \tag{5.12}$$

with $m$ given by

$$m = \left[ \frac{k^2 N^2}{(k + \omega_0 n)^2} - k^2 \mu^2 \right]^\frac{1}{2}. \tag{5.13}$$

In (5.12), $C_1$ and $C_2$ are functions of $k$ and $n$. They are determined from (5.9) and the top boundary condition.

The form of the upper boundary condition for each $k$ and $n$ depends on whether (5.12) is exponential or oscillatory in nature. It is exponential if $N^2/(k + \omega_0 n)^2 - \mu^2 < 0$. In this case, the portion of (5.12) that grows with height is physically unrealistic and it is neglected by setting $C_2 = 0$. The solution of (5.12) then becomes

$$W = Ce^{imz} \quad \text{with} \quad m = i \left| \frac{k}{k + \omega_0 n} \right| \sqrt{\mu^2 (k + \omega_0 n)^2 - N^2} \tag{5.14}$$

with $C = C_1$. If $N^2/(k + \omega_0 n)^2 - \mu^2 > 0$, (5.12) is oscillatory. In this regime, (5.10) indicates that $\psi$ is the superposition of plane waves with the frequency $\omega = -(k + \omega_0 n)$ and the vertical wavenumber that is either a positive or negative quantity (5.13). The proper sign of $m$ for each $k$ and $n$ is determined from the radiation condition, which states that $\partial \omega / \partial m > 0$. In terms of $\omega_0$, $k$ and $n$, this can be expressed as

$$\frac{\partial \omega}{\partial m} = \frac{m(k + \omega_0 n)}{m^2 + \mu^2 k^2} > 0. \tag{5.15}$$

Due to the fact that the denominator of this inequality is always greater than zero, the group velocity is positive and all energy is radiated outwards if the sign of $m$ coincides with that of $k + \omega_0 n$. Consequently, in the oscillatory regime, (5.12) may
be written as

\[ W = C e^{imz} \quad \text{with} \quad m = \frac{|k|}{k + \omega_0 n} \sqrt{N^2 - \mu^2(k + \omega_0 n)^2} \]  

(5.16)

where \( C \) is either \( C_1 \) or \( C_2 \) depending on the sign of \( k + \omega_0 n \). In (5.14) and (5.16), \( C \) is determined from (5.9) to be

\[ C = -\frac{k + \omega_0 n}{k} . \]  

(5.17)

The linear steady-in-the-mean solution for constant \( N \) and \( U \) given by (5.6) may now be written as

\[
\psi = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \overline{\tilde{h}} \sum_{n=-\infty}^{+\infty} W J_n \left( \frac{k \Delta}{\omega_0} \right) e^{i[kx + mz - k\Delta \cos(\omega_0 t + \omega_0 nt)]} dk \]  

(5.18)

where \( W \) is determined from (5.14) and (5.16).

We have verified the validity of this result by foremost confirming that for a uniform basic velocity (\( \Delta = 0 \)) and \( N = 1 \), the hydrostatic limit (\( \mu \to 0 \)) of (5.18) is the linear solution of Long (2.32). This can be easily seen by noting that for \( \Delta = 0 \), \( J_n \) is zero for \( n \neq 0 \) and one for \( n = 0 \). Moreover, (5.12) is always oscillatory with \( m = \text{sign}(k) \). With these simplifications, (5.18) becomes

\[
\psi = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\tilde{h}} e^{i[kx + \text{sign}(k)z]} dk  
\]  

(5.19a)

\[
= -\frac{\cos(z)}{2\pi} \int_{-\infty}^{+\infty} \overline{\tilde{h}} e^{ikx} dk - \frac{\sin(z)}{2\pi} \int_{-\infty}^{+\infty} i \text{sign}(k) \overline{\tilde{h}} e^{ikx} dk  
\]  

(5.19b)

\[
= -h \cos(z) - \mathcal{H}[\overline{\tilde{h}}] \sin(z). \]  

(5.19c)

We have also confirmed that in the absence of the time-invariant component of the free stream (5.6), (5.18) is equivalent to the equation governing the model of Bell [6]. Lastly, we have checked that (5.18) is analogous to the solution of Bannon and Zehnder [2] when the Coriolis acceleration in their model is neglected.

The solution of (5.18) can simply be implemented by using Gaussian quadrature

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to computationally evaluate the integral for each \( n \). In that regard, it is important to note that \( k = 0 \) is an integrable singularity. In fact, using l'Hôpital's rule, one can demonstrate that the limit \( k \to 0 \) of the integrand is finite for all \( n \) regardless of stratification.

