

Problem Set 5 Solutions

1. Let's denote each player's horizontal and vertical actions by A and D respectively, and Player 1's additional action at bottom-right node by E . And let's write the strategies in the form $\{t_{11}, t_{21}, t_{31}; t_{12}, t_{32}\}$ for Player 1, where t_{ij} indicates the action taken by type i at the j -th move, and $\{x\}$ for the strategy of Player 2 who has only one information set and is to move only once.

We can easily see that $t_{12} = D$, $t_{32} = E$, and $t_{31} = A$. Out of several possibilities, only the following case can be an equilibrium - Player 1 with the top type is mixing, and middle type is choosing D .

Suppose that Player 1 (top) is choosing A with probability p , then by Bayes' rule Player 2's beliefs are from top to bottom $(\frac{8p}{8p+1}, 0, \frac{1}{8p+1})$. Since she will be indifferent between A and D , it must be that $p = 1/4$. Also Player 1 (top) will be indifferent when Player 2 is playing A with probability $1/3$. In this case, there is no incentive to deviate. Therefore, PBE is $\{1/4A + 3/4D, D, A; D, E\}$, $\{1/3A + 2/3D\}$ with belief $(2/3, 0, 1/3)$

For other cases (pooling, separating, other hybrid..), you will end up with contradiction, or impossible probabilities. Hence, they cannot be an equilibrium.

2. In the second stage of the game, it is clear that each type will always play the static best response: $a = 2$ type will fight and $a = -1$ type will accommodate. Then for the first stage, with $\pi = 0.9$, the entrant will always enters since

$$EU(Enter) = 2\pi - 1 = 0.8 > EU(out) = 0,$$

and $a = 2$ type fights and $a = -1$ type accommodates.

The PBE is $\{(F, A; F, A), (E, O, E)\}$ - here we are denoting the strategy for the incumbent as $(t_{11}, t_{21}; t_{12}, t_{22})$, where t_{ij} indicates the action taken by type i in the stage j , and for the entrant as (a_1, a_{21}, a_{22}) , where a_1 refers to the first information set at $t = 1$, and the other two refer to the information sets at $t = 2$. Player 2's beliefs are updated such that we have $\mu(a = -1 | A) = 1$ at a_{22} , and $\mu(a = 2 | F) = 1$ at a_{21} .

3. (a) At $t = 1$, the buyer of type 1 (2) will accept if and only if $p_1 \leq 1$ ($p_1 \leq 2$). And the seller will offer $p_1 \geq 0$.

Let's write the seller's belief as $\mu(v = 2 | \text{history at } t = 1)$. Then if $\mu > 1/2$, the seller sets $p_1 = 2$; if $\mu < 1/2$, he sets $p_1 = 1$; if $\mu = 1/2$, he is indifferent between 1 and 2. If the seller observes $p_0 > 1$, then $\mu = 1$ and he will accept $p_0 \geq 0.9 \cdot 2$. And if he observes $p_0 \leq 1$, then $\mu = 0.8$ (the same as his prior) and this offer is rejected since only $p_0 \geq 0.9 \cdot 2 \cdot 0.8$ will be accepted. Then, type 1 (2) would set $p_0 \leq 1$ ($p_0 \geq 0.9 \cdot 2$). In this case, if the seller observes $p_0 \leq 1$, he will set $p_1 = 1$. But then type 2 buyer will not want to set $p_0 \geq 0.9 \cdot 2$ and deviate. Therefore, there is no pure strategy equilibrium.

Now consider mixed strategy equilibrium: type 1 buyer sets $p_0 = 0.9$ and type 2 buyer is mixing $p_0 = 1.8$ with probability $1 - q$ and $p_0 = 0.9$ with probability q , and the seller is accepting with probability p . Then, type 2 buyer is indifferent between 0.2 (when sets $p_0 = 1.8$) and $p(2 - 0.9)$ (when $p_0 = 0.9$) only if $p = 2/11$. Also we must have type 2 buyer is mixing such that the seller is indifferent between accepting and rejecting – that is, $0.9 = 1.8 \cdot \frac{0.8 \cdot q}{0.2 + 0.8 \cdot q}$, or $q = 1/4$.

Hence, in equilibrium - type 1 buyer sets 0.9, and type 2 buyer is mixing 1.8 and 0.9 with probability $3/4$ and $1/4$ at $t = 0$. And the seller accepts if he observes $p_0 = 1.8$, and if he observes $p_0 = 0.9$, he accepts with probability $2/11$ and rejects with probability $9/11$ then sets $p_1 = 2$.

(b) At $t = 1$, both types of buyer will offer $p_1 = 0$, and type 1 (2) will get 1 (2).

At $t = 0$, type 1 (2) buyer will accept offer by the seller if $(1 - p_0) \geq 0.9 \cdot 1$, or $p_0 \leq 0.1$ (if $(2 - p_0) \geq 0.9 \cdot 2$, or $p_0 \leq 0.2$). If the seller offers $0 \leq p_0 \leq 0.1$, then both types of buyer will accept, and his expected payoff is

$$EU_s = (1 - \pi) \cdot p_0 + \pi \cdot p_0 = p_0,$$

or 0.1, which is the optimal value for the seller.

If the seller offers $0.1 < p_0 \leq 0.2$, only type 2 buyer will accept, and his expected payoff is

$$EU_s = (1 - \pi) \cdot 0 + \pi \cdot p_0 = 0.8p_0,$$

or $0.8 \cdot 0.2 = 0.16$, which is the optimal value for the seller.

Hence in equilibrium, the seller sets $p_0 = 0.2$ at $t = 0$, and accepts offer $p_1 = 0$ at $t = 1$.

