Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2002)

Problem Set 8

1. Determine the expected value, variance and the transform of the PMF for the total number of trials from the start of a Bernoulli process (with parameter p) up to and including the nth success after the mth failure.

2. Al performs an experiment comprising a series of independent trials. On each trial, he simultaneously flips a set of three fair coins.

   (a) What is the probability that a trial with 3 heads occurs before a trial with at least 2 tails?

   (b) Given that Al has just had a trial with 3 *tails,* what is the probability that both of the next two trials will also have this result?

   (c) Whenever all three coins land on the same side in any given trial, Al calls the trial a success.

      i. Find the PMF for K, the number of trials up to, but *not* including, the second success.

      ii. Find the expectation of L, the number of successes in the first 100 trials.

      iii. Find the expectation and variance of M, the number of tails that occur *before* the first success.

      iv. Find the PMF for N, the number of trials with 3 heads that occur in 100 trials.

      v. Find the conditional PMF for N given $L = l$.

   (d) Bob conducts an experiment like Al's, except that he uses 4 coins for the first trial, and then he obeys the following rule: Whenever all of the coins land on the same side in a trial, Bob permanently removes one coin from the experiment and continues with the trials. He follows this rule until the *third* time he removes a coin, at which point the experiment ceases.

      i. Find $E(N)$, where N is the number of trials in Bob's experiment.

      ii. Find $M_N(s)$.

3. A store opens at $t = 0$ and *potential* customers arrive in a Poisson manner at an average arrival rate of λ potential customers per hour. As long as the store is open, and independently of all other events, each particular potential customer becomes an *actual* customer with probability p. The store closes as soon as ten actual customers have arrived.

   (a) What is the probability that exactly three of the first five potential customers become actual customers?

   (b) What is the probability that the fifth potential customer to arrive becomes the third actual customer?

   (c) What is the PDF and expected value for L, the duration of the interval from store opening to store closing?

   (d) Given only that exactly three of the first five potential customers became actual customers, what is the conditional expected value of the *total* time the store is open?

(e) Considering only customers arriving between $t = 0$ and the closing of the store, what is the probability that no two *actual* customers arrive within τ time units of each other?

4. A student observes buses at the intersection of road A and road B. The buses each arrive from one of two city bus depots. From bus depot A (B, respectively), buses arrive with exponentially distributed mutually independent interarrival times, each with mean $\frac{1}{\lambda_A}$ ($\frac{1}{\lambda_B}$, respectively). Assume the bus depots have effectively an unlimited supply of buses and that buses along each road cross the intersection without ever colliding. All buses out of depot A (depot B, respectively) travel at the same constant speed v_A (v_B, respectively) toward the intersection. The intersection is ℓ_A distance away from depot A and ℓ_B distance away from depot B.

   (a) If we pick at random a bus arriving at the intersection, what is the probability the bus came from depot A?

   (b) Given that the interarrival time between the third and fourth buses, X_4, is greater than t, what is the PDF *or* the transform describing random variable X_4.

   (c) What is the probability that there is at least one bus on its way to the student (i.e, travelling between one of the depots and the intersection)?

   (d) The students observes three consecutive buses pass by. What is the PDF *or* the transform describing the time duration between the arrival of the first bus and that of the third bus?

   (e) Out of boredom, the student decides to toss a fair coin every time a bus passes by, where the outcome of each toss is independent of all else. What is the PDF *or* the transform describing the time duration between two consecutive "heads" tosses?

5. Shem, a local policeman, drives from intersection to intersection in time X, which is exponentially distributed with parameter λ. At each intersection he observes (and reports) a car accident with probability p. (This activity does not slow his driving at all.) Independently of all else, Shem receives extremely brief radio calls in a Poisson manner with an average rate of μ calls per hour.

   (a) Determine the PMF for N, the number of intersections Shem visits up to and incluing the one where he reports his first accident.

   (b) Determine the PDF for q, the length of time Shem drives between reporting accidents.

   (c) What is the PMF for M, the number of accidents which Shem reports in two hours?

   (d) What is the PMF for K, the number of accidents Shem reports between his receipt of two successive radio calls?

   (e) We observe Shem at a random instant long after his shift has begun. Let W be the total time from Shem's last radio call until his next radio call.

      i. What is the PDF of W?

      ii. What is the transform of the PDF of W?

   (f) Again we observe Shem at a random instant. Determine the transform of the PDF for V, the time from our observation until he receives his first radio call *after* his next accident report?