The solution of (5.11) for a varying buoyancy frequency must be determined numerically. While the bottom boundary condition is given by (5.9) regardless of the stratification, the upper boundary condition for constant \( N \) can still be used if, following Davis [8], we assume that above some height \( z = z_\infty \), \( N = N_\infty \). For the matter of numerical simplicity, we additionally restrict our considerations to the hydrostatic response \( (\mu \to 0) \). With these two assumptions, the upper boundary condition for constant \( N \) is given by

\[
W = Ce^{ikNz_\infty},
\]  

(5.20)

and its \( z \)-derivative is

\[
W_z = C\frac{ikNz_\infty}{k + \omega_0 n}e^{ikNz_\infty}. 
\]  

(5.21)

We solve (5.11) by first guessing the value of \( C \) in (5.20) and (5.21). From the knowledge of \( W \) and \( W_z \) at \( z = z_\infty \), we then use the fourth order Runge–Kutta method to attain \( W \) at \( z = 0 \). The difference between this computed value, denoted here as \( W_0 \), and that required by the bottom boundary condition (5.9) gives the error associated with our guess, namely,

\[
E_{err}(C) = W_0(C) + \frac{k + \omega_0 n}{k}. 
\]  

(5.22)

We update this guess via Newton–Rhapson scheme. In particular, the new guess is given by

\[
C_{new} = C - \frac{E_{err}(C)}{E_{err}'(C)}. 
\]  

(5.23)

Here, \( E_{err}' \) is the approximation of the error function's derivative at \( C \) obtained from

\[
E_{err}'(C) = \frac{E_{err}(C + \delta) - E_{err}(C)}{\delta} 
\]  

(5.24)
where $\delta$ is a small perturbation to $C$. Once we obtain $W$ for each $k$ and $n$, we evaluate (5.18) using Gaussian quadrature analogous to the case of constant $N$.

In all simulations presented in this document, we use $k \in [-15, 15]$ with the grid spacing of 0.025. Moreover, we choose the range of $n$ in such a way that for all values of $n$ outside this range $J_n(k\Delta/\omega_0) < 10^{-5}$. In all computations involving a varying stratification, we use $\delta = 10^{-5}$ and the initial guess $C = -(k + \omega_0 n)/k$. We iterate the Newton–Rhapson scheme until $|E_{err}| < 10^{-3}$.

The numerical method for varying $N$ was tested in two ways. First, we verified that for $N = 1$ and $\Delta = 0$, the solution coincides with that of Long [23] given by (2.32). Moreover, for $\Delta = 0$ and the buoyancy frequency (2.41) with $c = 2$ and $d = 3$, we confirmed the agreement with the linear ($\epsilon \to 0$) limit of the response of Davis [8] (2.37).

### 5.3 Extension of the Unsteady Numerical Model

The modification that enables the unsteady numerical model to simulate flows with a time dependent free stream is quite simple. In chapter 4, we have computationally implemented the impulsive flow acceleration by adding an artificial body force, $b(t)$, to the right-hand side of the horizontal component of our momentum equation (2.8a). As explained in detail in Sec. 3.2, this term physically denotes the local time rate of change of a spatially uniform and the horizontally incident background velocity. Therefore, accommodating temporal changes in the basic flow, from the numerical standpoint, merely entails modifying this forcing function. In this study, we are concerned with the free stream, $U$, that is the sum of a homogeneous profile and a monochromatic time-varying component of the amplitude $\Delta$ and the frequency $\omega_0$. We computationally achieve this variation via the forcing function

$$b = \begin{cases} \frac{\pi (1 + \Delta)}{2 t_a} \sin \left( \frac{\pi t}{t_a} \right) & \text{if } t \leq t_a, \\ -\Delta \omega_0 \sin[\omega_0 (t - t_a)] & \text{if } t > t_a. \end{cases} \quad (5.25)$$
If \( U(0) = 0 \) then (5.25) yields

\[
U = \begin{cases} 
\frac{1 + \Delta}{2} \left[ 1 - \cos \left( \frac{\pi t}{t_a} \right) \right] & \text{if } t \leq t_a, \\
1 + \Delta \cos \left[ \omega_0 (t - t_a) \right] & \text{if } t > t_a.
\end{cases}
\] (5.26)

The first portion of (5.25), as in Chapter 4, increases \( U \) sinusoidally from zero to some finite value, which in this particular study is \( 1 + \Delta \); thereupon, \( U \) is varied harmonically. The initial region, \( t \leq t_a \), is necessary to ensure the continuity of the forcing function and therefore provide the numerical stability of our scheme.

