Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2002)

Final Exam Review Solutions
December 15, 2002

1. (a) Let L_i be the event that Joe played the lottery on week i, and let W_i be the event that he won on week i. We are asked to find

$$P(L_i \mid W_i^c) = \frac{P(W_i^c \mid L_i)P(L_i)}{P(W_i^c \mid L_i)P(L_i) + P(W_i^c \mid L_i^c)P(L_i^c)} = \frac{(1-q)p}{(1-q)p + 1 \cdot (1-p)} = \boxed{\frac{p-pq}{1-pq}}$$

(b) Conditioned on X, Y is binomial

$$p_{Y|X}(y \mid x) = \begin{cases} \binom{x}{y} q^y (1-q)^{(x-y)} & 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

(c) Realizing that X has a binomial PMF, we have

$$\begin{aligned} p_{X,Y}(x,y) &= p_{Y|X}(y \mid x) p_X(x) \\ &= \begin{cases} \binom{x}{y} q^y (1-q)^{(x-y)} \binom{n}{x} p^x (1-p)^{(n-x)} & 0 \le y \le x \le n \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(d) Using the result from (c), we could compute

$$p_Y(y) = \sum_{x=y}^{n} p_{X,Y}(x,y) \ ,$$

but the algebra is messy. An easier method is to realize that Y is just the sum of n independent Bernoulli random variables, each having a probability pq of being 1. Therefore Y has a binomial PMF:

$$p_Y(y) = \begin{cases} \binom{n}{y} (pq)^y (1-pq)^{(n-y)} & 0 \le y \le n \\ 0 & \text{otherwise.} \end{cases}$$

(e)

$$\begin{aligned} p_{X|Y}(x \mid y) &= \frac{p_{X,Y}(x,y)}{p_Y(y)} \\ &= \begin{cases} \dfrac{\binom{x}{y} q^y (1-q)^{(x-y)} \binom{n}{x} p^x (1-p)^{(n-x)}}{\binom{n}{y} (pq)^y (1-pq)^{(n-y)}} & 0 \le y \le x \le n \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(f) Given $Y = y$, we know that Joe played y weeks with certainty. For each of the remaining $n - y$ weeks that Joe did not win there are $x - y$ weeks where he played. Each of these events occured with probability $P(L_i \mid W_i^c)$ (the answer from part (a)). Using this logic we see that that X conditioned on Y is binomial:

$$p_{X|Y}(x \mid y) = \begin{cases} \begin{pmatrix} n-y \\ x-y \end{pmatrix} \left(\frac{p-pq}{1-pq}\right)^{x-y} \left(1 - \frac{p-pq}{1-pq}\right)^{n-x} & 0 \le y \le x \le n \\ 0 & \text{otherwise} \end{cases}$$

After some algebraic manipulation, the answer to (e) can be shown to be equal to the one above.

2. (a) We know that the total length of the edge for red interval is two times that for black inteval. Since the ball is equally likely to fall in any position of the edge, probability of falling in a red interval is $\frac{2}{3}$.

(b) Conditioned on the ball having fallen in a black interval, the ball is equally likely to fall anywhere in the interval. Thus, the PDF is

$$f_{Z|\text{black interval}}(z) = \begin{cases} \frac{15}{\pi r} & , \quad z \in [0, \frac{\pi r}{15}] \\ 0 & , \quad \text{otherwise} \end{cases}$$

(c) Since the ball is equally likely to fall on any point of the edge, we can see it is twice as likely for $z \in [0, \frac{\pi r}{15})$ than $z \in \left[\frac{\pi r}{15}, \frac{2\pi r}{15}\right]$. Therefore, intuitively, let

$$f_Z(z) = \begin{cases} 2h & , \quad z \in [0, \frac{\pi r}{15}) \\ h & , \quad z \in \left[\frac{\pi r}{15}, \frac{2\pi r}{15}\right] \\ 0 & , \quad \text{otherwise} \end{cases}$$

