Nash Bargaining Solution and Alternating Offer Games

MIT 14.126-Game Theory

Nash Bargaining Model

Formulation

Axioms & Implications

Elements

- The bargaining set: $S$
  - Utility pairs achievable by agreement
  - When? Immediate agreement?
- Disagreement point: $d \in \mathbb{R}^2$
  - Result of infinitely delayed agreement?
  - Payoff during bargaining?
  - Outside option?
- Solution: $f(S,d) \in \mathbb{R}^2$ is the predicted bargaining outcome

Impossibility of Ordinal Theory

- Fix $(S,d)$ as follows
  $$ d = (0,0), S = \{ x \geq 0 \mid x_1 + x_2 \leq 1 \} $$
- Represent payoffs "equivalently" by $(u_1, u_2)$ where
  $$ u_1 = x_1^5, u_2 = 1 - (1 - x_2)^5 $$
- Then, the bargaining set is:
  $$ d' = (0,0), S' = \{ u \geq 0 \mid u_1 + u_2 \leq 1 \} $$
- Ordinal preferences over bargaining outcomes contain too little information to identify a unique solution.
Nash’s Initial Assumptions

- Cardinalization by risk preference
  - Why?
  - What alternatives are there?
- Assume bargaining set $S$ is convex
  - Why?

Nash’s Axioms

- Independence of Irrelevant Alternatives (IIA)
  - If $f(S,d) \in T \subset S$, then $f(T,d) = f(S,d)$
- Independence of positive linear transformations (IPLT)
  - Let $h_i(x_i) = \alpha_i x_i + \beta_i$, where $\alpha_i > 0$, for $i=1,2$.
  - Suppose $a = f(S,d)$. Let $S' = h(S)$ and $d' = h(d)$. Then, $f(S',d') = h(a)$.
- Efficiency
  - $f(S,d)$ is on the Pareto frontier of $S$
- Symmetry
  - Suppose $d' = (d_2,d_1)$ and $x \in S \Leftrightarrow (x_2,x_1) \in S'$. Then, $f_1(S,d) = f_2(S',d')$ and $f_2(S,d) = f_1(S',d')$.

Independence of Irrelevant Alternatives (IIA)

- Statement of the IIA condition
  - If $f(S,d) \in T \subset S$, then $f(T,d) = f(S,d)$
- Definitions.
  - $\text{Vex}(x,y,d) = \text{convex hull of} \{x,y,d\}$.
  - $xP_d y$ means $x = f(\text{Vex}(x,y,d),d)$.
  - $xPy$ means $x = f(\text{Vex}(x,y,0),0)$
- By IIA, these are equivalent
  - $x = f(S,d)$
  - $xP_d y$ for all $y$ in $S$

Efficiency

- Statement of the efficiency condition
  - $f(S,d)$ is on the Pareto frontier of $S$
- Implications
  - The preference relations $P_d$ are “increasing”
Positive Linear Transformations

Statement of the IPLT condition
- Let \( h_i(x_i) = \alpha_i x_i + \beta_i \), where \( \alpha_i > 0 \), for \( i = 1, 2 \).
- Suppose \( a = f(S,d) \). Let \( S' = h(S) \) and \( d' = h(d) \). Then, \( f(S', d') = h(a) \).

Implications
- \( x \text{P}_d y \) if and only if \( (x - d) \text{P}(y - d) \)
- Suppose \( d = 0 \) and \( x_1 x_2 = 1 \).
  - If \( (x_1, x_2) \text{P}(1,1) \) then \( (1,1) \text{P}(1/x_1, 1/x_2) = (x_2, x_1) \)

Symmetry

Statement of the symmetry condition
- Suppose \( d' = (d_2, d_1) \) and \( x \in S \Rightarrow (x_2, x_1) \in S' \). Then, \( f_1(S,d) = f_2(S',d') \) and \( f_2(S,d) = f_1(S',d') \).

Implication
- When \( d = (0,0) \), \( (x_1, x_2) \) is indifferent to \( (x_2, x_1) \).

IPLT + Symmetry imply
- \( x_1 x_2 = 1 \) \( \Rightarrow \) \( x \) is indifferent to \( (1,1) \).
- \( x_1 x_2 = y_1 y_2 \) \( \Rightarrow \) \( x \) is indifferent to \( y \).

A Nash Theorem

Theorem. The unique bargaining solution satisfying the four axioms is given by:
\[
 f(S,d) \in \arg\max_{x \in S} (x_1 - d_1)(x_2 - d_2)
\]

Question: Did we need convexity for this argument?

