(Optional Evening) Recitation 6 Solutions

1. Following the usual technique for finding the density of a function of a random variable, we first find the distribution, and then differentiate to find the density.
a) $Y = 4X + 3$.

$$F_Y(y) \quad = \quad P(Y \leq y) = P(X \leq \frac{y-3}{4}) = \frac{y-3}{4}$$

and therefore we have:

$$f_Y(y) = \frac{1}{4}, \text{for } 3 \leq y \leq 7.$$

b) $Y = 4X^2 + 3$

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq \frac{y-3}{4}) = P\left(-\sqrt{\frac{y-3}{4}} \leq X \leq \sqrt{\frac{y-3}{4}}\right) = \sqrt{\frac{y-3}{4}}$$

and thus:

$$f_Y(y) = \frac{1}{8} \cdot \sqrt{\frac{4}{y-3}}, \text{for } 3 \leq y \leq 7.$$

2. Let X and Y be the distances of Xavier's and Yolanda's throws, respectively. The throws are independent, so the joint PDF is

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \begin{cases} \frac{1}{100}\frac{1}{60}e^{-\frac{y}{60}}, & 0 \leq X \leq 100, Y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a)

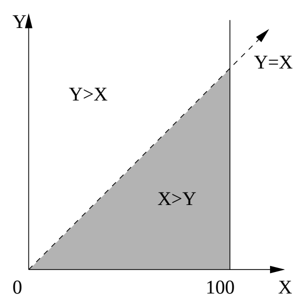
$$P(X = 75) = \int_{75}^{75} \frac{1}{100} dx = 0.$$

(b)

$$P(Y > 100) = \int_{100}^{\infty} \frac{1}{60} e^{-\frac{y}{60}} dy = e^{-\frac{100}{60}} \approx 0.1889.$$

(c) $E[X] = 50, E[Y] = 60$.

(d) We want to find $P(X > Y)$ and compare it to $P(X < Y)$. To do this, we look at the joint sample space:



Thus,

$$P(X > Y) = \int_0^{100} \int_0^x f_{X,Y}(x,y) dy dx = \int_0^{100} \int_0^x \frac{1}{100}\frac{1}{60} e^{-\frac{y}{60}} dy dx \approx 0.5133.$$

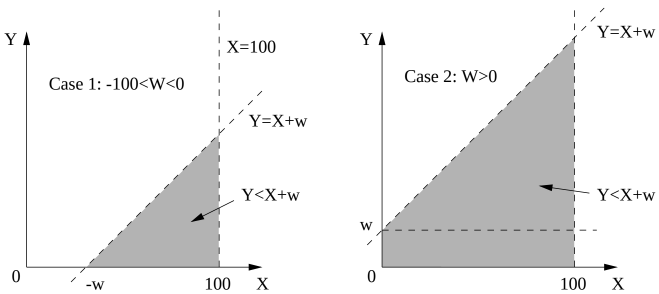
We also have that $P(Y > X) = 1 - P(X > Y) \approx 0.4867$.
Looking just at the expected values of the individual random variables might suggest that Yolanda is more likely to throw further, but we have found that the probability of Xavier throwing further is slightly higher.

(e) $f_{Y|X}(y|75) = f_Y(y)$ because X and Y are independent.

(f) Let $W = Y - X$. First find the CDF of W:

$$F_W(w) = P(W \le w) = P(Y - X \le w) = P(Y \le X + w)$$

The event is shown below.



For $-100 \le W \le 0$, we have

$$\begin{aligned} F_W(w) &= \int_{-w}^{100}\int_0^{x+w} f_{X,Y}(x,y)\,dy\,dx = \int_{-w}^{100}\int_0^{x+w} \frac{1}{100}\frac{1}{60}e^{-\frac{y}{60}}\,dy\,dx \\ &= \frac{1}{100}\left(60e^{-\frac{1}{60}(w+100)} + w + 40\right) \end{aligned}$$

For $W \ge 0$, we have

$$\begin{aligned} F_W(w) &= \int_0^{100}\int_0^{x+w} f_{X,Y}(x,y)\,dx\,dy = \int_0^{100}\int_0^{x+w} \frac{1}{100}\frac{1}{60}e^{-\frac{y}{60}}\,dy\,dx \\ &= \frac{3}{5}e^{-\frac{w}{60}}\left(e^{-\frac{5}{3}} - 1\right) + 1 \end{aligned}$$

Now we differentiate to obtain the PDF:

$$f_W(w) = \frac{d}{dw}F_W(w) = \begin{cases} \frac{1}{100}\left(1 - e^{-\frac{1}{60}(w+100)}\right), & -100 \le w \le 0 \\ \frac{1}{100}e^{-\frac{w}{60}}\left(1 - e^{-\frac{5}{3}}\right), & w \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

3. Let the random variable $S = \sqrt{X^2 + Y^2}$. The values for which $f_S(s)$ is positive correspond to the set of distances from the origin over which $f_{X,Y}(x,y)$ is non-zero. To find the PDF for S, first find the CDF and then differentiate it.

Because the PDF for $f_{X,Y}(x,y)$ is uniform, the CDF of S for any particular value s corresponds to the volume of the cylinder with base area equal to πs^2 and height $\frac{1}{\pi r^2}$. Thus:

$$P(S \le s) = \begin{cases} 0, & s < 0 \\ \frac{\pi s^2}{\pi r^2}, & 0 \le s \le r \\ 1, & s > r \end{cases}$$

Differentiating gives the PDF for S:

$$f_S(s) = \frac{d}{ds}(F_S(s)) = \begin{cases} \frac{2s}{r^2}, & 0 \le s \le r \\ 0, & \text{otherwise} \end{cases}$$

Substituting back in for $S = \sqrt{X^2 + Y^2}$, we get the final answer:

$$f_{g(X,Y)}(g(x,y)) = \begin{cases} \frac{2\sqrt{x^2+y^2}}{r^2}, & 0 \le \sqrt{x^2+y^2} \le r \\ 0, & \text{otherwise} \end{cases}$$