Unless indicated otherwise, in all simulations in this chapter, we use the same resolution as in modeling flows with \( U = 1 \), described in Sec. 3.6. The computational approach was tested in three ways. We first examined the performance of viscous layers by comparing the behavior for the inviscid domain bounded by \( x = \pm 10 \) and \( z = 50 \) to that obtained with boundaries positioned at \( x = \pm 20 \) and \( z = 60 \). The contrast between responses with \( \epsilon = 0.6, \mu = 1/6, t_a = 0.05\mu \) and \( \omega_0 = 12 \) is presented in Fig. 5-1. Results suggest that even with \( \Delta = 0.75 \), layers at \( x = \pm 10 \) are effectively dissipating disturbances by negligibly influencing streamline patterns within the inviscid region. This comparison is conducted at \( t = 100 \), which is the largest time associated with our simulations involving the varying free stream.

As the second test, we confirmed that the computation in Fig. 5-1 reaches the steady state and holds it for a substantial amount of time. The comparison between responses at \( t = 83 \) and \( t = 100 \) is shown in Fig. 5-2. It is evident from the figure that steady conditions are reached by \( t = 83 \) and that the model is holding this state extremely well.

As the last check, we verified that in the limit \( \epsilon \to 0 \) the flow field with \( \Delta = 0.75 \) and \( \omega_0 = 12 \) approaches the steady-in-the-mean solution predicted by the theory derived in the previous section. The agreement for \( \epsilon = 0.2, \mu = 1/6, \) and \( t_a = 0.05\mu \) is presented in Fig. 5-3(a). With an increase in \( \epsilon \) to 0.6, nonlinearity becomes important and the two responses grow apart as illustrated in 5-3(b).

We end the discussion of unsteady numerics by reporting that our formulation of
Figure 5-1: Comparison of the response for $\epsilon = 0.6$, $\mu = 1/6$, $\Delta = 0.75$, $\omega_0 = 12$, $t_a=0.05\mu$ and $t=100$ when absorbing layers are positioned at $x=\pm10$ (dashed lines) with that obtained for the same set of parameters when the inviscid region extends to $x=\pm18$ (solid lines).

Figure 5-2: Evolution of the flow field for $\epsilon = 0.6$, $\mu = 1/6$, $\Delta = 0.75$, $t_a = 0.05\mu$ and $\omega_0 = 12$: dashed lines $t=83$, solid lines $t=100$. 
Figure 5-3: Comparison of the behavior predicted by the unsteady model at $t = 100$ (solid lines) and that forecasted by the linear steady-in-the-mean theory (dashed lines) for $\mu = 1/6$, $\Delta = 0.75$, $\omega_0 = 12$ and $t_a = 0.05\mu$ (a) $\epsilon = 0.2$; (b) $\epsilon = 0.6$. 
the dissipative layer technique did not work in the absence of the uniform stream, that is, when \( U = \cos(\omega_0 t) \). Under these circumstances, our computations were blowing up. We did not explore reasons behind this malfunction because, as pointed out by Lott and Teitelbaum [24], the case of reversing background flow is not extremely relevant to atmospheric conditions that we are concerned with in this study.

5.4 Gradually Accelerated Background Velocity

We gain an insight into the significance of nonlinearity in flows featuring temporal variations in the free stream through the study of the field produced by a gradually increasing basic velocity according to (5.26) with \( \Delta = 0 \). Here, \( t_a \) then physically denotes the time required to accelerate \( U \) from rest to its terminal value \( U = 1 \). Due to the fact that following the acceleration the background flow is kept constant, we expect the response to approach the steady-state solution for \( U = 1 \), which is different from that of Long given by (2.30) and (2.31) because we conduct our study for the finite value of \( \mu \). With \( \mu \) fixed, the problem under consideration is governed by two parameters, namely, \( \epsilon \) and \( t_a \). As it is customary for a two-variable problem, we conduct the analysis by first holding \( t_a \) constant and varying \( \epsilon \). Subsequently, we fix \( \epsilon \) and allow for changes in \( t_a \). In each case, we contrast the resulting response to that obtained with the same value of \( \epsilon \) and \( t_a = 0.05 \mu \), which simulates the impulsive startup.

Our fully nonlinear simulations indicate that in the limit of the vanishing topography height (\( \epsilon \to 0 \)), there is a negligible difference between the impulsive and the slowly accelerated behavior. This is concretely evident from the comparison of responses for \( \epsilon = 0.2 \) and \( \mu = 1/6 \) presented in Fig. 5-4. The contrast at \( t = 40 \) in Fig. 5-4(a), an instant when the gradually increased free-stream velocity has just reached its terminal value, reveals negligible difference in streamline amplitudes and steepness directly above the barrier. The patterns differ, however, in the lee of the obstacle and this discrepancy is due to transients of the slowly accelerated response, which at this time are still present in the domain of interest. They depart by \( t = 60 \).
(Fig. 5-4(b)) and after this point the two flow fields coincide.

