

Quantum Physics III (8.06) Spring 2003

Assignment 1

Readings

The reading assignment for the first **three** weeks of 8.06 is:

- Supplementary notes on Natural Units.
- Supplementary notes on Canonical Quantization and Application to a Charged Particle in a Magnetic Field.
- Griffiths Section 10.2.4 is an excellent treatment of the Aharonov-Bohm effect, but ignore the connection to Berry's phase for now. We will come back to this later.
- Quite remarkably, given its length, Cohen-Tannoudji never mentions the Aharonov-Bohm effect. It does have a nice treatment of Landau levels, however, in Complement E-VI.
- Those of you reading Sakurai should read pp. 130-139.

Useful Facts

- 1 atmosphere = 1.01×10^6 dynes/cm³ (cgs)
- $M_{\text{sun}} = 1.99 \times 10^{33}$ grams
- 1 MeV = 1.602×10^{-6} ergs (cgs)

Problem Set 1

1. cgs Units for Mechanical and Electromagnetic Quantities (16 points)

There are several slightly different ways for generalizing cgs units to electromagnetism. The differences revolve around factors of c and 4π . We use units where Gauss' Law reads $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ and where a factor of c is introduced into the Lorentz force law so that \vec{E} and \vec{B} have the same units. This system is known as "Gaussian units". We can just think of it as cgs extended to include electromagnetism.

Find the cgs units — which must take the form $\text{gm}^a\text{cm}^b\text{sec}^c$ — for the following quantities:

- (a) Force
- (b) Energy
- (c) Surface tension
- (d) Pressure
- (e) Momentum density (momentum per unit volume)
- (f) Energy flux (energy per unit area per unit time)
- (g) Electric charge
- (h) Electric current
- (i) Magnetic field
- (j) Magnetic flux
- (k) Voltage
- (l) Conductance
- (m) Conductivity
- (n) Resistance
- (o) Resistivity
- (p) Inductance

You should be able to deduce the units of electromagnetic quantities from the physical laws that define these quantities in the cgs system:¹

$$\begin{aligned} \vec{F}_{12} &= \frac{e_1 e_2}{r_{12}^2} \hat{r}_{12} && \text{Coulomb's Law} \\ \vec{F} &= e\vec{E} + \frac{e}{c} \vec{v} \times \vec{B} \\ V &= \int \vec{dl} \cdot \vec{E} \\ V &= IR && \text{Ohm's law} \\ R &= \rho \times \text{length/area} \\ V &= -\frac{L}{c} \frac{dI}{dt} && \text{Faraday's law} \\ \vec{\nabla} \times \vec{B} &= \frac{4\pi}{c} \vec{j} && \text{Ampere's Law} \end{aligned}$$

¹If you are curious about the factor of $1/c$ that appears in Faraday's law, look at Jackson page 820-821 for further discussion.

2. **Electromagnetic energy density, momentum density and energy flux (4 points)**

If \vec{E} and \vec{B} are electric and magnetic fields, show that:

- (a) \vec{E}^2 and \vec{B}^2 have the units of energy density,
- (b) $\frac{1}{c}\vec{E} \times \vec{B}$ has units of momentum density,
- (c) $c\vec{E} \times \vec{B}$ has units of energy flux.

3. **Natural Units (10 points)**

Verify Equation (12) in the notes on Natural Units, reproduced here:

$$\begin{aligned} [\text{mass}] &= \text{eV}c^{-2} \\ [\text{time}] &= (\text{eV})^{-1}\hbar \\ [\text{length}] &= (\text{eV})^{-1}\hbar c \\ [\text{momentum}] &= \text{eV}c^{-1} \\ [\text{force}] &= (\text{eV})^2\hbar^{-1}c^{-1} \\ [\text{pressure}] &= (\text{eV})^4\hbar^{-3}c^{-3} \\ [\text{charge}^2] &= \hbar c \\ [\text{magnetic field}] &= (\text{eV})^2\hbar^{-3/2}c^{-3/2} \end{aligned} \quad (1)$$

and work out the natural units for magnetic flux and for power.

4. The Bag Pressure (12 points)

The mass, force and energy scales that characterize the physics of quarks are gigantic when expressed in terms of every days units. Here is a problem to show the size of the forces that are at work inside protons and neutrons.

The proton and neutron (“nucleons” for short) are made of three quarks. The mass of the nucleon is approximately 940 MeV. The quarks are confined to the interior of nucleons because it takes work to “open up” a region of space in which they can be present. (Another way of saying this: the presence of the quarks disturbs the vacuum, and thus costs energy.) The work that must be done in order to make a space in which quarks can live is parametrized by a constant known as the “bag constant”, B , which has units of energy per unit volume, and the value approximately $B = 60\text{MeV}/\text{fm}^3$. Here, $1\text{ MeV} = 10^6\text{ eV}$ and $1\text{ fm} = 10^{-13}\text{ cm}$. Thus, the work needed to make a “quark bag” of volume V is BV .

A simple description of this dynamics is obtained by studying the quantum dynamics of fermions (quarks are fermions) under constant pressure. This defines the “bag model” — invented here at MIT by Prof. Jaffe and his collaborators in the mid 1970’s.