Using the fact that $\int_{-\infty}^{\infty} f_Z(z) = 1$,

$$(2h)(\frac{\pi r}{15}) + (h)(\frac{\pi r}{15}) = 1 \Rightarrow h = \frac{5}{\pi r}$$

$$f_Z(z) = \begin{cases} \frac{10}{\pi r} & , \quad z \in [0, \frac{\pi r}{15}) \\ \frac{5}{\pi r} & , \quad z \in \left[\frac{\pi r}{15}, \frac{2\pi r}{15}\right] \\ 0 & , \quad \text{otherwise} \end{cases}$$

(d) The total gains (or losses), T, equals to the sum of all X_i, i.e. $T = X_1 + X_2 + \cdots + X_n$, Since all the X_i's are independent of each other, and they have the same Gaussian distribution, the sum will also be a Gaussian with

$$E[T] = E[X_1] + E[X_2] + \cdots + E[X_n] = 0$$

$$var(T) = var(X_1) + var(X_2) + \cdots + var(X_n) = n\sigma^2$$

Therefore, the standard deviation for T is $\sqrt{n}\sigma$.

(e)

$$\begin{aligned} P(|T| > 2\sqrt{n}\sigma) &= P(T > 2\sqrt{n}\sigma) + P(T < -2\sqrt{n}\sigma) \\ &= 2P(T > 2\sqrt{n}\sigma) \\ &= 2(1 - \Phi(\frac{2\sqrt{n}\sigma - E[T]}{\sigma_T})) \\ &= 2(1 - \Phi(2)) \simeq 0.0454. \end{aligned}$$

3. (a) $Y = g(X) = \sin(\frac{\pi}{2}X)$. Because $g(x)$ is a monotonic function for $-1 < x < 1$, we can define an inverse function $h(y) = \frac{2\arcsin y}{\pi}$ and use the PDF formula given in lecture:

$$\begin{aligned} f_Y(y) &= f_X(h(y))\left|\frac{dh(y)}{dy}\right| \quad \text{for } -1 < y < 1 \\ &= f_X\left(\frac{2\arcsin y}{\pi}\right)\left|\frac{2}{\pi\sqrt{1-y^2}}\right| \\ &= \frac{1}{2}\cdot\frac{2}{\pi\sqrt{1-y^2}} \\ &= \frac{1}{\pi\sqrt{1-y^2}} \end{aligned}$$

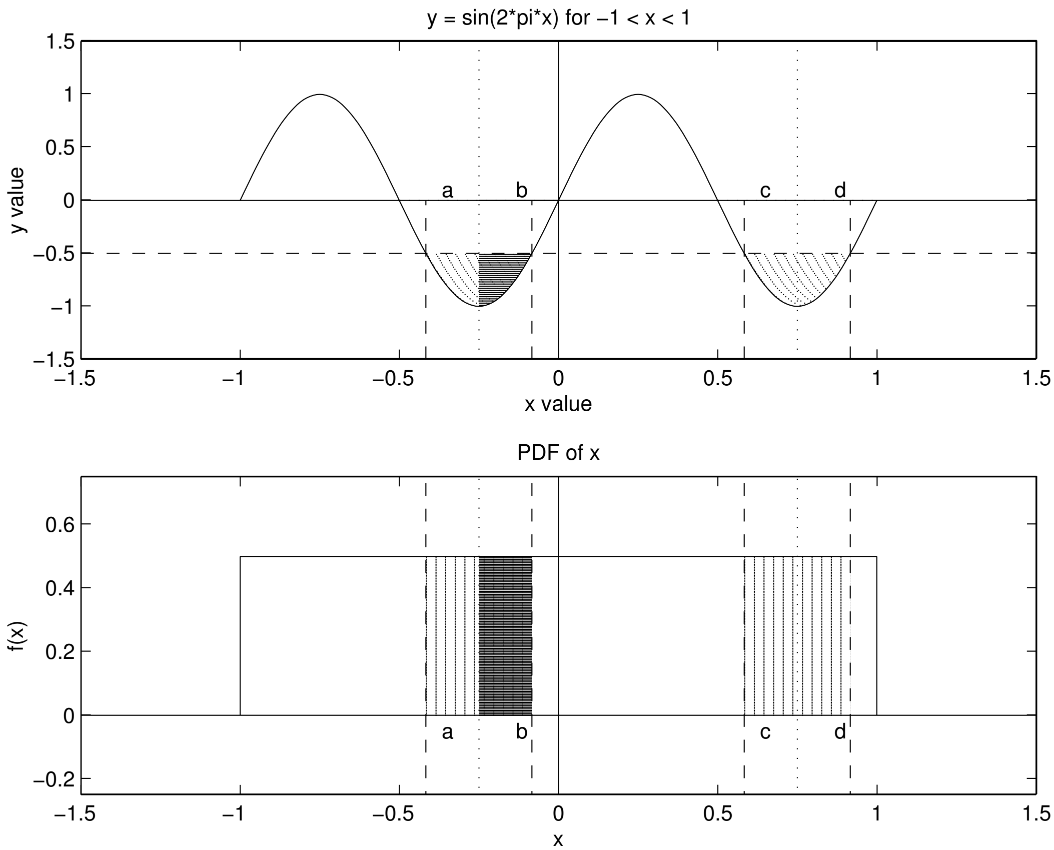
The final answer is:

$$f_Y(y) = \begin{cases} 0 & y < -1 \\ \frac{1}{\pi\sqrt{1-y^2}} & -1 \le y < 1 \\ 0 & 1 \le y \end{cases}$$

(b) $Y = \sin(2\pi X)$. In rearranging the equation, we find $X = \frac{\arcsin Y}{2\pi}$. This is a many-to-one mapping of X to Y. That is, given a value of Y, it is not possible to pinpoint the corresponding value of X. For example, if $Y = 1$, X could have been $\frac{1}{4}$ or $\frac{-3}{4}$. This means we cannot use the formula used in part (a).

The CDF is still the means to achieving the PDF, but we take a graphical approach instead. First, let's consider the extreme cases. By virtue of the $\sin X$ function, the value of Y varies from -1 to 1. It is obvious that no value of the random variable Y can be less than -1, so $F_Y(y \le -1) = 0$. Also, every value of Y is less than 1, so $F_Y(y \ge 1) = 1$.

For $-1 \le y \le 0$, consider the following diagram:



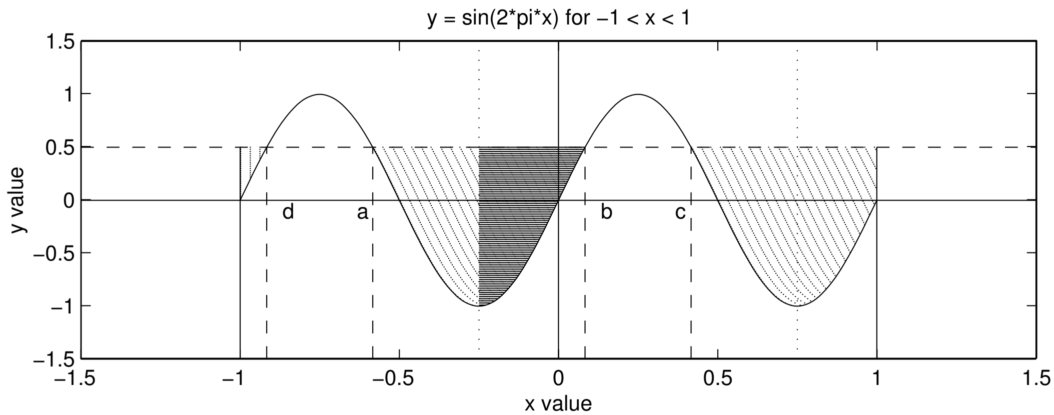
The value b indicates the conventional response of $\arcsin y$ which is between $[-\frac{\pi}{2}, \frac{\pi}{2}]$. In other words, by rearranging the original equation:

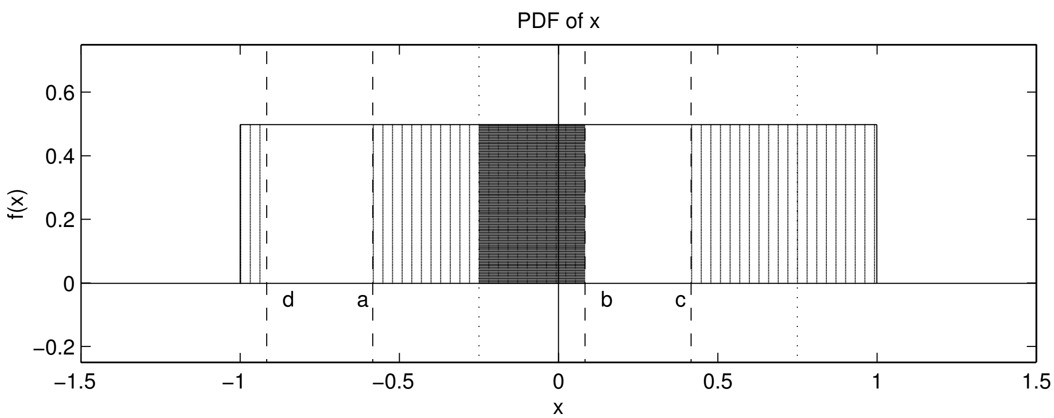
$$b = \frac{\arcsin y}{2\pi}$$
$$-\frac{1}{4} \leq b \leq \frac{1}{4}$$

So, we now have $F_Y(y) = P[(a \leq X \leq b) \cup (c \leq X \leq d)]$, where the event on the right is the shaded region in the above PDF of X. By symmetry, and mutual exclusivity, this shaded region can be expressed as four times the more darkly shaded region of the PDF.

$$\begin{aligned} F_Y(y| -1 \leq y \leq 0) &= P[(a \leq X \leq b) \cup (c \leq X \leq d)] \\ &= 4P(-\frac{1}{4} \leq X \leq b) \\ &= 4(0.5)[b - (-\frac{1}{4})] \\ &= \frac{\arcsin y}{\pi} + \frac{1}{2}. \end{aligned}$$

For $0 \leq y \leq 1$, consider the following diagram:





By an analogous argument to the previous case, we find that $F_Y(y|0 \leq y \leq 1)$ is represented by the shaded region from -1 to d, a to b, and c to 1. Once again, however, this is exactly four times the more darkly shaded region from $-\frac{1}{2}$ to b. So, the expression for the CDF is the same as in the case $-1 \leq y \leq 0$.

$$F_Y(y|0 \leq y \leq 1) = \frac{\arcsin y}{\pi} + \frac{1}{2}$$

To summarize:

$$F_Y(y) = \begin{cases} 0 & y \leq -1 \\ \frac{\arcsin y}{\pi} + \frac{1}{2} & -1 \leq y < 1 \\ 1 & 1 \leq y \end{cases}$$

Now use the identity $\frac{d}{dy}\arcsin y = \frac{1}{\sqrt{1-y^2}}$ and differentiate with respect to y:

$$f_y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} 0 & y < -1 \\ \frac{1}{\pi\sqrt{1-y^2}} & -1 \leq y < 1 \\ 0 & 1 \leq y \end{cases}$$

4. **(a)** Let K = the number of arrivals in the three-hour period in consideration, and M_i = the size of the ith arrival group, which includes the user of the mailbox plus his or her accompanying friends. Thus, $M_i = N_i + 1$. We can express Y as

$$Y = M_1 + M_2 + \cdots + M_K.$$

Y is a random sum of random variables. Using iterated expectations:

$$E[Y] = E[E[Y|K]] = E[K]E[M] = E[K](E[N]+1).$$

Using the moment-generating property of transforms:

$$\begin{aligned} E[N] &= \left.\frac{dM_N(s)}{ds}\right|_{s=0} \\ &= 1/3 + 2/6 \\ &= 2/3, \end{aligned}$$