Figure 5-4: Comparison of slowly (solid lines) and impulsively (dashed lines) accelerated streamline patterns for $\epsilon = 0.2$, $\mu = 1/6$ and $t_a = 40$ at (a) $t = 40$; (b) $t = 60$. 
An increase in $\varepsilon$ to 0.6 while keeping $\mu$ and $t_a$ constant, generates a profound difference between the two responses. Particularly, at $t = 40$, as depicted in Fig. 5-5(a), streamline amplitudes of the slowly initiated flow are now nearly twice as large as their rapidly actuated analogues. Although in this simulation wave breaking did not occur, we note that streamlines corresponding to the gradually accelerated field are much steeper and are nearly at the point of overturning. As transients move out, the progressively excited behavior once again approaches its impulsive counterpart. In comparison to the previously discussed case with $\varepsilon = 0.2$, however, the accord is reached at a much later time, $t = 140$, as indicated in Fig. 5-5(b).

The described behavior for the fixed $\varepsilon$ and two different acceleration times, $0.05\mu$ and 40, suggests that for each value of $t_a$ there is a distinct nonlinearity parameter for which wave breaking takes place. Correspondingly, we have further utilized our unsteady numerical model to build the diagram in Fig. 5-6 that illustrates this relationship for $\mu = 1/6$. As anticipated, the chart indicates that as the value of $\varepsilon$ decreases, it takes larger $t_a$ to produce breaking. It is important to point out that we did not pursue computations for values of $\varepsilon$ that are less than 0.6 as their corresponding acceleration times are physically meaningless. This is because they are of the order of a day(s) and, as mentioned in the introductory segment of this chapter, we expect these flows to typically last for several hours or less. We lastly stress that the attained acceleration times are accurate within ±2 of the plotted values. This tolerance is indicated by the error bars surrounding each point on the graph. They were obtained by refining the resolution of the simulation for $\varepsilon = 0.6$ until the breaking time did not significantly change any longer. The computed points are also dependent on the long-wave parameter although for $0 < \mu < 1/6$ this dependency is expected to be mild. This conclusion is based on the fact that the solution of Long wave-breaks for $\varepsilon = 0.85$ while its counterpart with $\mu = 1/6$ breaks at the slightly lower value, $\varepsilon = 0.82$.

The underlying conclusion of this investigation is that temporal variations in the incident flow in combination with nonlinearity may give rise to large amplitude transients, which are unaccounted by the previously constructed models. As a result,
Figure 5-5: Comparison of slowly (solid lines) and impulsively (dashed lines) accelerated streamline patterns for $\epsilon = 0.6$, $\mu = 1/6$ and $t_a = 40$ at (a) $t = 40$; (b) $t = 140$. 

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these earlier efforts underestimate the size of the topography generated atmospheric buoyancy waves.

5.5 High-Frequency Sinusoidal Modulations

We devote the reminder of this chapter to the investigation of a flow field generated when the background velocity, in accordance with (5.6), is a superposition of a uniform stream and monochromatic temporal variations. In discussing these flows, we limit our considerations to oscillation amplitudes that are smaller than the steady component \( \Delta < 1 \), so that the basic velocity does not experience the change in direction. Moreover, in this section, we concentrate on the high-frequency regime \( (\omega_0 \to \infty) \) of these modulations and we postpone the analysis of general frequencies to Sec. 5.6.

In the limit \( \omega_0 \to \infty \), the steady-in-the-mean solution (5.18) for a uniform stratification and \( \mu \to 0 \), as analytically demonstrated by Bannon and Zehnder [2], is
identical to the behavior with the incident wind \( U = 1 \). In other words, the transience in the background velocity does not impact the response; hence, at time instances when the basic flow is unity, (5.18) coincides with the linear solution of Long [23] given by (2.32). Our numerical simulations reveal that with \( \Delta = 0.75 \), the two hydrostatic streamline patterns essentially coincide when \( \omega_0 \) is roughly 200. As the value of \( \mu \) increases, however, the frequency at which the agreement between the behavior with \( \Delta = 0 \) and that with \( \Delta = 0.75 \) takes place rapidly diminishes. In particular, according to our computations, for \( \mu = 1/6 \), the accord is reached when \( \omega_0 = 6 \), which upon the conversion to the dimensional form corresponds to the atmospheric Brunt-Väisälä frequency. We also find that aforementioned conclusions are unaffected by variations in the buoyancy frequency as the steady-in-the-mean solution (5.18) for \( \Delta = 0.75 \) and \( \omega_0 = 6 \) matches the linear (\( \epsilon \to 0 \)) response of Davis [8] (2.37) when the stratification is given by (2.41) with \( c = 2 \) and \( d = 3 \).

