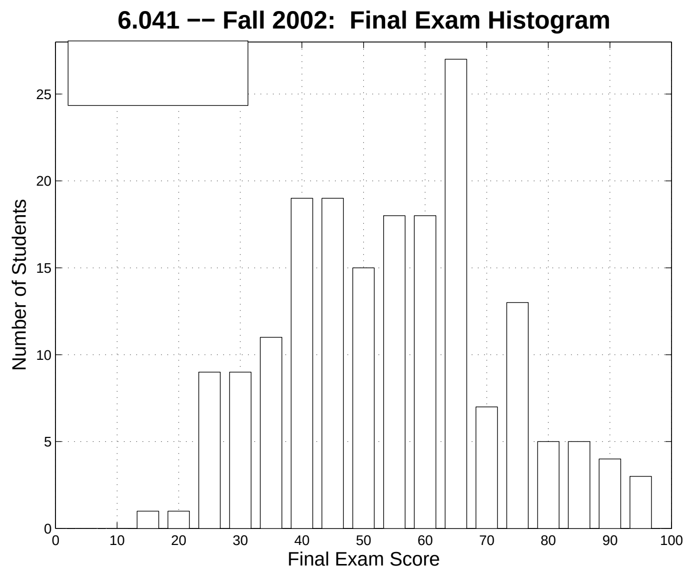
# Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
**6.041/6.431: Probabilistic Systems Analysis**
(Fall 2002)

**Final Exam Results and Solutions**




**Problem 1.** (44 points) The given PDF for $X$ conveniently corresponds to a random choice between two equally-likely standard random variables: event $A$ corresponding to $X$ being exponential and event $B$ corresponding to $X$ being Erlang of order 2, both cases with parameter $\lambda$.

(a) (4 pts) Using the Law of Iterated Expectations,

$$\mathbf{E}[X] = \mathbf{E}[X \mid A]\mathbf{P}(A) + \mathbf{E}[X \mid B]\mathbf{P}(B) = \frac{1}{\lambda}\cdot\frac{1}{2} + \frac{2}{\lambda}\cdot\frac{1}{2} = \frac{3}{2\lambda}$$

(b) (4 pts) We could use the Law of Total Variance, but we also know $\text{var}(X) = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2$ where we've already found $\mathbf{E}[X]$ above. Again using the Law of Iterated Expectations,

$$\mathbf{E}\left[X^2\right] = \mathbf{E}\left[X^2 \mid A\right]\mathbf{P}(A) + \mathbf{E}\left[X^2 \mid B\right]\mathbf{P}(B) = \left(\frac{1}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2\right)\frac{1}{2} + \left(\frac{2}{\lambda^2} + \left(\frac{2}{\lambda}\right)^2\right)\frac{1}{2} = \frac{4}{\lambda^2}$$

$$\Rightarrow \text{var}(X) = \frac{4}{\lambda^2} - \left(\frac{3}{2\lambda}\right)^2 = \frac{7}{4\lambda^2} \qquad .$$

(c) (5 pts) The event $\{N_{10} = 0\}$, or zero arrivals during the time interval $[0, 10)$, is equivalent to the first arrival time $X_1$ being greater than 10. Using the hint to aid with the integration,

$$\begin{aligned}\mathbf{P}(N_{10} = 0) &= \mathbf{P}(X_1 > 10) = \int_{10}^{\infty} f_X(x)dx = \frac{1}{2}\left(\int_{10}^{\infty} \lambda e^{-\lambda x}dx + \int_{10}^{\infty} \lambda^2 x e^{-\lambda x}dx\right) \\ &= \frac{1}{2}\left(-e^{-\lambda x} + (-\lambda x - 1)e^{-\lambda x}\right)\Big|_{10}^{\infty} = (5\lambda + 1)e^{-10\lambda}\end{aligned}$$

where we have concluded that $\lim_{x\to\infty} \lambda x e^{-\lambda x} = 0$ by L'Hopital's Rule.

(d) (i) (6 pts) No. While the process does renew itself at each arrival instant, in the sense that *at the instant of an arrival* the PDF $f_X(x)$ characterizes the time until the next arrival, the PDF $f_X(x)$ does not characterize the time of the next arrival starting from *any* arbitrary time instant $t$.

Of all distributions discussed in this class, only the exponential and geometric distributions for first-order interarrival times yield a memoryless process, and it's unlikely that there are others.