In the following, you will want to use the conversion factors given in (9) and (13) in the notes on Natural Units.

- (a) Show that energy density and pressure have the same units. (You can use results from Problem 1.)
- (b) Calculate the pressure equivalent to the bag constant in dynes/cm² and in atmospheres. This is the pressure that quarks must fight against to open up a bag.
- (c) Express B in natural units. The conventional way of expressing B in natural units is $B = (xxx\text{ MeV})^4/(\hbar c)^3$. So, find xxx .
- (d) Lets now see how the value of the B can be estimated.

A massless, relativistic quark has a zero point energy $E_0 = |p_0|c$, where $|p_0|$ is the momentum that the uncertainty principle tells you is the price that must be paid for being confined within a domain of size R . Suppose that $|p_0| = 2.04\hbar/R$ for a quark confined to a sphere of radius R . (Deriving the numerical factor 2.04 requires solving the relativistic analog of the Schrödinger equation for a fermion in a spherical bag. We shall not do this in 8.06.) You can now determine the size of a proton, R_p . R_p is the value of the radius R that minimizes the sum of the zero point energy of three quarks plus the “bag energy” BV . Taking the bag to be a sphere, find the radius of the proton in terms of the bag constant. What is it in fm?

[You've just estimated R_p beginning from a given value of B . A slightly better way to think of this is that we know R_p from experimental measurements, and then use this plus your calculation to estimate the value of B . This is still an oversimplification, however. In reality, the proton does not have a sharp edge, and so R_p cannot be defined literally as we did in this problem, and the bag model cannot be used to describe all of the properties of the proton. Still, a reasonable characterization of much experimental data is that the radius of the proton is about 1 fm, or 1×10^{-13} cm. If I made a reasonable choice for B in posing the problem, the value of R_p that you have obtained should be comparable to 1 fm.]

5. Quantum Gravity (12 points)

When we consider gravitational forces in the cgs system, we introduce a proportionality factor, Newton's constant G_N , defined via

$$\vec{F}_{12}(r) = G_N \frac{m_1 m_2}{r^2} \hat{r} \quad (2)$$

Then G_N measures the gravitational force in dynes between two objects each of mass 1 gm, separated by a distance of 1 cm. Experimental measurement tells us that $G_N = 6.70711(86) \times 10^{-8} \text{ cm}^3/\text{gm sec}^2$.

- (a) Compute the value of G_N in natural units (see the lecture notes).
- (b) Calculate the gravitational interaction energy between two electrons separated by their Compton wavelength. Express the result as a fraction of their rest mass. Give the result in terms of fundamental constants and also numerically. This is a measure of the strength of gravity which can be compared with the strength of electromagnetism (measured by α).
- (c) If you get very close to a very massive pointlike object, relativistic effects modify the predictions of Newtonian gravity and it is necessary to use general relativity. Estimate the length scale of relativistic gravity for a point mass m by combining m with G_N and c to obtain a length. This length is one half the "Schwarzschild radius", r_s , of the mass m . In general relativity if a mass m is concentrated inside its Schwarzschild radius, it forms a black hole. Evaluate the Schwarzschild radius for an object of mass 1 gm. Evaluate it for the sun.
- (d) Quantum effects become important when we look at distances of order the Compton wavelength of a particle. If a particle's Compton wavelength was the same order as its Schwarzschild radius, then it would be necessary to understand quantum gravity to describe physics at those distances. Find the mass of such a particle. What is its Compton wavelength? We have no satisfactory theory of relativistic quantum gravity, so we don't know

how to describe physics at these distance scales. String theory is a possible description of relativistic quantum gravity. You can learn about it in Prof. Zwiebach's course this semester.

6. Landau Levels: a Prelude (6 points)

When we analyze the problem of a charged particle in a magnetic field, we shall find that the energy eigenvalues are separated by a spacing $\hbar\omega_L$, where ω_L (the Larmor or cyclotron frequency) is proportional to the magnetic field B and is given by

$$\omega_L = \frac{eB}{mc},$$

with m the mass of the electron.

Furthermore, we shall find that the length

$$\ell_0 = \sqrt{\frac{\hbar}{m\omega_L}}$$

and the area

$$A_B = 2\pi\ell_0^2$$

play an important role.

- (a) Suppose B is a field of 10 Tesla. (This is a very strong magnetic field but is certainly one which can be created in the laboratory. I am actually not sure what the strongest laboratory magnetic fields achieved to date are, but I believe they are less than 100 Tesla.) In a 10 Tesla magnetic field, what is $\hbar\omega_L$ in eV? What is ℓ_0 in natural units? What is ℓ_0 in cm?
- (b) Lets see whether we can get some sense of how the area A_B may arise in this problem. In a magnetic field of strength B , the flux Φ through an area A perpendicular to B is $\Phi = BA$. What is the flux through the area A_B ? Express your answer in units of hc/e . (Note: hc/e , not $\hbar c/e$.)

Useful facts: 1 Tesla = 10^4 gauss. The gauss is the cgs unit of B . This turns out to mean that if B is 1 gauss, then the force eB is 300 eV/cm. Also, $\hbar c = 197 \times 10^{-7}$ eV cm. And, the mass of the electron is $m = 0.511\text{MeV}/c^2$.