The behavior exhibited by the linear response also persists in the nonlinear regime. This is concretely evident from the comparison of the resonant flow field in Fig. 4-1(a) with that obtained for the same stratification profile and the background velocity (5.26) when \( t_a = 0.05 \mu, \Delta = 0.75 \) and \( \omega_0 = 12 \). The plot of the two flow fields at \( t = 100 \), illustrated in Fig. 5-7, reveals that streamline patterns are nearly identical.

We lastly point out that in agreement with conclusions of the previous section, we find that the amplitude of transients produced by temporal variations in the basic velocity increases with the decrease in \( \omega_0 \). This is the quality that dominates the nature of flows at lower frequencies of temporal free-stream oscillations, which we consider next.

### 5.6 General Frequencies

In this section, we discuss the flow field produced when the modulation frequency in (5.6) is of the order of few hours. Similar to section Sec. 5.5, we conduct the analysis by first examining properties of the linear steady-in-the-mean solution (5.18) and then we consider effects of the nonlinearity and transients.
Figure 5-7: Comparison between the response with the uniform (dashed lines) and monochromatically varying (solid lines) basic velocity for $\epsilon = 0.6$, $\mu = 1/6$, $\Delta = 0.75$, $\omega_0 = 12$, $t_a = 0.05\mu$, $F = 1$ and $\phi = 0$ at $t = 100$.

When the frequency of monochromatic modulations in (5.6) is $\omega_0 = 0.35$ and $\mu = 1/6$, the steady-in-the-mean solution (5.18) qualitatively retains properties of the response with the uniform incident velocity. Particularly, in the presence of the resonant buoyancy frequency given by (4.2) with $F = 1$, $q_1 = 0.25$ and $q_2 = 0$, the solution for $\Delta = 0.75$ exhibits the resonant behavior in a sense that the output of our numerical algorithm does not converge to a physically realistic solution. We were unable to verify if the resonance persists at lower values of $\omega_0$ as these simulations necessitate a larger range of $n$, which in turn increases the cost of our simulations.

In addition, our computations indicate that the hydrostatic limit of (5.18) for $\omega_0 = 0.35$ and $\Delta = 0.75$, analogous to models of Davis [8] and Durran [9], can be tuned by adjusting the tropopause height. For the stratification profile (2.41) with $c = 2$ and $d = 3$, we find that in the range $d \in [2, 2\pi]$ the response is maximized when $d = 3\pi/2$ and minimized for $d = 3\pi/4$ (Fig. 5-8). These values are in contrast to the $\omega_0 \to \infty$ limit where the linear flow field is tuned or detuned depending on whether $d$ is an even or an odd multiple of $\pi/2$ respectively.
Figure 5-8: Linear steady-in-the-mean solution for $\epsilon = 0.2$, $\mu = 1/6$, $\Delta = 0.75$ and $\omega_0=0.35$ when the buoyancy frequency is given by (2.41) with $c=2$: $d=3\pi/2$ (solid lines) and $d=3\pi/4$ (dashed lines).

In accordance with our findings in Sec. 5.4, gradual monochromatic temporal free-stream modulations in the presence of nonlinearity produce transients whose amplitude grows with time. This is concretely demonstrated in Fig. 5-9, which compares the response for $\epsilon = 0.6$, $\mu = 1/6$ and $t_a = 0.05\mu$ at two different times, namely, 2.5 and 12.5, when the background flow in Fig. 5-9(a) modulates with the amplitude $\Delta=0.75$ and the frequency $\omega_0=0.63$. The two times are therefore exactly one period of the incident flow oscillations apart. It is evident from Fig. 5-9(b) that streamlines corresponding to $t = 12.5$ are much larger in amplitude and steeper than those associated with $t = 2.5$. This in turn suggests that similar to the resonant buoyancy frequency oscillations in the previous chapter, gradual variations in the basic velocity may produce overturning of density contours well below the critical amplitude of the topography predicted by the model of Long [23]. Unfortunately, we were unable to carry this simulation to the point of wave breaking as transients generated by the gradual basic flow variations eventually become so large that our viscous layers are unable to absorb them even with an increase in the value of $\nu$. 