*Aside:* To appreciate this, a rather sophisticated argument is as follows. Memoryless implies that, given an arrival has not occurred by time $t > 0$, the distribution describing the time beyond $t$ until the next arrival is identical to the original interarrival distribution—mathematically, letting $Y = X - t$, we've just stated that memoryless requires $f_{Y|X>t}(x|X > t) = f_X(x)$ for any $t > 0$. Working first with CDFs to express the conditional distribution in terms of the original distribution, and then differentiating with respect to $x$ to express the relationship in terms of PDFs, we obtain

$$\begin{aligned} F_{Y|X>t}(x \mid X > t) &= \mathbf{P}(X - t \leq x \mid X > t) = \frac{\mathbf{P}(\{X \leq x + t\} \bigcap \{X > t\})}{\mathbf{P}(X > t)} \\ &= \frac{F_X(x+t)}{\mathbf{P}(X > t)} \qquad \Rightarrow \qquad f_{Y|X>t}(x \mid X > t) = \frac{f_X(x+t)}{\mathbf{P}(X > t)} \end{aligned}$$

Hence, if $f_X(x)$ is a functional form where a shift by $t$ is cancelled when dividing by the probability that $X > t$, then the process is memoryless (a property arguably unique to exponential functional forms). For example, consider $f_X(x)$ of this problem and $t = 10$ to leverage the answer of part (c):

$$\begin{aligned} \frac{f_X(x+10)}{\mathbf{P}(X > 10)} &= \frac{\frac{1}{2}\left(\lambda e^{-\lambda(x+10)} + \lambda^2(x+10)e^{-\lambda(x+10)}\right)}{(5\lambda+1)e^{-10\lambda}} = \frac{\frac{1}{2}\left(\lambda e^{-\lambda x} + \lambda^2(x+10)e^{-\lambda x}\right)}{(5\lambda+1)} \\ &= \frac{\frac{1}{2}\left(\lambda e^{-\lambda x} + \lambda^2 x e^{-\lambda x} + \lambda^2 10 e^{-\lambda x}\right)}{(5\lambda+1)} = \frac{f_X(x) + \lambda^2 5 e^{-\lambda x}}{(5\lambda+1)} \neq f_X(x) \end{aligned}$$

So it follows that, because $f_{Y|X>t}(x|X > t) \neq f_X(x)$ for at least one value of $t$, the arrival process is not memoryless.

(ii) (5 pts) For a memoryless process, we know (by the arguments in the text) that $\mathbf{E}[W] = 2\mathbf{E}[X]$. For a process that is not memoryless, we rely on the general formulas (see solutions to Recitation 10) relating $W$ to $X$ and use the answers from parts (a) and (b):

$$f_W(w) = \frac{w f_X(w)}{\mathbf{E}[X]} \qquad \Rightarrow \qquad \mathbf{E}[W] = \frac{\mathbf{E}\left[X^2\right]}{\mathbf{E}[X]} = \frac{\frac{4}{\lambda^2}}{\frac{3}{2\lambda}} = \frac{8}{3\lambda}$$

(e) We are given that $T_K = X_1 + X_2 + \ldots X_K$ where $K$ is the sum of six independent and identically-distributed Bernoulli trials, each with success probability $\frac{1}{2}$.

(i) (4 pts) Thus, $K$ is described by a binomial distribution with parameters $n = 6$ and $p = \frac{1}{2}$, and therefore $\mathbf{E}[K] = np = 3$ and $\text{var}(K) = np(1-p) = \frac{3}{2}$.

(ii) (4 pts) $T_K$ is a sum of a random number of independent random variables, so

$$\mathbf{E}[T_K] = \mathbf{E}[X]\mathbf{E}[K] = \frac{9}{2\lambda} \quad \text{and} \quad \text{var}(T_K) = \text{var}(X)\mathbf{E}[K] + (\mathbf{E}[X])^2\text{var}(K) = \frac{69}{8\lambda^2}$$

(f) Define the sample mean by $M_n = (X_1 + X_2 + \ldots + X_n)/n$ and let $\mu = \mathbf{E}[X]$.

(i) (6 pts) Note that $\mathbf{P}(A_n) = \mathbf{P}(|M_n - \mu| \geq 10^{-6})$ and so, by the WLLN where $\epsilon = 10^{-6}$, $\lim_{n\to\infty} \mathbf{P}(A_n) = 0$.

(ii) (6 pts) Note that $\mathbf{P}(\lim_{n\to\infty} B_n) = \mathbf{P}(\lim_{n\to\infty} M_n \neq \mu) = 1 - \mathbf{P}(\lim_{n\to\infty} M_n = \mu) = 0$ because, by the SLLN, $\mathbf{P}(\lim_{n\to\infty} M_n = \mu) = 1$.

**Problem 2.** (54 points) A state transition (not necessarily a change-of-state) occurs every hour, on the hour. Though self-transitions are not explicitly shown, because the sum of all transition probabilites from a state $i$ must be one, we deduce $p_{00} = 0.50$, $p_{11} = 0.30$ and $p_{22} = 0.36$.