K is Poisson with rate λ over an interval of length 3, so $E[K] = 3\lambda$. Combining everything:

$$\boxed{E[Y] = 5\lambda}$$

(b) We use the Pascal PMF:

$$\begin{aligned} & P(\text{Exactly 3 parcels arrive by the time the 5th letter arrives}) \\ = \; & P(\text{the fifth letter arrives at time 8}) \\ = \; & P(\text{the first 7 arrivals consists of exactly 3 parcels, and 4 letters, \textit{and} the 8th arrival is a letter}) \\ = \; & \binom{7}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^4\frac{2}{3} \\ = \; & \boxed{\binom{7}{3}\left(\tfrac{1}{3}\right)^3\left(\tfrac{2}{3}\right)^5} \end{aligned}$$

(c) We use the binomial PMF:

$$\begin{aligned} & P(\text{Exactly 3 out of next 8 users will mail parcels}) \\ = \; & \boxed{\binom{8}{3}\left(\tfrac{1}{3}\right)^3\left(\tfrac{2}{3}\right)^5} \end{aligned}$$

(d) Let M be the size of the group to which any person belongs. $M = N+1$, the number of accompanying friends plus the person himself).

$$M_M(s) = E[e^{s(N+1)}] = e^s M_N(s) = \frac{1}{2}e^s + \frac{1}{3}e^{2s} + \frac{1}{6}e^{3s}$$

Note that, using part b), $E[M] = E[N] + 1 = 5/3$. Also $p_M(1) = \frac{d}{de^s}M_M(s)|_{e^s=0} = 1/2$. The key observation to this problem is that we are actually dealing with *random incidence*. Imagine we line everyone up (both users and non-users), with people coming from the same group standing next to their accompanying friends. Suppose now that we pick a person i at random from this line. Let W be the size of the group to which i belongs. The probability that $W = w$ (i.e., the probability that she is from a group of size w) is the same as the probability that the gap into which we enter a process by random incidence is of duration w. This problem is similar to question 1d) from problem set 9. The probability that a group of a certain size, w, is chosen is

$$p_W(w) = \frac{w p_M(w)}{E[M]}$$

In particular, $p_W(1)$ is the probability that a randomly selected person i is the only person in the group, i.e., he or she is unaccompanied. Therefore:

$$\begin{aligned} & P(\text{a randomly selected person is accompanied}) \\ = \; & 1 - P(\text{a randomly selected person is } \textit{not} \text{ accompanied}) \\ = \; & 1 - p_W(1) \\ = \; & \boxed{7/10}. \end{aligned}$$

(e) As in part (a), define Y = the total number of people arriving in one hour, K = the number of arrivals in *one* hour, and Q_i = the size of the ith arrival group. We can express Y as:

$$Y = Q_1 + Q_2 + \cdots + Q_K.$$

where the Q_i's are independent and identically distributed. The transform of Y is:

$$\begin{aligned} M_Y(s) &= E[e^{sY}] \\ &= E[e^{sQ_1} e^{sQ_2} \dots e^{sQ_K}] \\ &= E[E[e^{sQ_1} e^{sQ_2} \dots e^{sQ_K} | K]] \\ &= E[(M_Q(s))^K] \\ &= M_K(s)|_{e^s = M_Q(s)} \\ &= M_K(s)|_{e^s = \frac{1}{2}e^s + \frac{1}{3}e^{2s} + \frac{1}{6}e^{3s}} \end{aligned} \qquad (1)$$

K is a Poisson random variable with parameter λ, so:

$$M_K(s) = e^{\lambda(e^s - 1)}.$$

Plugging in:

$$\boxed{M_Y(s) = e^{\lambda(\frac{1}{2}e^s + \frac{1}{3}e^{2s} + \frac{1}{6}e^{3s} - 1)}}$$

To get the actual closed-form PMF, if it exists at all, one must invert the transform $M_Y(s)$ above – a very difficult task. The lesson here is that for random sums of random variables it is usually much easier to get their transforms than their PMFs.