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Figure 5-9: Transients produced by monochromatic gradual temporal modulations in the free stream (a) background flow characterized by $t_a = 0.05\mu$, $\Delta = 0.75$ and $\omega_0 = 0.63$; (b) evolution of the response for $\epsilon = 0.6$ and $\mu = 1/6$: dashed lines $t = 2.5$, solid lines $t = 12.5$. In this simulation, $\nu = 0.2$. 
Chapter 6

Concluding Remarks

In this work, we have constructed a computational model for forecasting the dynamics of infinitely deep air rising over a two-dimensional and locally constricted topographic obstacle. The most challenging aspect of the numerical work has been the implementation of open boundary conditions, which for the Boussinesq form of Euler equations cannot be analytically formulated. In that regard, we have demonstrated that artificial layers of varying viscosity may be successfully employed to dissipate internal gravity waves emerging and evolving away from the barrier. In the ensuing effort, we have utilized the model to investigate the influence of vertical fluctuations in buoyancy frequency and temporal variations in free-stream velocity on the nature of these flows.

Although derived for an unlimited number of infinitesimal periodic oscillations superposed on a uniform Brunt–Väisälä frequency, our simulations reveal that the asymptotic theory of Prasad and Akylas [26] is reliable under more realistic environmental conditions, namely, a limited number of finite amplitude modulations. Specifically, computations indicate that, in accord with the theory, the presence of fluctuations causes vertical trapping of internal gravity waves, which becomes stronger as the amplitude of variations increases. Moreover, the PA resonant condition (4.12) correctly predicts the occurrence of resonance under these more genuine atmospheric circumstances. Resonant streamline amplitudes, as forecasted by Prasad and Akylas [26], grow with time and steady-state solution is not reached. Furthermore, in
agreement with the theory, resonant flow field features upstream propagating distur-
bances, which disappear once (4.12) is detuned. This in turn suggests that changes in
properties of buoyancy frequency oscillations can profoundly influence the character
of the generated response. Lastly, we find that all aforementioned conclusions persist
in the presence of the tropopause.

A question that naturally arises at this point is that of likelihood that one may
observe the resonance phenomenon in nature. With our model, we have succeeded in
producing the resonant flow field with only three wavelengths of periodic oscillations
whose amplitude is only 11% of the otherwise uniform buoyancy frequency. In nature,
as depicted in Fig. 1-1, modulations are not entirely periodic, which weakens the
gravity-wave trapping and therefore resonant effects. On the other hand, there are
many of them and, on average, they are larger than 11%, which strengthens the
trapping mechanism.

In analyzing the effect of temporal changes in the free stream, we have restricted
our considerations to monochromatic variations. Here, we have found that in the
nonlinear regime, gradual background velocity oscillations generate large transients
and cause breaking well below the critical amplitude of the topography predicted
by models that assume constant basic flow. Governing equations (2.13), however,
are nonlinear in the incident velocity. Accordingly, in future developments it could
be of interest to closely consider a free stream that is a superposition of multiple
monochromatic components as in view of this nonlinearity situations may arise where
the interaction between modes may even further enlarge the response.

Although our dissipative layers adequately absorb internal gravity waves when the
background velocity exhibits monochromatic temporal modulations, this numerical
imitation of open boundary conditions possesses several deficiencies, which need to
be addressed before conducting more realistic future investigations. Particularly, for
the free stream featuring shear in the form of vertical sinusoidal oscillations whose
amplitude is only 8% of their otherwise constant average, the comparison between
the response with layers at \( x = \pm 10 \) and that with sponges at \( x = \pm 20 \) did not yield
satisfactory results. Generated streamline patterns were substantially different in the
inviscid region after a relatively short time, $t = 200$, indicating that lateral viscous zones are unable to damp out gravity waves in the presence of shear. From the computational standpoint, on the other hand, the approach is extremely costly as nearly two thirds of our numerical domain are devoted to absorbers. Moreover, the inclusion of viscous terms in the second-order projection scheme requires solving an additional linear system at each time-step to obtain the auxiliary velocity field within dampers, as explained in Sec. 3.2. This further increases the length of simulations and resources required to conduct them. As a result, with the presently available computational power of a commercially available desktop workstation, this method may also not be suitable for modeling atmospheric flows over a three-dimensional ridge.