4. As proposed in the question, suppose there is an equilibrium in which the Sender's strategy is (R, R) , where the first letter is type t_1 's strategy and the second is t_2 's.

Then the Receiver's information set corresponding to R is on the equilibrium path, and her belief at this information set is determined by Bayes' rule and the Sender's strategy - $(0.5, 0.5)$. Note that this is the same as the prior distribution since there is no updating in belief for pooling equilibrium.

Given this belief, the Receiver's best response following R is to play d since

$$EU_R(d | R) = 0.5 \cdot 0 + 0.5 \cdot 2 = 1 > EU_R(u | R) = 0.5 \cdot 1 + 0.5 \cdot 1 = 0.5.$$

Now consider the Receiver's beliefs and best response at information set following L . To determine whether both types of Sender are willing to choose R , we need to specify how the Receiver would react to L . In particular, we need to pin down her beliefs such that $EU_R(u | L) > EU_R(d | L)$ in order for both types of Sender to play R . In order for u to be the best response for the Receiver, her belief at the information set following L should be $q \geq 1/3$ (where q is the probability for t_1). Thus, if there is an equilibrium in which the Sender's strategy is (R, R) , then the Receiver's strategy must be such that he plays d following R , and a following L .

Concluding, pooling Perfect Bayesian Nash equilibrium is

$$\{(R, R), (d, u); p = 0.5, q \geq 1/3\}$$

5. (a) We can consider the following several cases.

* Pooling on R :

- the Receiver's belief on the equilibrium path is $\mu(t_1 | R) = \mu(t_2 | R) = 0.5$
- the Receiver's best response is u since

$$EU_R(u | R) = 0.5 \cdot 2 + 0.5 \cdot 0 = 1 > EU_R(d | R) = 0.5 \cdot 0 + 0.5 \cdot 1 = 0.5$$

- each type of Sender's payoff is $U_{t_1}(R) = 2$ and $U_{t_2}(R) = 1$.

- any play by the Receiver after observing L sustains (R, R) .

Hence PBE in this case are:

$$\begin{aligned} & \{(R, R), (u, u); q > 1/2\}, \\ & \{(R, R), (u, d); q < 1/2\}, \\ & \{(R, R), (u, \alpha u + (1 - \alpha)d); q = 1/2\}. \end{aligned}$$

* Pooling on L :

This cannot be an equilibrium since the Sender (t_2) will always want to play R .

* Separating equilibrium in which the Sender (t_1) plays R and the Sender (t_2) plays L :

- both of the Receiver's information sets are on the equilibrium path and beliefs are $\mu(t_1 | R) = 1$ and $\mu(t_2 | L) = 1$.

- based on this belief, the Receiver's best responses are from left to right (u, d). However, the Sender (t_2) will want to play R . Thus, this case cannot be an equilibrium.

* Separating equilibrium in which the Sender (t_2) plays L and the Sender (t_1) plays R :

- the Receiver's beliefs are $\mu(t_1 | L) = 1$ and $\mu(t_2 | R) = 1$

- the Receiver's best responses are (u, d)

- the Sender's payoff: $U_{t_1}(L) = 1$ and $U_{t_2}(R) = 1$. And there is no incentive to deviate

Therefore, PBE is $\{(L, R), (u, d)\}$ with specified beliefs above.

(b) Again we consider the following four cases.

* Pooling on R :

- the Receiver's belief on the equilibrium path is $\mu(t_1 | R) = \mu(t_2 | R) = 0.5$

- the Receiver's best response is u since

$$EU_R(u | R) = 0.5 \cdot 2 + 0.5 \cdot 0 = 1 > EU_R(d | R) = 0.5 \cdot 0 + 0.5 \cdot 1 = 0.5$$

- each type of Sender's payoff is $U_{t_1}(R) = 0$ and $U_{t_2}(R) = 1$.

However, this case cannot be an equilibrium since the Sender (t_1) will always want to play L .

* Pooling on L :

- the Receiver's belief on the equilibrium path is $\mu(t_1 | R) = \mu(t_2 | R) = 0.5$

- the Receiver's best response is u since

$$EU_R(u | L) = 0.5 \cdot 3 + 0.5 \cdot 0 = 1.5 > EU_R(d | R) = 0.5 \cdot 1 + 0.5 \cdot 1 = 1$$

- each type of Sender's payoff is $U_{t_1}(L) = 3$ and $U_{t_2}(L) = 3$.

- the Receiver's beliefs and the best response at information set following R are (p is the probability for t_1)

$$EU_R(u | R) = 2(1 - p) > EU_R(d | R) = p, \text{ when } p < 2/3$$

Therefore, PBE is $\{(R, L), (d, u); p < 2/3\}$.

* Separating equilibrium in which the Sender (t_1) plays R and the Sender (t_2) plays L :

- in this case, both of the Receiver's information sets are on the equilibrium path and beliefs are $\mu(t_1 | R) = 1$ and $\mu(t_2 | L) = 1$.

- based on this belief, the Receiver's best responses are (d, u) .

- And no deviation is profitable for both types of Sender.

Hence, PBE is $\{(R, L), (d, u)\}$ with beliefs specified above.

* Separating equilibrium in which the Sender (t_2) plays L and the Sender (t_1) plays R :

- the Receiver's beliefs are $\mu(t_1 | L) = 1$, and $\mu(t_2 | R) = 1$

- the Receiver's best responses are (d, u)

- No deviation is profitable for both types of the Sender.

Thus, PBE is $\{(L, R), (d, u)\}$ with beliefs specified above.