Problem Set 10

1. For each of the following definitions of state X_k at time k ($k = 1, 2, \ldots$), determine whether the Markov property is satisfied and, when it is, specify the transition probabilities p_{ij}:

   (a) A six-sided die is rolled repeatedly.

      i. Let X_k denote the largest number rolled in the first k rolls.

      ii. Let X_k denote the number of sixes in the first k rolls.

      iii. At time k, let X_k be the number of rolls since the most recent six.

   (b) Let Y_k be the state of some discrete-time Markov process at time k (i.e., it is known Y_k satisfies the Markov property) with known transition probabilities q_{ij}.

      i. For a fixed integer $r > 0$, let $X_k = Y_{r+k}$.

      ii. Let $X_k = Y_{2k}$.

      iii. Let $X_k = (Y_k, Y_{k+1})$; that is, the state X_k is defined by the sequence of state *pairs* in a given Markov process.

2. A discrete-time Markov chain is known to be a birth-death process with three states—the probabilities $b_0 = b_1 = d_1 = 0.2$ but d_2 is unspecified. Let X_k denote the state at time $k = 0, 1, \ldots$. For each of the following statements, *either* determine the possible value(s) of probability d_2 *or* explain why no such value exists:

   (a) Upon entering state 2, the expected time it takes to first exit state 2 is equal to 100.

   (b) The probability $\mathbf{P}(X_2 = 2 \mid X_0 = 2)$ is equal to 0.28.

   (c) Given no self-transitions ever occur, the probability $\mathbf{P}(X_4 = 0 \mid X_0 = 0)$ is equal to 0.5.

   (d) The steady-state probability associated with state 1 is equal to 0.25.

   (e) The long-term expected frequency of deaths is equal to 0.3.

   (f) State 2 is an absorbing state.

   (g) The mean recurrence time of state 2 is equal to 6.

3. **Signal-to-Noise Ratio:** If random variable X has mean $\mu \neq 0$ and standard deviation $\sigma > 0$, the ratio $r = |\mu|/\sigma$ is called the measurement *signal-to-noise ratio*, or *SNR*, of X. The idea is that X can be expressed as $X = \mu + (X - \mu)$, with μ representing a deterministic, constant-valued "signal" and $(X - \mu)$ the random, zero-mean "noise." If we define $|(X - \mu)/\mu| = D$ as the relative deviation of X from its mean μ, show that for $\alpha > 0$,

$$\mathbf{P}(D \leq \alpha) \geq 1 - \frac{1}{r^2\alpha^2} \quad .$$

4. **Chernoff Bound:** For any continuous random variable X and an arbitrary constant c, proceed in the steps below to prove that

$$\mathbf{P}(X \geq c) \leq \min_{s \geq 0} \left[e^{-sc} M_X(s) \right]$$

   where $M_X(s)$ denotes the transform of the PDF $f_X(x)$.

(a) The *unit-step function* $u(x)$ is defined by

$$u(x) = \begin{cases} 1 & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases} \quad .$$

For what real values of s will $u(x) \leq e^{sx}$ over all real values of x?

(b) For $u(x)$ as defined in part (a), show that $\quad \mathbf{P}(X \geq c) = \int_{-\infty}^{\infty} u(x-c) f_X(x) dx \quad$.

(c) Combine the answers to parts (a) and (b) to conclude the proof.

5. Let X be a Poisson random variable with mean 20 and let $\quad p = \mathbf{P}(X \geq 26) \quad$.

   (a) Use the Markov inequality to obtain an upper bound on p.

   (b) Use the Chebyshev inequality to obtain an upper bound on p.

   (c) Use the Chernoff bound (see problem 4) to obtain an upper bound on p.

   (d) Approximate p by making use of the central limit theorem.

   (e) Determine the value of p by running a suitable computer program.

G1†. Sam and Pat are playing foosball. When they begin, the score is 0-0. To make things interesting, if the score ever becomes tied, it is instantly reset to 0-0. Starting from any score, the probability that Sam gets the next point is $\frac{1}{3}$.

(a) Suppose the game stops when one player's score reaches 2.

   i. Draw an appropriate Markov chain that describes the game.

   ii. Identify all transient, recurrent, and periodic states.

   iii. Find $\mathbf{P}$(Pat wins).

(b) Now suppose instead that the game stops when a total of 3 points have been scored (note that this stopping condition does not explicitly depend on the score). The player with the most points when the game ends wins. Draw an appropriate Markov chain that describes the game.

G2†. Just as in analysis, in probability theory there are various kinds of convergence of random variables. We have defined in lecture "convergence in probability", and in this exercice we will define "convergence in mean of order p" (In the case $p = 2$, it is called "mean square convergence") as follows:

The sequence of random variables $Y_1, Y_2, \ldots$ of random variables converges in mean of order p ($p > 0$) to the real number a, if

$$\mathbf{E}[|Y_n - a|^p] \xrightarrow[n\to\infty]{} 0$$

(a) Prove that convergence in mean of order p (for any given positive value of p) implies convergence in probability.

(b) Give a counter example that shows that the other implication is not true.

†Required for 6.431; optional for 6.041