The aforementioned weaknesses of our sponge layer technique could be potentially eliminated via use of Perfectly Matched Layers (PML). The PML methodology is based on finding a proper space-time transformation, so that all waves supported by Euler equations have the same sign of phase and group velocities. Due to the fact that the mountain is the only source of buoyancy disturbances, in the transformed coordinates, all waves entering these layers have outgoing group velocity and as such must be dissipated. The technique was recently developed for linear Euler equations with a uniform mean flow by Hu [16] who in the subsequent effort [18] extended it to encompass a nonuniform free-stream velocity. In his survey of absorbing boundary conditions, Hu [17] contrasts benefits of his PML procedure to a number of other existing wave-absorbing formulations. In the shear flow through a channel test case [18], discussed by the same author, Perfectly Matched Layers effectively dissipate waves using only 10 grid points, which is in sharp contrast to our simulations that use 150 points along lateral boundaries. While the PML approach in comparison to our effort is significantly less expensive and is simultaneously capable (at least in theory) of treating any free-stream variation, the method, nevertheless, possesses several disadvantages. Specifically, the space-time transformation introduces new terms to linear Euler equations; consequently, one must ensure that the numerical scheme can solve the resulting equations. Furthermore, when applied to topographic
flows, Perfectly Matched Layers must be positioned sufficiently far away from the mountain where the flow is linear. Nevertheless, the computational cost introduced by these deficiencies is still expected to by far smaller than that associated with our methodology.

In future attempts, the implementation of Perfectly Matched Layers could allow for the study of flows that entail shear. For atmospheric considerations of special interest here are 1–2 km oscillations in the basic velocity (portrayed in Fig. 1-1), which, analogous to those in buoyancy frequency, may give rise to resonance, as prompted by Phillips [25] and later PA [26]. Our preliminary computations suggest that the gravity-wave trapping generated by sinusoidal fluctuations superposed on a homogeneous free stream is much stronger than that arising from the buoyancy frequency modulations. With lateral layers positioned at $x = \pm 20$ and resonance condition (4.12) satisfied, we observe a rapid initial ($t < 75$) streamline amplification even when the amplitude of these fluctuations is only 8% of the otherwise uniform background velocity. We did not include these results into this monograph, however, because we could not reproduce them for longer times with absorbers positioned at $x = \pm 10$. A detailed exploration of the behavior with these background conditions then still remain to be conducted. Ultimately, it could be of interest to study a nonlinear response generated by sinusoidal variation in both Brunt–Väisälä frequency and the incident velocity.

We conclude by mentioning that with an implementation of open boundary conditions that is less computationally expensive than the one employed here, it may be reasonable to examine the flow over a three-dimensional locally confined obstacle. Due to the fact that under these circumstances a portion of the incident air would circulate around the barrier rather than rising over it, we intuitively anticipate conclusions of this investigation to be less pronounced. Concretely, we expect short-scale buoyancy frequency oscillations to be less effective in trapping internal gravity waves. Therefore, at resonant conditions we anticipate that it would take longer for wave breaking to take place. In a similar fashion, given the value of the nonlinearity parameter, we postulate that longer periods would be necessary for temporally
oscillating background flow to produce overturning of density contours.
Appendix A

Numerical Method for Steady-State Computations

We describe a numerical formulation for solving the Helmholtz equation

$$\psi_{zz} + N^2\psi = 0$$  \hspace{1cm} (A.1)

in a two-dimensional space with top and bottom boundary conditions

$$\psi = -h(x) \quad \text{at} \quad z = \epsilon h(x),$$  \hspace{1cm} (A.2)

$$\psi_z = N_{\infty} \mathcal{H}[\psi] \quad \text{at} \quad z = z_{\infty}.$$  \hspace{1cm} (A.3)

Here, $\mathcal{H}$ signifies the Hilbert transform with respect to $x$, namely,

$$\mathcal{H}[\psi(x, z)] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\psi(x', z)}{x' - x} dx'.$$  \hspace{1cm} (A.4)

Physically, as demonstrated in Sec. 2.3, the solution of the problem at hand is the perturbation streamline pattern, $\psi$, for a two-dimensional, steady, hydrostatic flow of an inviscid and incompressible Boussinesq fluid over a localized topography, $h(x)$. While (A.1)–(A.3) assume a uniform free-stream velocity, the fluid can be arbitrarily stratified as the buoyancy frequency, $N$, in (A.1) is a function of $z + \epsilon \psi$ where $\epsilon$ is
a constant that controls nonlinear effects and it is defined in Sec. 2.1. The bottom boundary condition ensures zero fluid velocity in a direction perpendicular to the barrier while (A.3) is the radiation condition.

