Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2002)

Problem Set 5

1. Let x_1, x_2 and x_3 be three independent experimental values of a particular continuous random variable X. Given that x_1 is greater than x_2, what is the conditional probability that x_1 is also greater than x_3?

2. You may find the result in part (a) useful to show the result for part (b):

   (a) Let Y be any continuous random variable. Show that

   $$\mathbf{E}[Y] = \int_0^\infty \mathbf{P}(Y > y)dy - \int_0^\infty \mathbf{P}(Y < -y)dy \quad .$$

   (b) Let X be any continuous, *nonnegative* random variable. Show that

   $$\mathbf{E}[X^n] = \int_0^\infty nx^{n-1}\mathbf{P}(X^n > x)dx \quad .$$

   Hint: Consider a suitably-chosen change of variables.

3. **Signal Classification:** A wire connecting two locations serves as the transmission medium for ternary-valued messages; in other words, any message between locations is known to be one of three possible symbols, each occurring equally-likely. It is also known that any numerical value sent over this wire is subject to distortion; namely, if the value X is transmitted, the value Y received at the other end is described by $Y = X + N$ where the random variable N represents additve noise, assumed to be normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 4$.

   (a) Suppose the transmitter encodes the three types of messages with the values -1, 0 and 1. At the other end, the received message is decoded according to the following rules:

   - If $Y > \frac{1}{2}$, then conclude the value 1 was sent.
   - If $Y < -\frac{1}{2}$. then conclude the value -1 was sent.
   - If $-\frac{1}{2} \leq Y \leq \frac{1}{2}$, then conclude the value 0 was sent.

   Determine the probability of error for this encoding/decoding scheme. Reduce your calculations to a single numerical value.

   (b) In an effort to reduce the probability of error, the following modifications are made. The transmitter encodes the three types of messages with the values -2, 0 and 2 while the receiver's decoding rules are:

   - If $Y > 1$, then conclude the value 2 was sent.
   - If $Y < -1$. then conclude the value -2 was sent.
   - If $-1 \leq Y \leq 1$, then conclude the value 0 was sent.

   Repeat part (a) for this modified encoding/decoding scheme.

4. The random variable X is exponentially distributed with parameter $\lambda_X = 1$; in other words, continuous random variable X is defined by the PDF

   $$f_X(x) = e^{-x}, \quad x \geq 0 \qquad .$$

Given that $X = x$, the random variable Y is exponentially distributed with parameter $\lambda_Y = x$; in other words, the conditional PDF of Y given $X = x$ is

$$f_{Y|X}(y|x) = xe^{-xy}, \quad y \geq 0 \quad .$$

(a) Leting A denote the event that $X \geq 2$, calculate the conditional PDF $f_{X|A}(x|A)$. Explain the importance of how the answer compares with the unconditional PDF $f_X(x)$.

(b) Find $f_{X,Y}(x,y)$, the joint PDF of X and Y.

(c) Find $f_Y(y)$, the marginal PDF of Y.

(d) Find the conditional PDF $f_{X|Y}(x|2)$.

5. Random variables X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c & , \text{ if } 0 \leq y < x \leq 2 \\ 0 & , \text{ otherwise} \end{cases} \quad .$$

Let A be the event $\{X \geq 0.5\}$ and let B be the event $\{Y < 1 - X\}$. We suggest you rely on sketches to answer the following questions:

(a) Are X and Y independent? Explain.

(b) Calculate the numerical value of c.

(c) Calculate the probability $\mathbf{P}(B|A)$.

(d) Make a clearly labelled sketch of the conditional PDF $f_{X|Y}(x|0.5)$.

(e) Given that $Y = 0.5$, evaluate the conditional expectation and the conditional variance of X.

(f) Make a clearly labelled sketch of the conditional PDF $f_{X|B}(x|B)$.

(g) Calculate $\mathbf{E}[XY]$.

G1†. Let X be a random variable that takes on values between 0 and c; that is, $\mathbf{P}(0 \leq X \leq c) = 1$. Show that

$$\text{var}(X) \leq \frac{c^2}{4}.$$

G2†. (a) Let X be a zero-mean Gaussian random variable with variance σ^2. Derive an expression for $\mathbf{E}[X^n]$, the nth moment, that is valid for all $n \geq 0$. *Hint*: Consider the two cases of n odd and n even.

(b) A *central Chi-Square* random variable Y, with n degrees of freedom, is defined as

$$Y = X_1^2 + X_2^2 + \cdots + X_n^2,$$

where the X_i's are *independent and identically distributed* zero-mean Gaussian random variables with variance σ^2. Find the expectation and variance of Y.

†Required for 6.431; optional for 6.041