MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2002)

QUIZ 1 ANNOUNCEMENTS

Quiz 1: (closed-book, with a ___handwritten___ 1-sided 8.5 x 11 formula sheet and calculator permitted)

Date: Wednesday, October 9, 2002
Time: 12:05pm - 12:55pm
Location: 6.431: 34-101
6.041: Top-floor of Walker Memorial (Bldg 50)
Content: All topics discussed in
Lectures 1 through 8
Reading through all of Chapter 2
Recitations 1 through 4
Tutorials 1, 2 and 4
Problem Sets 1 through 4

Marathon Office Hours: The TAs will jointly hold office hours on Tuesday, October 4 from 9am to 5pm (as usual, in Room 24-312 & 24-314). A schedule will be posted on the web by Monday.

Optional Quiz Review Session: There will be two identical, two-hour quiz review sessions administered by the TAs. The session will consist of two parts. In the first hour, a concise overview of the theory will be presented. In the second hour, a set of practice problems will be solved. The quiz review is completely optional, but it is usually a good idea to attend and reinforce your understanding of the material, and perhaps gain some insight you did not have before. (Solving the attached practice quiz *before* you look at the solutions may also be a good idea, as this may indicate areas which need additional study.) Details for the quiz review sessions are:

Date: Tuesday, October 8, 2002
Time: 5-7pm and 7-9pm (identical sessions)
Location: 54-100

The problems to be solved in the quiz review are attached, as is a practice quiz with solutions. We strongly recommend working the quiz review problems before coming to the quiz review.

QUIZ 1 REVIEW PROBLEMS

1. The newest invention of the 6.041/6.431 staff is a three-sided die. On any roll of this die, the outcome x is:

$$p_X(x) = \begin{cases} \frac{1}{2}, & \text{if } x = 1 \\ \frac{1}{4}, & \text{if } x = 2 \\ \frac{1}{4}, & \text{if } x = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Consider a sequence of six independent rolls of this die, and let x_i be the random variable corresponding to the ith roll.

(a) What is the probability that exactly three of the rolls have an outcome equal to 3?

(b) What is the probability that the first roll is 1, given that exactly two of the six rolls had an outcome of 1?

(c) We are told that exactly three of the rolls resulted in 1 and exactly three resulted in 2. Given this information, what is the probability that the sequence 121212 resulted?

(d) Conditioned on the event that at least one roll resulted in 3, find the conditional PMF of the number of 3's.

2. The president of a company discovers that one of her two vice presidents, A and B is embezzling money from the company. In order to determine who is guilty, she decides to hire a private detective to investigate. If she chooses to investigate VP A she will have to pay D_A to the detective, and if A turns out to be guilty, the president will have to pay R_A to replace A. Similarly, investigating B has costs D_B and R_B. Furthermore, if the detective decides that one of the VP's is innocent, the president will have to pay the detective to investigate the other VP. If the a priori probability that A is guilty is p, and that B is guilty is $1-p$, find the conditions on p, D_A, D_B, R_A, R_B for which investigating A first would minimize the expected cost of the procedure.

3. Professor May B. Right often has her science facts wrong, and answers each of her students' questions incorrectly with probability 1/4, independently of other questions. In each lecture May is asked either 1 or 2 questions with equal probability.

(a) What is the probability that May gives wrong answers to all the questions she gets in a given lecture?

(b) Given that May gave wrong answers to all the questions she got in a given lecture, what is the probability that she got two questions?

(c) Let X and Y be the number of questions May gets and the number of questions she answers correctly in a lecture, respectively. What is the mean and variance of X and the mean and the variance of Y?

(d) Give a neatly labeled sketch of the joint PMF $p_{X,Y}(x, y)$.

(e) Let $Z = X + 2Y$. What is the expectation and variance of Z?

For the remaining parts of this problem, assume that May has 20 lectures each semester and each lecture is independent of any other lecture.

(f) The university where May works has a peculiar compensation plan. Each lecture May gets paid a base salary of $1,000 plus $40 for each question she answers and an additional $80 for each of these she answers correctly. In terms of random variable Z, she gets paid $\$1000 + \$40Z$ per lecture. What is the expected value and variance of her *semesterly* salary?

(g) Determined to improve her reputation, May decides to teach an additional 20-lecture class in her specialty (math), where she answers questions incorrectly with probability 1/10 rather than 1/4. What is the expected number of questions that she will answer wrong in a randomly chosen lecture (math or science).