We seek the solution of (A.1) through (A.4) on a physical domain \( x \in [x_{\min}, x_{\max}] \) and \( z \in [h(x), z_{\infty}] \), the size of which is insignificant as long as lateral boundaries are sufficiently far away from the topography to ensure that the computed \( \psi \) is invariant with a decrease in \( x_{\min} \) and an increase in \( x_{\max} \). This numerical procedure, originally developed by Davis [8], also requires that above \( z_{\infty} \) the Brunt–Väisälä frequency is constant and has value \( N_{\infty} \). From the numerical standpoint, a significant benefit of (A.1) is that it does not involve \( x \)-derivatives of \( \psi \); accordingly, for a fixed value of \( x \) it becomes an ordinary differential equation (ODE) in \( z \). We take advantage of this property and solve the equation by foremost guessing the value of \( \psi(x, z_{\infty}) \). We then take the Hilbert transform of the guess to obtain its \( z \)-derivative, \( \psi_z(x, z_{\infty}) \). From the knowledge of \( \psi \) and \( \psi_z \) at the upper boundary, we utilize fourth-order Runge-Kutta method to solve the second order ODE for each value of \( x \). The difference between the computed solution at \( z = \epsilon h(x) \) and \( -h(x) \) represents the error, \( E(x) \), associated with the initial guess. We update the guess via multidimensional Newton-Rhapson scheme, that is, \( \psi(x_i, z_{\infty})^{n+1} = \psi(x_i, z_{\infty})^n - J^{-1} E(x_i)^n \) where \( n \) is the iteration number, \( J^{-1} \) is the inverse of Jacobian

\[
J = \begin{pmatrix}
\frac{\partial E_1}{\partial \psi_1} & \frac{\partial E_1}{\partial \psi_2} & \cdots & \frac{\partial E_1}{\partial \psi_I} \\
\frac{\partial E_2}{\partial \psi_1} & \frac{\partial E_2}{\partial \psi_2} & \cdots & \frac{\partial E_2}{\partial \psi_I} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial E_I}{\partial \psi_1} & \frac{\partial E_I}{\partial \psi_2} & \cdots & \frac{\partial E_I}{\partial \psi_I}
\end{pmatrix}
\] (A.5)

and \( x_i \) is the value of \( x \) at degreed grid points \( 1, 2, \ldots, I \). The construction of the Jacobian is computationally the most expensive segment of the procedure as it requires a separate previously described error evaluation for each column \( J \). Particularly, the column \( i \) of the Jacobian is obtained by foremost perturbing the element \( i \) of \( \psi(x_i, z_{\infty})^n \) and then calculating the resulting error. The difference between this error and \( E^n \) divided by the magnitude of the perturbation yields the column \( i \) of (A.5).
In our simulations, we perturb each element of $\psi(x_i, z_\infty)^n$ by 0.001 and we iterate the numerical scheme until the magnitude of the error vector, $E^n$, is less than 0.001.

We make the physical domain discrete in such a way that grid points are equidistant in the horizontal direction while their vertical spacing is a function of $x$ and is computed by dividing the difference of $z_\infty$ and $h$ by the total number of grid points in the $z$-direction. Unless otherwise indicated, all presented results are generated with 900 points in the vertical direction, the upper boundary positioned at $z_\infty = 15\pi$, lateral boundaries at $x = \pm 25$, and the horizontal grid-point distance of 0.05. As the initial guess, in each case we use the solution of (A.1)-(A.4) with $\epsilon = 0$, which can be easily obtained numerically as explained by Davis [8].

The method of solution was tested in various ways. Foremost, we confirmed that for the uniformly stratified flow our results agree with those of Lilly and Klemp [22]. Moreover, we ensured that for two-layer buoyancy frequency our flow field coincides with that in Fig. 2.6 of Davis [8]. Lastly, for all of our simulations, we verified that computational domain is large enough so that results do not vary with an increase in its size. In that regard, we observe that when $z_\infty$ is not a multiple of $\pi$, a greater width of the numerical domain is required to ensure invariance of results with respect to decrease in $x_{\min}$ and increase in $x_{\max}$ thus making the problem more computationally expensive. This is evident from the comparison of flow fields obtained with $z_\infty = 15\pi$ and $z_\infty = 45$ (Fig. A-1). For both runs, $N = 1$ and $\epsilon = 0.6$ while horizontal boundaries are located at $x = \pm 25$. The response obtained for $z_\infty = 15\pi$ at this value of the domain width is indistinguishable from the solution in Fig. 2-1, attained using the approach of Lilly and Klemp [22]. With an increase of the domain length, while keeping the horizontal grid spacing at 0.05, $z_\infty = 45$ pattern approaches its $z_\infty = 15\pi$ counterpart as illustrated in Fig. A-2.
Figure A-1: Long’s solution for $\epsilon = 0.6$ and $x \in [-25, 25]$ when $z_\infty = 15\pi$ (solid) and $z_\infty = 45$ (dashed).

Figure A-2: Long’s solution for $\epsilon = 0.6$. Solid line represents the run for $z_\infty = 15\pi$ and $x \in [-25, 25]$ while the dashed line is the result for $z_\infty = 45$ and $x \in [-50, 50]$. 
Bibliography