(a) (6 pts) A self-transition in state 0 corresponds to the cafe being empty and no passengers arriving on the next shuttle, which occurs with probability $p_K(0) = p = p_{00} = 0.5 \Rightarrow p = 0.5$. A transition from state 1 to state 0 corresponds to the single customer ending the session and no passengers arriving on the next shuttle, which by independence occurs with probability $q \cdot p_K(0) = qp = p_{10} = 0.4 \Rightarrow q = 0.8$.

*Aside:* While it is possible to deduce from the problem description that

$$\begin{aligned}
p_{01} &= p(1-p), \quad p_{02} = \sum_{k=2}^{\infty} p(1-p)^k = (1-p)^2, \quad p_{11} = qp(1-p) + (1-q)p = p(1-qp), \\
p_{12} &= q\sum_{k=2}^{\infty} p(1-p)^k + (1-q)\sum_{k=1}^{\infty} p(1-p)^k = 1 - p(1+q-qp), \quad p_{20} = q^2 p, \\
p_{21} &= q^2 p(1-p) + \binom{2}{1} q(1-q)p = qp(2 - q(1+p)), \\
p_{22} &= q^2 \sum_{k=2}^{\infty} p(1-p)^k + \binom{2}{1} q(1-q) \sum_{k=1}^{\infty} p(1-p)^k + (1-q)^2 = 1 - qp(2-qp)
\end{aligned}$$

and then equate any subset of the above transition probabilities to the values in the graph to solve for parameters $p$ and $q$, it is clearly more effort than simply using $p_{00}$ and $p_{10}$.

(b) (6 pts) The chain forms a single recurrent class and is aperiodic; thus, the steady-state probabilities satisfy the equations

$$\begin{aligned}
\pi_0 &= 0.50\pi_0 + 0.40\pi_1 + 0.32\pi_2 \\
\pi_1 &= 0.25\pi_0 + 0.30\pi_1 + 0.32\pi_2 \\
\pi_2 &= 0.25\pi_0 + 0.30\pi_1 + 0.36\pi_2
\end{aligned}$$

Combining any two of these equations with $\pi_0 + \pi_1 + \pi_2 = 1$ yields, after some algebra, $\pi_0 = \frac{176}{421}$, $\pi_1 = \frac{120}{421}$, and $\pi_2 = \frac{125}{421}$.

(c) (8 pts) Let $X_5$ and $X_6$ denote the state just after 5am and 6am, respectively, and note that after months of operation we can safely assume that $\mathbf{P}(X_5 = i) \approx \pi_i$:

$$\begin{aligned}
\mathbf{P}(X_6 > X_5 \mid X_6 \neq X_5) &= \frac{\mathbf{P}(X_6 > X_5 \bigcap X_6 \neq X_5)}{\mathbf{P}(X_6 \neq X_5)} = \frac{\mathbf{P}(X_6 > X_5)}{\mathbf{P}(X_6 \neq X_5)} \\
&= \frac{\sum_{i=0}^{2} \mathbf{P}(X_6 > X_5 \mid X_5 = i)\pi_i}{\sum_{i=0}^{2} \mathbf{P}(X_6 \neq X_5 \mid X_5 = i)\pi_i} = \frac{0.50\pi_0 + 0.30\pi_1}{0.50\pi_0 + 0.70\pi_1 + 0.64\pi_2} = \frac{31}{63}
\end{aligned}$$

(d) During an hour when the cafe is empty (i.e., state 0), zero messages are generated; during an hour when the cafe has a single customer (i.e., state 1), messages are generated at a Poisson rate of $\lambda$ per hour; during an hour when the cafe has two customers (i.e., state 2), messages are generated at a Poisson rate of $2\lambda$ per hour.

(i) (8 pts) Let $X_{10}$ denote the state just after 10am and, again assuming the process is in steady-state, we have $\mathbf{P}(X_{10} = i) \approx \pi_i$. Letting event $A_i = \{X_{10} = i\}$ and noting that we are interested in the number of Poisson arrivals in $\tau = 0.5$ hours,

$$p_N(n) = \sum_{i=0}^{2} \pi_i \mathbf{P}(N = n \mid A_i) = \begin{cases} \pi_0 + \pi_1 \mathbf{P}(N=0 \mid A_1) + \pi_2 \mathbf{P}(N=0 \mid A_2) & , \quad n = 0 \\ \pi_1 \frac{\mathbf{P}(N=n|A_1)}{\mathbf{P}(N\neq 0|A_1)} + \pi_2 \frac{\mathbf{P}(N=n|A_2)}{\mathbf{P}(N\neq 0|A_2)} & , \quad n = 1, 2, \ldots \end{cases}$$

$$= \begin{cases} \pi_0 + \pi_1 e^{-0.5\lambda} + \pi_2 e^{-\lambda} & , \quad n = 0 \\ \pi_1 \frac{(0.5\lambda)^n e^{-0.5\lambda}}{1-e^{-0.5\lambda}} + \pi_2 \frac{\lambda^n e^{-\lambda}}{1-e^{-\lambda}} & , \quad n = 1, 2, \ldots \end{cases}$$

