MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
**6.041/6.431: Probabilistic Systems Analysis**
(Fall 2002)

# Problem Set 7

1. A coin is tossed repeatedly, heads appearing with probability $q$ on each toss. Let random variable $T$ denote the number of tosses when a run of $n$ consecutive heads has appeared for the first time.

   (a) Show that the PMF for $T$ can be expressed as

   $$p_T(k) = \begin{cases} 0 & , \quad k < n \\ q^n & , \quad k = n \\ \left( \sum_{i=k-n}^{\infty} p_T(i) \right)(1-q)q^n & , \quad k \geq n+1 \end{cases} .$$

   (b) Determine the transform $M_T(s)$ associated with random variable $T$.

   (c) Compute $\mathbf{E}[T]$, the expectation of random variable $T$.

2. Let $X$, $Y$ and $Z$ denote any three discrete random variables. You may find the result of part (a) useful to show the result of part (b).

   (a) **Pull-Through Property:** For any suitable function $g$, show that

   $$\mathbf{E}[Yg(X) \mid X] = g(X)\mathbf{E}[Y \mid X] \quad .$$

   (b) **Tower Property:** Show that

   $$\mathbf{E}\left[\mathbf{E}[Y \mid X, Z] \mid X\right] = \mathbf{E}[Y \mid X] = \mathbf{E}\left[\mathbf{E}[Y \mid X] \mid X, Z\right] \quad .$$

3. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables, where each $X_i$ is distributed according to the *logarithmic* PMF with parameter $p$ i.e.,

   $$p_X(k) = \frac{(1-p)^k}{k \ln(1/p)}, \quad k = 1, 2, 3, \ldots ,$$

   where $0 < p < 1$. Discrete random variable $N$ has the Poisson PMF with parameter $\lambda$ i.e.,

   $$p_N(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots ,$$

   where $\lambda > 0$.

   (a) Determine the transform $M_X(s)$ associated with each random variable $X_i$. *Hint:* You may find the following identity useful:

   $$\text{Provided } -1 < a \leq 1, \quad \ln(1+a) = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \ldots$$

   (b) Defining $Y = \sum_{i=1}^{N} X_i$, determine the transform $M_Y(s)$.

4. **Swallowed Buffon's Needle:** A doctor is treating a patient who has accidentally swallowed a needle. The key factor in whether to operate on the patient is the true length of the needle, which is not known exactly but is assumed to be uniformly distributed between 0 and $\ell > 0$. While the needle may show up on an X-ray, the doctor recognizes that the random orientation of the needle within the patient's stomach implies the needle's length on film could be misleading. The doctor has asked you to analyze this scenario and form an estimate of the needle's true length based on its projected length in the X-ray.

   Oscar, a sloppy but otherwise reliable colleague, begins to model the problem and states that it involves four random variables he calls $W$, $X$, $Y$ and $Z$. He then claims that based on geometry and the symmetry inherent to the problem, the key equation is $Y = X \cos W$ and that random variable $Z$ turns out to be irrelevant. Oscar also states that the density of each of the three relevant random variables is nonzero only in an interval with lower limit of 0 and an upper limit that depends on the random variable; in other words, for some choice of $b_W, b_X, b_Y > 0$, Oscar is stating that

   $$f_W(w) = 0 \text{ for } w \notin [0, b_W], f_X(x) = 0 \text{ for } x \notin [0, b_X] \text{ and } f_Y(y) = 0 \text{ for } y \notin [0, b_Y] \quad .$$

   Unfortunately, Oscar dashes out the door before explaining to you what the four variables physically represent in his model and how he reached each of his conclusions.

   (a) Determine the probabilistic model Oscar has in mind,
      i. identifying what the random variables $W$, $X$, $Y$ and $Z$ physically represent,
      ii. explaining the main symmetry arguments, and
      iii. providing the values for parameters $b_W$, $b_X$ and $b_Y$.

   (b) Determine the *least-squares estimate* of the needle's true length given the value of the needle's projected length in a single X-ray. In other words, if $X$ were to denote the needle's true length and $Y$ its length as measured from the resulting X-ray, determine the mathematical formula for $\mathbf{E}[X|Y]$. Oscar leaves a message recommending that you proceed in the order implied by the steps below:
      i. Derive $f_{Y|X}(y \mid x)$, the conditional PDF of $Y$ given $X = x$.
      ii. Determine $f_Y(y)$, the unconditional PDF of $Y$.
      iii. Determine $f_{X|Y}(x \mid y)$, the conditional PDF of $X$ given $Y = y$, and then compute $\mathbf{E}[X|Y = y]$ directly.

