Problem Set 4

1. Random variables X and Y have the joint PMF

$$p_{X,Y}(x,y) = \begin{cases} cxy & , \quad x = 1,2,4; y = 1,3 \\ 0 & , \quad \text{otherwise} \end{cases} \quad .$$

   (a) What is the value of the constant c?
   (b) What is $\mathbf{P}(Y < X)$?
   (c) What is $\mathbf{P}(Y > X)$?
   (d) What is $\mathbf{P}(Y = X)$?
   (e) What is $\mathbf{P}(Y = 3)$?
   (f) Find the marginal PMFs $p_X(x)$ and $p_Y(y)$?
   (g) Find the expectations $\mathbf{E}[X]$ and $\mathbf{E}[Y]$?
   (h) Find the variances $\text{var}(X)$ and $\text{var}(Y)$?

2. Let X be a random variable having expectation μ and variance σ^2, where both μ and σ are known constants. Find the expectation and variance of random variable Y defined as

$$Y = \frac{X - \mu}{\sigma} \quad .$$

3. Suppose $p_{X,Y}(x,y)$, the joint PMF in random variables X and Y, is given by

$$p_{X,Y}(x,y) = \begin{cases} \frac{1}{8} & , \quad (x,y) \in \{(0,1),(0,3),(2,1),(2,2),(4,0),(4,1),(4,2),(4,3)\} \\ 0 & , \quad \text{otherwise} \end{cases} \quad .$$

   Note that it may be helpful to first make a 3-D sketch of this joint PMF.

   (a) Which experimental value(s) x of random variable X maximize(s) $\mathbf{E}[Y|X = x]$, the conditional expectation of random variable Y?
   (b) Which experimental value(s) y of random variable Y maximize(s) $\text{var}(X|Y = y)$, the conditional variance of random variable X?
   (c) Let A denote the event $X^2 \geq Y$. Determine numerical values for the quantities $\mathbf{E}[XY]$ and $\mathbf{E}[XY|A]$.

4. At Tony's pizza, the following four toppings are available: (1) mushroom, (2) sausage, (3) pepperoni and (4) onion. A random pizza has topping i with probability $p_i = 2^{-i}$ independent of whether that pizza has any other topping and each pizza is ordered independently of every other pizza. On a day in which the number of pizzas sold is n, let N_i equal the number of pizzas sold with topping i. What is the joint PMF $p_{N_1,N_2,N_3,N_4}(n_1,n_2,n_3,n_4)$?

5. In this problem, the results established in part (a) will be helpful to answer parts (b) and (c).

   (a) Suppose a random variable X can have only non-negative integer values.

i. Show that

$$\mathbf{E}[X] = \sum_{x=0}^{\infty} \mathbf{P}(X > x) \quad .$$

ii. Show that

$$\sum_{x=0}^{\infty} x\mathbf{P}(X > x) = \frac{1}{2}\left(\mathbf{E}\left[X^2\right] - \mathbf{E}[X]\right) \quad .$$

(b) An urn contains b blue and r red balls. Balls are removed at random until the first blue ball is drawn.

i. Show that the expected number of balls that are drawn is $(b + r + 1)/(b + 1)$.

ii. Calculate the variance of the number of balls that are drawn given $b = 3$ and $r = 2$.

Hint: You may find the following combinatorial identity useful, which can be proved (though you need not do so) by using the binomial theorem or repeated application of the simple identity $\binom{x}{r-1} + \binom{x}{r} = \binom{x+1}{r}$:

$$\sum_{x=0}^{r} \binom{x+b}{b} = \binom{r+b+1}{b+1} \quad .$$

(c) Assume the balls removed in part (b) are replaced so that the urn again contains b blue balls and r red balls. Now, instead of removing balls until the first blue ball is drawn, consider the experiment where balls are removed at random until all the balls remaining in the urn are of the same color.

i. Find the expected number of balls remaining in the urn.

ii. Calculate the variance of the number of balls remaining in the urn given $b = 3$ and $r = 2$.

G1†. Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a vector of independent and identically distributed random variables, each having a Bernoulli distribution with parameter p. Let $f : [0, 1]^n \to \Re$ be *increasing*, which is to say that $f(\mathbf{x}) \le f(\mathbf{y})$ whenever $x_i \le y_i$ for each i.

(a) Let $e(p) = \mathbf{E}[f(\mathbf{X})]$. Show that $e(p_1) \le e(p_2)$ if $p_1 \le p_2$.

(b) **FKG inequality** (named after C. Fortuin, P. Kasteleyn and J. Ginibre (1971), but due in this form to T.E. Harris (1960)). Let f and g be increasing functions from $[0, 1]^n$ into $\Re$. Show by induction on n that $\text{cov}(f(\mathbf{X}), g(\mathbf{X})) \ge 0$.

G2†. Let X and Y be two discrete random variables. Prove the following:

(a) **Jensen's inequality.** If $f(x)$ is convex, $\mathbf{E}[f(X)] \ge f(\mathbf{E}[X])$.

(b) **Cauchy-Schwarz inequality.** $(\mathbf{E}[XY])^2 \le \mathbf{E}[X^2]\mathbf{E}[Y^2]$.

(c) If X and Y are positive, independent and identically distributed,

$$\mathbf{E}\left[\frac{X}{Y}\right] \ge 1 \quad .$$

(This statement may seem paradoxical: X and Y are identically distributed and therefore, in some sense, of the "same size." But, the result $\mathbf{E}[X/Y] \ge 1$ suggests that X tends to be "larger" than Y; note also that $\mathbf{E}[Y/X] \ge 1$ follows from your proof!)

†Required for 6.431; optional for 6.041