(ii) (10 pts) By definition of a Poisson process, the time until each individual customer generates a first message is an exponential random variable with parameter $\lambda$. Given also that each customer generates at least one message in the hour, an event with probability $1 - e^{-\lambda}$, the conditional PDF characterizing the arrival time $Z$ of a customer's first message becomes

$$f_Z(z) = \frac{\lambda e^{-\lambda z}}{1 - e^{-\lambda}} \quad , \quad 0 < z < 1 \quad .$$

Note that random variable $Y = \max\{Z_1, Z_2\}$, where $Z_1$ and $Z_2$, denoting the time until customer 1 and 2, respectively, generate their first messages, are independent and identically distributed with PDF $f_Z(z)$. We now derive the PDF for $Y$ by first relating its CDF to random variable $Z$ and then taking the derivative:

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \le y) = \mathbf{P}(\max\{Z_1, Z_2\} \le y) = \mathbf{P}(Z_1 \le y \bigcap Z_2 \le y) \\ &= \mathbf{P}(Z_1 \le y)\mathbf{P}(Z_2 \le y) = \mathbf{P}(Z \le y)^2 = [F_Z(y)]^2 \\ \Rightarrow f_Y(y) &= \frac{d}{dy}[F_Z(y)]^2 = 2F_Z(y) \cdot f_Z(y) = 2\left(\frac{1 - e^{-\lambda y}}{1 - e^{-\lambda}}\right)\left(\frac{\lambda e^{-\lambda y}}{1 - e^{-\lambda}}\right) \\ &= \frac{2\lambda e^{-\lambda y}(1 - e^{-\lambda y})}{(1 - e^{-\lambda})^2}, \quad 0 < y < 1 \quad . \end{aligned}$$

(e) (i) (8 pts) Let $L$ be the number of prizes awarded during the promotion. The owner's requirement states $\mathbf{P}(L = 150) \ge 0.8 = \frac{4}{5}$. We view the promotion as a Bernoulli process, where a success on the $w$th trial corresponds to a prize being awarded in the $w$th week. Thus, each Bernoulli trial $X_w$ has success probability $\alpha$. The promotion lasts $W = \min\{Y_{150}, 200\}$ weeks, where $Y_{150}$ denotes the number of trials until the 150th success (characterized by a Pascal PMF of order 150). It follows that

$$L = X_1 + X_2 + \ldots + X_W \le X_1 + X_2 + \ldots + X_{200} \quad \Rightarrow \mathbf{E}[L] \le 200\mathbf{E}[X_w]$$

so, combining the boss's requirement with the Markov inequality (and noticing that $L$ can be at most 150),

$$\frac{4}{5} \le \mathbf{P}(L = 150) = \mathbf{P}(L \ge 150) \le \frac{\mathbf{E}[L]}{150} \le \frac{200\mathbf{E}[X_w]}{150} = \frac{4\alpha}{3} \quad \Rightarrow \alpha \ge \frac{3}{4} \cdot \frac{4}{5} = \frac{3}{5}$$

(ii) (8 pts) In a full week, there will be $24 \cdot 7 = 168$ shuttle arrivals, with the $i$th shuttle delivering $K_i$ passengers where the $K_i$s are independent and identically distributed. The total number of passengers in any week is then

$$N = K_1 + K_2 + \ldots K_{168} \qquad \Rightarrow \mathbf{E}[N] = 168\mathbf{E}[K_i] \quad \text{and} \quad \text{var}(N) = 168\text{var}(K_i),$$

where $\mathbf{E}[K_i] = \frac{1}{p} - 1 = \frac{1-p}{p}$ and $\text{var}(K_i) = \frac{1-p}{p^2}$. We wish to choose $n$ no greater than the value at which $\mathbf{P}(N \ge n) = \alpha$. Using a CLT approximation, with the DeMoivre-Laplace correction to account for $N$ being discrete,

$$\mathbf{P}(N \ge n) = \mathbf{P}\left(\frac{N - \mathbf{E}[N]}{\sqrt{\text{var}(N)}} \ge \frac{n - \mathbf{E}[N]}{\sqrt{\text{var}(N)}}\right) \approx 1 - \Phi\left(\frac{n - \frac{1}{2} - \mathbf{E}[N]}{\sqrt{\text{var}(N)}}\right)$$

and so, in terms of parameters $p$ and $\alpha$, we choose $1 \le n \le n_0$ where $n_0$ is no greater than the value of $n$ that satisfies

$$\Phi\left(\frac{p(n - \frac{1}{2}) - 168(1 - p)}{\sqrt{168(1 - p)}}\right) = 1 - \alpha$$