Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2002)

Problem Set 9

1. Consider the manufacture of Grandmother's Fudge Nut Butter Cookies. Grandmother has noted that the number of nuts in a cookie is a random variable with a Poisson mass function and that the average number of nuts per cookie is 2.5.
   (a) What is the probability of having at least two nuts in a randomly selected cookie?
   (b) Determine the variance of the number of nuts per cookie.
   (c) Determine the probability that a box of exactly q cookies contains exactly the expected value of the number of nuts for a box of p cookies. ($q = 1, 2, 3, \ldots\ ;\ p = 1, 2, 3, \ldots$)
   (d) What is the probability that a nut selected at random goes into a cookie containing exactly k nuts?
   (e) Grandmother instructs her inspectors to discard each cookie which contains less than two nuts. Determine the mean and variance of the number of nuts per cookie for the remaining cookies.

2. A discrete-time Markov chain with seven states has the following transition probabilities:

$$p_{ij} = \begin{cases} 0.5 & , \quad (i,j) = (3,2), (3,4), (5,6) \text{ and } (5,7) \\ 1 & , \quad (i,j) = (1,3), (2,1), (4,5), (6,7) \text{ and } (7,5) \\ 0 & , \quad \text{otherwise} \end{cases} .$$

   In the questions below, we let X_k be the state of the Markov process at time k.
   (a) For what values of n is the probability $r_{14}(n) = \mathbf{P}(X_n = 4 \mid X_0 = 1) > 0$?
   (b) What are the set of states $A(i)$ that are accessible from state i, for each $i = 1, 2, \ldots, 7$?
   (c) Identify which states are transient and which states are recurrent. For each recurrent class, state whether it is periodic (and give the period) or aperiodic.
   (d) What is the minimum number of transitions with nonzero probability that must be added so that all seven states form a single recurrent class?

3. A digital mobile phone transmits one packet in every time slot over a wireless connection. With probability p, a packet is received in error, independent of any other packet. To avoid wasting transmitter power when the link quality is poor, the transmitter enters a timeout state whenever five consecutive packets are received in error. During such a timeout, the mobile terminal performs an independent Bernoulli trial with success probability q in every slot. When a success occurs, the mobile terminal starts transmitting in the next slot as though no packets had been in error.
   (a) Construct a discrete-time Markov chain for this system, which includes
      i. defining an appropriate state space and
      ii. drawing the transition probability graph.
   (b) Solve for the steady-state probabilities in terms of parameters p and q.

4. Oscar goes for a run each morning. When he leaves his house for his run, he is equally-likely to go out either the front or the back door; and similarly, when he returns, he is equally likely to go to either the front or back door. Oscar owns only five pairs of running shoes which he takes off immediately after the run at whichever door he happens to be. If there are no shoes at the door from which he leaves to go running, he runs barefooted. We are interested in determining the long-run proportion of time that he runs barefooted.
   (a) Set the scenario up as a Markov chain, specifying the states and transition probabilities.
   (b) Determine the long-run proportion of time Oscar runs barefooted.

5. **Partially-Observable Markov Processes:** In applications where a decision at any time k depends on the true state X_k of a Markov process, it may sometimes be too difficult or too costly to observe the state directly. Thus, the decision-maker must act based on a prediction of the otherwise unknown state. For Markov process models with a finite number m of states, an intuitive representation of such a prediction is called the *probabilistic state*, defined for any k by a PMF over the original state space: $p_{X_k}(i) = \mathbf{P}(X_k = i)$ for $i = 1, 2, \ldots m$.

   (a) Consider the two-state Markov process model with transition probabilities

   $$p_{11} = 1 - \theta, \quad p_{12} = \theta, \quad p_{21} = \phi, \quad \text{and} \quad p_{22} = 1 - \phi \quad .$$

   Find the probabilistic state at time $k = 1000$ (justified approximations are acceptable).

   In a *partially-observable* Markov process, a measurement device is available that generates "noisy" observations of the otherwise unknown state. The dependence of each such observation Z on the true state X_k is defined by a *sensor model*, typically specified as a conditional distribution of Z given X_k. The conditional PMF $\mathbf{P}(X_k = i|Z)$ for $i = 1, 2, \ldots m$ can be viewed as an estimator $g(Z)$ of the probabilistic state based on observation Z; then, for any observed value z of Z, the *probabilistic state estimate* $p_{X_k|Z}(i|z) = \mathbf{P}(X_k = i|Z = z)$ summarizes the available state information in a manner consistent with the realized observation as well as all process model and sensor model parameters.

   (b) Suppose that just after the process defined in part (a) enters state X_{1000}, a single observation $Z = z$ is realized according to the following sensor model

   $$f_{Z|X_{1000}}(z|i) = \mathcal{N}(\mu_i, \sigma_i^2) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left(-\frac{(z-\mu_i)^2}{2\sigma_i^2}\right), \; -\infty < z < \infty \qquad \text{for } i = 1, 2 \quad .$$

   i. Find the probabilistic state estimate at time $k = 1000$. *Hint:* Use Bayes' rule.
   ii. What conditions on parameters $(\mu_1, \sigma_1, \mu_2, \sigma_2)$ of the given sensor model make observation Z *uninformative*, meaning $p_{X_{1000}|Z}(i|z) = p_{X_{1000}}(i)$ for all i and all z?
   iii. Assume sensor model parameters $(\mu_1, \sigma_1, \mu_2, \sigma_2) = (-1, 1, 1, 1)$ and two different instances of the process model parameters: $(\theta, \phi) = (0.5, 0.5)$ and $(\theta, \phi) = (0.1, 0.9)$. Use a computer to plot and clearly label the probability $\mathbf{P}(X_{1000} = 1|Z = z)$ over the domain $[-4, 4]$ for z, plotting the distinct curve associated with either instance of the process model on the same axes for ease of comparison.
   iv. Repeat part (iii) but assuming sensor model parameters $(\mu_1, \sigma_1, \mu_2, \sigma_2) = (0, 1, 0, 2)$.

G1†. The first order interarrival times for a renewal process are either 1.0 or 1.5 hours with probabilities 1/3 and 2/3, respectively. Let T denote the time from an instant of random incidence until the 3rd following arrival. Find $\mathbf{E}[T]$ and $\text{var}(T)$.

G2†. Given an irreducible and aperiodic Markov chain $\{X_n\}$ with steady state probabilities π_i and transition probabilities p_{ij}, consider a second Markov chain whose state at time n is $\{X_{n-1}, X_n\}$.

   (a) Show that the steady-probabilities are $\eta_{ij} = \pi_i p_{ij}$.
   (b) Generalize part (a) to the case of the Markov chain $\{X_{n-k}, X_{n-k+1}, \ldots, X_n\}$.
   (c) Consider an infinite sequence of independent coin tosses, where the probability of a head is p. Each time the coin comes up heads twice in a row you get \$1 (so the sequence $THHHHT$ gets you \$3). What is the (approximate) expected amount of money that you will get per coin toss, averaged over a large number of coin tosses?

†Required for 6.431; optional for 6.041