Alternating Offer Bargaining

Two models
- Both models have two bargainers, feasible set \( S \)
- Multiple rounds: bargainer #1 makes offers at odd rounds, #2 at even rounds
- An offer may be
  - Accepted, ending the game
  - Rejected, leading to another round
- Possible outcomes
  - No agreement is ever reached
  - Agreement is reached at round \( t \)
Model #1: Risk of Breakdown

- After each round with a rejection, there is some probability $p$ that the game ends and players receive payoff pair $d$.
- Best equilibrium outcome for player one when it moves first is a pair $(x_1, x_2)$ on the frontier of $S$.
- Worst equilibrium outcome for player two when it moves first is a pair $(y_1, y_2)$ on the frontier of $S$.
- Relationships:
  \[
  x_2 = (1 - p)y_2 + pd_2 \\
  y_1 = (1 - p)x_1 + pd_1 
  \]

The Magical Nash Product

- Manipulating the equations:
  \[
  x_2 - d_2 = (1 - p)(y_2 - d_2) \\
  (1 - p)(x_1 - d_1) = y_1 - d_1 \\
  (x_1 - d_1)(x_2 - d_2) = (y_1 - d_1)(y_2 - d_2) 
  \]
- Taking $d=(0,0)$, a solution is a 4-tuple $(x_1, y_1, x_2, y_2)$ such that $x_1y_1 = x_2y_2$, as follows:

Main Result

- Theorem. As $p \to 0$, $(x_1, x_2)$ and $(y_1, y_2)$ (functions of $p$) converge to $f(S, d)$.
- Proof. Note: $y_1 = (1-p)x_1$ and $x_2 = (1-p)y_2$ and...

Commentary

- Facts and representations
  - Cardinal utility enters because risk is present
  - The risk is that the disagreement point $d$ may be the outcome.
  - Comparative statics (risk aversion hurts a bargainer) is interpretable in these terms.
Outside Options

- Modify the model so that at any time t, either bargainer can quit and cause the outcome \( z \in S \) to occur.
  - Is \( z \) a suitable threat point?
  - Two cases:
    - If \( z_1 \leq y_1 \) and \( z_2 \leq x_2 \), then the subgame perfect equilibrium outcome is unchanged.
    - Otherwise, efficiency plus
      \[
      x_2 = \max[z_2, (1 - p)y_2]
      \]
      \[
      y_1 = \max[z_1, (1 - p)x_1]
      \]

Model #2: Time Preference

- An outcome consists of an agreement \( x \) and date \( t \).
- Assumptions to model time preference
  - A time indifferent agreement \( n \) exists
  - Impatience: \((x,0)P(n,0)\) and \( t < t' \) imply \((x,t)P(x,t')\)
  - Stationarity: \((x,t)P(x',t')\) implies \((x,t+s)P(x',t'+s)\).
  - Time matters (+continuity): \((x,0)P(y,0)P(n,0)\) implies there is some \( t \) such that \((y,t)I(x,0)\).

- **Theorem.** For all \( \delta \in (0,1) \), there is a function \( u \) such that \((x,t)P(x',t')\) if and only if \( u(x)\delta^t > u(x')\delta^{t'} \). In particular, \( u(n) = 0 \).

Proof Exercise

- **Insight:** Same axioms imply that preferences can be written as:
  \[ v(x) - t \ln(\delta) \]
- **Exercise:** Interpret \( t \) as cash instead of time.
  - State similar axioms about preferences over (agreement, payment) pairs.
  - Use these to prove the quasi-linear representation that there exists a function \( v \) such that \((x,0)\) is preferred to \((y,t)\) if and only if \( v(x) > v(y) + t \).

Representing Time Preference

- **Theorem.** Suppose that \( u \) and \( v \) are positive functions with the property that \( v(x) = [u(x)]^A \) for some \( A > 0 \). Then \( u(x)\delta^t \) and \( v(x)\epsilon^t \) represent the same preferences if and only if \( \epsilon = \frac{\delta^t}{\delta^{t'}} \).

- **Proof.** Exercise.
Comparative Statics

The following changes in preferences are equivalent:
- From $u(x)\delta^t$ to $u(x)\varepsilon^t$
- From $u(x)\delta^t$ to $v(x)\delta^t$, where $v(x) = u(x)^A$ and $A = \ln(\delta)/\ln(\varepsilon)$.

Hence, for fixed $\delta$, greater impatience is associated with “greater concavity” of $u$.

Bargaining with Time Preference

This model is identical in form to the risk preference model, but has a different interpretation.
- Fix $\delta \in (0,1)$ and corresponding utility functions $u_1$ and $u_2$ such that bargainer j’s preferences over outcome $(z,t)$ are represented by $x_j = \delta^t u_j(z)$.
- Best equilibrium outcome for player one when it moves first is a pair $(x_1, x_2)$ on the frontier of $S$.
- Worst equilibrium outcome for player two when it moves first is a pair $(y_1, y_2)$ on the frontier of $S$.
- Relationships:
  \[ x_2 = \delta y_2, y_1 = \delta x_1 \]
  \[ x_1 x_2 = y_1 y_2 \]

General Conclusions

Cardinalization principle
- The proper way to cardinalize preferences depends on the source of bargaining losses that drives players to make a decision.

Outside option principle
- Outside options are not “disagreement points” and affect the outcome only if they are better for at least one party than the planned bargaining outcome.