(f) Let us select a time in this Poisson process by random incidence. The time from our random entry to the arrival of the future fifth user is:

$$X_1 + \cdots + X_5$$

where the X_i's are i.i.d. exponential random variable with mean $1/\lambda$. Also, the time from the random entry to the previous third user would be:

$$X_{-3} + X_{-2} + X_{-1}$$

where each X_i's are once again independent exponential random variables with mean $1/\lambda$. So, T, the duration between the arrival of third previous users and the fifth future user would be

$$T = X_{-3} + X_{-2} + X_{-1} + X_1 + \cdots X_5$$

Hence, we have

$$\begin{aligned} M_T(s) &= (M_X(s))^8 \\ &= \boxed{\left(\frac{\lambda}{\lambda - s}\right)^8}. \end{aligned}$$

Note that $p_T(t)$ corresponds to the PDF of an Erlang random variable of order 8, i.e. the time until 8 arrivals in a Poisson process.

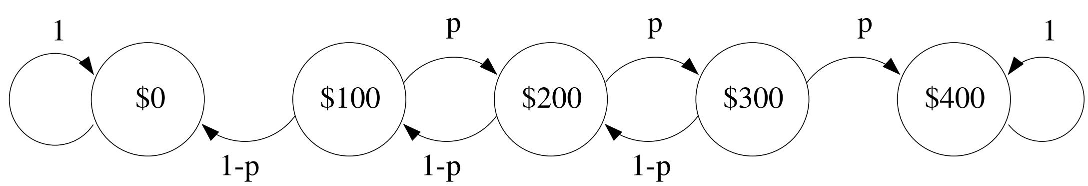
5. a) Assume Mary's goal is not to maximize her total profit but to simply maximize the probability she acquires \$400. We calculate the simple cases first. The easiest decision is when Mary has \$100. She must bet \$100 because she can't bet \$200.

   When Mary has \$300, she should bet \$100. Whether she bets \$100 or \$200, she will meet her goal if she wins the next game. If she bets \$100 and loses, she will then have \$200. If she bets \$200 and loses, she will then have \$100. Everything else being equal, it's more advantageous to have \$200 than to have \$100.

   The more difficult decision is how much to bet when Mary has \$200. We'll investigate both possible strategies and decide which is preferable.

   First, Mary can bet \$200. This case is easy to analyze. She will either win the desired amount or go "bust" on the next game. The probability that she will win is p.

   Second, Mary can bet \$100. In this case, we have the following state transition diagram.



   Winning and going "bust" are the two absorption states, and we want to find the probability of eventually winning, given that she starts with \$200. We will denote the probability that Mary wins given that she starts with j hundred dollars by a_j.

$$\begin{aligned} a_1 &= a_2 p \\ a_2 &= a_1(1-p) + a_3 p \\ a_3 &= a_2(1-p) + p \end{aligned}$$

   A few simple substitutions yield the following.

$$a_2 = \frac{p^2}{1 - 2p + 2p^2}$$

   We need to compare p and a_2 to determine which strategy is optimal. Solving for the condition such that $p > a_2 = \frac{p^2}{1-2p+2p^2}$ yields that betting \$200 is advantageous when $p < 1/2$ and betting \$100 is advantageous when $p > 1/2$. When $p = 1/2$ neither strategy is better than the other.

b) With $p = .75$, the optimal strategy is for Mary to bet \$100 when she has \$200. We need to find the expected time until absorption, given that Mary started with \$200.

Let μ_i = E(number of transitions to absorption starting from i(\$100)). We know that $\mu_0 = 0$ and $\mu_4 = 0$ because these are absorption states. We have the following relationship to determine the other μ_i's.

$$\mu_i = 1 + \sum_{j=1}^{3} p_{ij}\mu_j$$

So, we get the following three equations.

$$\begin{aligned}\mu_1 &= 1 + p\mu_2 \\ \mu_2 &= 1 + (1-p)\mu_1 + p\mu_3 \\ \mu_3 &= 1 + (1-p)\mu_2\end{aligned}$$

We need to solve for μ_2. Inserting $p = .75$, we get the following values for μ_i.

$$\begin{aligned}\mu_1 &= 3.4 \\ \mu_2 &= 3.2 \\ \mu_3 &= 1.8\end{aligned}$$

Therefore, the answer is $\boxed{3.2}$.