6.041 Spring 2002 Quiz 1
Wednesday 13th of March, 12:05-12:55pm

DO NOT TURN THIS QUIZ OVER UNTIL YOU ARE TOLD TO DO SO

- You have 50 minutes to complete the quiz.
- Write your solutions in the exam booklet. We will not consider any work not in the exam booklet.
- There is a total of 100 possible points for 6.041 students and 113 points for 6.431 students.
- This quiz has 3 problems that are not necessarily in order of difficulty. Problem 2 part (d) is ONLY for students taking the graduate version 6.431. Students taking the undergraduate version, 6.041, do not need to do this part.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of <u>numbers</u> that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(p/2)^k\sqrt{q}$ are also fine.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- This is a closed-book exam except for one single-sided, handwritten, 8.5 by 11 formula sheet plus a calculator.
- Be neat! If we can't read it, we can't grade it.
- At the end of the quiz, turn in your solutions along with this quiz (these 2 pieces of paper).
- Quizzes will be handed back in recitation Monday. Quiz solutions will be handed out in lecture Friday.

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Problem 1: (2 points)
Write your TA's name on the front of the booklet.

Problem 2:
The probability that the MIT bobsled team wins a medal at any given race depends on the mood of the driver on the day of the race. If the driver is in a good mood, the probability of winning a medal is p, if the driver is in a bad mood, the probability of winning a medal is q. Conditioned on the mood of the driver on different days, the outcomes of different races are assumed independent. there was a total of 11 races this season.

(a) (12 points) If the driver was in a good mood for 5 out of the 11 races, find the expectation and variance of the number of medals won.

(b) (12 points) If the driver was in a good mood for each of the 11 races, give a formula for the probability that they won medals in AT LEAST half of the races.

(c) (13 points) Suppose that $p = q$. Given that the driver was in a good mood in 4 out of 11 races, and given that they won 5 medals, what is the probability that they won a medal on each day that the driver was in a good mood?

(d) (13 points) **Do this part, part (d), ONLY if you are taking the graduate version, 6.431.** If the team wins a medal in a particular race, they either get the gold medal with probability $\frac{1}{5}$, the silver medal with probability $\frac{2}{5}$, or the bronze medal with probability $\frac{2}{5}$, independently of other races. Given that the team won 7 medals, what is the probability that they won 3 gold, 3 silver, and 1 bronze?

(e) (13 points) The driver's mood was good for the first race. For subsequent races, the driver's mood was good if and only if they won a medal in the previous race. Given that they won a medal in the third race, what is the probability that they also won a medal in the second race? (Do not assume that $p = q$.)

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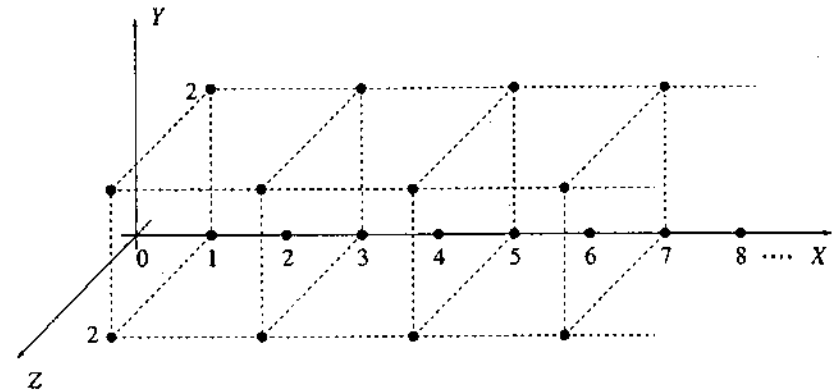
Problem 3:
Consider three random variables X, Y, and Z, associated with the same experiment. The random variable X is geometric with parameter p. If X is even, then Y and Z are equal to zero. If X is odd, Y and Z are uniformly distributed on the set $S = \{(0,0),(0,2),(2,0),(2,2)\}$. The figure below shows all the possible values for the triple (X,Y,Z).



(a) (12 points) Find the joint PMF $p_{X,Y,Z}(x,y,z)$.

(b) Answer with "yes" or "no" (no explanation needed).

(i) (4 points) Are Y and Z independent?

(ii) (4 points) Given that $Y = 2$, are X and Z independent?

(iii) (4 points) Given that $Y = 0$, are X and Z independent?

(c) (12 points) Give a formula that expresses $\mathrm{E}[X \mid Y = 2]$ as a function of p.
(If the formula is in the form of a series, you do not need to evaluate the series.)

(d) (12 points) Find $\mathrm{var}(Y + Z \mid X \text{ is odd})$.