Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2002)

Problem Set 2

1. This problem is an extension of the "three-curtain" Game Show example discussed in class.

   Dilbert's Ruin: Dilbert is put to the ultimate test, the *four curtain test*, by his pointy-haired boss. There are four curtains, one concealing a prize and the other three concealing junk. Dilbert picks a curtain and his boss, who knows where the prize actually is, opens one of the other curtains revealing junk. Dilbert is then given the choice of sticking to his original decision or switching to either of the other two concealed curtains. Whether Dilbert stays or switches, his boss opens *another* curtain once Dilbert has made his choice (so now the boss has revealed two curtains, one of which could have been Dilbert's first choice if Dilbert switched). Again, the boss offers Dilbert the option to switch.

   (a) Should Dilbert switch or stay put now (remember, at this stage there are two curtains that the boss has opened and Dilbert has two curtains left to choose between)?

   (b) How does the problem generalize to n curtains and one prize?

2. **Ternary Channel:** A communication system transmits one of three signals, s_1, s_2 and s_3, with equal probabilities. The reception is corrupted by noise, potentially causing the transmission to be changed according to the following table of conditional probabilities:

		Receive, j		
		s_1	s_2	s_3
Send, i	s_1	0.8	0.1	0.1
	s_2	0.05	0.9	0.05
	s_3	0.02	0.08	0.9

   For example, if s_1 is sent, the probability of receiving s_3 is 0.1. The entries of the table list the probability of s_j received, given that s_i is sent i.e., $P(s_j \text{ received}|s_i \text{ sent})$.

   (a) Compute the (unconditional) probability that s_j is received for $j = 1, 2, 3$.

   (b) Compute the probability $P(s_i \text{ sent}|s_j \text{ received})$ for $i, j = 1, 2, 3$.

3. A ball is in any one of n boxes. It is in the ith box with probability P_i. If the ball is in box i, a search of that box will uncover it with probability α_i. Show that the conditional probability that the ball is in box j, given that a search in box i did *not* uncover it, is

$$\frac{P_j}{1-\alpha_i P_i} \quad \text{if } j \neq i \qquad \text{and} \qquad \frac{(1-\alpha_i)P_i}{1-\alpha_i P_i} \quad \text{if } j = i.$$

4. You are lost in the campus of MIT, where the population is entirely composed of brilliant students and absent-minded professors. The students comprise two-thirds of the population, and any one student gives a correct answer to a request for directions with probability $\frac{3}{4}$. (Assume answers to repeated questions are independent, even if the question and the person asked are the same.) If you ask a professor for directions, the answer is always false.

   (a) You ask a passer-by whether the exit from campus is East or West. The answer is East. What is the probability this is correct?

   (b) You ask the same person again, and receive the same reply. Show that the probability that this second reply is correct is $\frac{1}{2}$.

   (c) You ask the same person again, and receive the same reply. What is the probability that this third reply is correct?

(d) You ask for the fourth time, and receive the answer East again. Show that the probability it is correct is $\frac{27}{40}$.

(e) Show that, had the fourth answer been West instead, the probability that East is nevertheless correct is $\frac{9}{10}$.

Your friend, Ima Nerd, happens to be in the same position as you are, only she has reason to believe a-priori that, with probability ϵ, East is the correct answer.

(f) Show that whatever answer is first received, Ima continues to believe that East is correct with probability ϵ.

(g) Show that if the first two replies are the same (that is, either WW or EE), Ima continues to believe that East is correct with probability ϵ.

(h) Show that after three like answers, Ima will calculate as follows (in the obvious notation):

$$P(\text{East correct}|EEE) = \frac{9\epsilon}{11-2\epsilon}, \qquad P(\text{East correct}|WWW) = \frac{11\epsilon}{9+2\epsilon}.$$

Evaluate these when $\epsilon = \frac{9}{20}$.

5. Anne, Betty, Chloe and Daisy were all friends in school. Subsequently each of the six subpairs meet up once; at each of the six meetings, the pair quarrels with some fixed probability p and otherwise the pair retains a firm friendship. Quarrels take place independently of each other.

   In the future, if any one of the four hears a rumour, she tells it to her firm friends only. Supposing that Anne hears a rumour, what is the probability that:

   (a) Daisy hears it?

   (b) Daisy hears it if Anne and Betty have quarrelled?

   (c) Daisy hears it if Betty and Chloe have quarrelled?

   (d) Daisy hears it if she has quarrelled with Anne?

G1†. Consider a community of m families and suppose, for i = 1, ..., k, that n_i of them have i children; note that $\sum_{i=1}^{k} n_i = m$. Consider the following two methods for choosing a child:

(i) Choose one of the families at random (uniformly) and then randomly (uniformly) choose a child from that family.

(ii) Choose one of the $\sum_{i=1}^{k} in_i$ children at random.

Show that method **(i)** is more likely to result in the choice of a first-born child.

Hint: You may need to show that

$$\left(\sum_{i=1}^{k} in_i\right)\left(\sum_{j=1}^{k} \frac{n_j}{j}\right) \geq \left(\sum_{i=1}^{k} n_i\right)\left(\sum_{j=1}^{k} n_j\right)$$

G2†. Find a sample space Ω, two probability laws P and Q, and three events A, B and C (subsets of Ω) such that *all* of the following are satisfied:

(a) A and B are independent for P,

(b) A and B are not independent for Q,

(c) A and B are conditionally (given C) independent for Q.

†Required for 6.431; optional for 6.041