6. (a) When we merge the arrival process of wombats, which is Poisson with rate 2, with the arrival process of dingos, which is Poisson with rate 4, we get a Poisson process with rate 6 that represents arrivals of animals. The expected number of arrivals in a 24-hour period is just $6(24) = 144$.

   (b) The Poisson process is memoryless, so the fact that a wombat is currently drinking doesn't matter. The probability that the next animal is a dingo is $\frac{4}{2+4} = \frac{2}{3}$.

   (c) Note that there is a possibility that Crocodile Dundee will arrive to the water hole at first and see a dingo. In the case, the amount of time he'll have to wait is 0. Otherwise, he'll have to wait until one arrives, and the amount of time he'll have to wait is exponential with rate 4. Let π_d denote the probability that when Crocodile Dundee arrives he sees a dingo at the water hole and π_w denote the probability he sees a wombat. Because the series of arrivals preceding the moment when Crocodile Dundee arrives also form a Poisson process (i.e. we can look at the arrivals in reverse and the interarrival times are still exponential), we have from part b) that $\pi_d = \frac{2}{3}$ and $\pi_w = \frac{1}{3}$.
   If we denote the amount of time until he sees a dingo as X, we have

$$\begin{aligned}E[X] &= E[X|\text{he arrives to hole occupied by a dingo}]P(\text{occupied by dingo}) + \\ &\quad E[X|\text{he arrives to hole occupied by a wombat}]P(\text{occupied by wombat}) \\ &= (0)\pi_d + E[exponential(4)]\pi_w \\ &= \frac{1}{12}\end{aligned}$$

(d) We again need to condition on who is occupying the hole when he arrives. Note that if Crocodile Dundee arrives at the water hole and a dingo is occupying the hole, the total amount of time that dingo spent at the hole is in fact Erlang order 2 by the random incidence phenomenon (there's an exponential amount of time left from when Crocodile Dundee arrives until another animal arrives, and there's an independent exponential amount of time from when the dingo first arrived to when Crocodile Dundee arrives). If Crocodile Dundee arrives when the hole is occupied by a wombat, then when the first dingo does come, the amount of time that dingo will spend will just be the interarrival time of a Poisson process of rate $2+4$, which is *exponential*(6). So if we call the amount of time spent by the first dingo seen by Crocodile Dundee Y, we have:

$$\begin{aligned}
E[Y] &= E[Y|\text{he arrives to hole occupied by a dingo}]P(\text{occupied by dingo}) + \\
&\quad E[Y|\text{he arrives to hole occupied by a wombat}]P(\text{occupied by wombat}) \\
&= E[Erlang-order2(6)]\pi_d + E[exponential(6)]\pi_w \\
&= \frac{5}{18}
\end{aligned}$$

(e) Let Y be the expected number of hours Crocodile Dundee spends at the water hole. Y is a sum of 900 independent interarrival times, each with an exponential distribution with parameter 6. The expected value of each interarrival times is $\frac{1}{6}$, and the variance of each is $\frac{1}{6^2} = \frac{1}{36}$. Therefore $E[Y] = 900(\frac{1}{6}) = 150$ and $\text{Var}(Y) = 900(\frac{1}{36}) = 25$.
$P(140 < Y < 160) = P(|Y - 150| < 10) = P(|Y - E[Y]| < 10)$.
By Chebyshev's inequality, $P(|Y - E[Y]| \geq 10) \leq \frac{\sigma^2}{10^2} = \frac{25}{100} = \frac{1}{4}$, so the probability that Crocodile stays at the water hole for between 140 and 160 hours is at least $1 - \frac{1}{4} = \frac{3}{4}$.

(f) Since Y is a sum of 900 independent, identically distributed random variables, the Central Limit Theorem implies that the cdf of $\frac{Y-150}{5}$ can be approximated by the cdf of $X \sim N(0,1)$, so

$$P(|Y - 150| < 10) = P\left(\frac{|Y-150|}{5} < 2\right) \approx P(-2 < X < 2) = 2(\Phi(2) - .5) = .954.$$

The probability that Crocodile stays at the water hole for between 140 and 160 hours is therefore approximately .954.