Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2002)

Tutorial 14 Answers

1. (a) $E[X_n] = \frac{1}{n}$, $var(X_n) = \frac{1}{n} - \frac{1}{n^2}$

   $E[Y_n] = 1$, $var(Y_n) = n - 1$

   (b) The sequence X_n converges in probability to 0. $P(|X_n| \geq \epsilon) = P(X_n = 1) = \frac{1}{n}$ which goes to 0 as $n \to \infty$.

   (c) The sequence Y_n converges in probability to 0. $P(|Y_n| \geq \epsilon) = P(Y_n = n) = \frac{1}{n}$ which goes to 0 as $n \to \infty$.

2. (a) We want to find the probability that there are at least 45 successes out of 50 total trials, where the probability of success is given to be .95. Using the Normal approximation to the binomial (where $\mu = 47.5$ and $\sigma \approx 1.54$), we find:

$$\begin{aligned}\mathbf{P}(45 \text{ to } 50 \text{ successes}) &\approx 1 - \Phi\left(\frac{44.5 - \mu}{\sigma}\right) \\ &\approx 1 - \Phi(-1.95) \\ &= \Phi(1.95) \\ &= 0.9744\end{aligned}$$

   (b) To be able to use the Poisson approximation p has to be small and n has to be relatively large. Therefore, using $p = 0.95$ will not give a good approximation. Instead, we define a new random variable, I, to be the number of incorrect predictions out of 50.

$$\begin{aligned}\mathbf{P}(45 \text{ to } 50 \text{ successes}) &= \mathbf{P}(I=0) + \mathbf{P}(I=1) + \mathbf{P}(I=2) + \mathbf{P}(I=3) + \mathbf{P}(I=4) + \mathbf{P}(I=5) \\ &\approx \sum_{k=0}^{5} \frac{2.5^k e^{-2.5}}{k!} \approx 0.9582\end{aligned}$$

   (c) The second method, although more tedious, is perhaps more appropriate. The Normal approximation works well with sums of symmetric distributions, which for the binomial is satisfied when p is close to .5. Here p is quite far from that. Of course, the Normal distribution makes it quite convenient to calculate, especially when the number of terms in the sum grows.

3. **Practice Problem:** Note that each Z_k is a Poisson random variable with parameter $1/k$.

   (a) Yes, because by the Markov Inequality

$$\mathbf{P}(|Z_k| \geq \epsilon) = P(Z_k \geq \epsilon) \leq \frac{E[Z_k]}{\epsilon} = \frac{1}{k\epsilon}$$

   where the bound goes to zero as $k \to \infty$.

   (b) No, because arrivals still occur as time goes to infinity and so, for any k_0, there is always some $k > k_0$ such that an arrival occurs and therefore some $Z_k > 0$.