      *Hint:* You may find the following integral formulas useful:

      $$\int_a^b \frac{1}{\sqrt{\alpha^2 - c^2}} d\alpha = \ln\left(\alpha + \sqrt{\alpha^2 - c^2}\right)\Big|_a^b \qquad \int_a^b \frac{\alpha}{\sqrt{\alpha^2 - c^2}} d\alpha = \sqrt{\alpha^2 - c^2}\Big|_a^b$$

   (c) Determine the *linear least-squares estimate* approximation to the answer of part (b). Also compute the resulting estimation error. *Hint:* Consider using the Law of Iterated Expectations to simplify the necessary calculations.

5. Let random variables $X$ and $Y$ have the bivariate normal PDF

   $$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{1 - \rho^2}} \exp\left\{ -\frac{(x^2 - 2\rho xy + y^2)}{2(1 - \rho^2)} \right\}, \quad -\infty < x, y < \infty \qquad ,$$

   where $\rho$ denotes the correlation coefficient between $X$ and $Y$.

(a) Determine the numerical values of $\mathbf{E}[X]$, $\text{var}(X)$, $\mathbf{E}[Y]$ and $\text{var}(Y)$.

(b) Show that $X$ and $Z = (Y - \rho X)/\sqrt{1-\rho^2}$ are independent normal random variables, and determine the numerical values of $\mathbf{E}[Z]$ and $\text{var}(Z)$.

(c) Deduce that

$$\mathbf{P}\left(\{X > 0\} \bigcap \{Y > 0\}\right) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho \quad .$$

G1†. When using a multiple access communication channel, a certain number of users $N$ try to transmit information to a single receiver. If the real-valued random variable $X_i$ represents the signal transmitted by user $i$, the received signal $Y$ is

$$Y = X_1 + X_2 + \cdots + X_N + Z,$$

where $Z$ is an additive noise term that is independent of the transmitted signals and is assumed to be a zero-mean Gaussian random variable with variance $\sigma_Z^2$. We assume that the signals transmitted by different users are mutually independent and, furthermore, we assume that they are identically distributed, each Gaussian with mean $\mu$ and variance $\sigma_X^2$.

(a) If $N$ is deterministically equal to 2, find the transform <u>or</u> the PDF of $Y$.

(b) In most practical schemes, the number of users $N$ is a random variable. Assume now that $N$ is equally likely to be equal to $0, 1, \ldots, 10$.

i. Find the transform <u>or</u> the PDF of $Y$.

ii. Find the mean and variance of $Y$.

iii. Given that $N \geq 2$, find the transform <u>or</u> PDF of $Y$.

(c) Depending on the time of day, the number of users $N$ has different PMF's. Between 7 and 10AM, and between 5 and 8PM, $N$ can be assumed to be Poisson distributed with parameter $\lambda = 10$. During all other hours, $N$ is equally likely to be $0, 1, \ldots, 10$. On a random time during the day, based on the reception of $Y$, the receiver estimates the transmitted signals. Determine the linear least-squared estimator of $X_k$ (where $k$ is a positive integer) given reception $Y$.

G2†. Let $\underline{V} = (X, Y)$ be a pair of zero mean jointly Gaussian random variables. Let $K$ be the *covariance matrix* of $\underline{V}$ defined as

$$K = \begin{pmatrix} \mathbf{E}[X^2] & \mathbf{E}[XY] \\ \mathbf{E}[XY] & \mathbf{E}[Y^2] \end{pmatrix}.$$

(a) Show that the joint PDF of $X$ and $Y$ can be written as

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{|K|}} e^{-\underline{v}^{\text{t}} K^{-1} \underline{v}} \quad ,$$

where $\underline{v}$ is the column vector $(x, y)^{\text{t}}$, $|K|$ denotes the determinant of the matrix $K$. and $K^{-1}$ its inverse.

(b) Let $Z = 2X + Y$ and $W = X - 2Y$.

i. Are $Z$ and $W$ jointly Gaussian?

ii. Find the joint PDF of $Z$ and $W$.

†Required for 6.431; optional for 6.041