Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2002)

Recitation 11 Answers

1. (a) The class {1,2} is recurrent and aperiodic (i.e. the class has period 1). The class {4,5,6} is recurrent with period 3. The state {3} is transient.
   (b) $r_{33}(n) = (0.2)^n$
   (c) Let T_{33} be the number of trials up to and including the first trial on which the process *leaves* state 3, given that it starts in state 3. Then $\mathbf{E}[T_{33}] = \frac{5}{4}$.
   (d) Let X_n denote the state after n trials. Then $\mathbf{P}(X_n \neq 1 \text{ for all } n \mid X_0 = 3) = \frac{3}{8}$.
   (e) $r_{34}(10) = (0.3) + (0.2)^3(0.3) + (0.2)^6(0.3) + (0.2)^9(0.3) \approx 0.3024$
   (f) $\mathbf{P}(X_1 = 4 \mid X_{10} = 4,\ X_0 = 3) = \frac{0.3}{r_{34}(10)} = 0.992$

2. The states of the Markov chain must capture all of the "pertinent" information in order that the process has the Markov property, i.e. in order to insure that:

$$\mathbf{P}(X_n = j|X_{n-1} = i, \ldots, X_1 = k) = \mathbf{P}(X_n = j|X_{n-1} = i).$$

Thus our Markov chain must have 3 states, where the state is the difficulty of the most recent exam:

$$\{S_1 = H, S_2 = M, S_3 = E\}.$$

We are given the transition probabilities:

$$[P] = \begin{pmatrix} P_{HH} & P_{HM} & P_{HE} \\ P_{MH} & P_{MM} & P_{ME} \\ P_{EH} & P_{EM} & P_{EE} \end{pmatrix} = \begin{pmatrix} 0 & .5 & .5 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{pmatrix}.$$

It is easy to see that our Markov chain has a single, aperiodic recurrent class, and is therefore ergodic. Thus we can use the fundamental theorem which tells us that if we can find $\{\pi_i\}$ that satisfy $\pi_j = \sum_i \pi_i p_{ij}$ and $\sum_i \pi_i = 1$ then the $\{\pi_i\}$ are in fact the steady state probabilities. Therefore we have:

$$\pi_j = \sum_i \pi_i p_{ij} \quad \Rightarrow \quad \begin{aligned} \frac{1}{4}(\pi_2 + \pi_3) &= \pi_1 \\ \frac{1}{2}(\pi_1 + \pi_2) + \frac{1}{4}\pi_3 &= \pi_2 \\ \frac{1}{2}(\pi_1 + \pi_3) + \frac{1}{4}\pi_2 &= \pi_3 \end{aligned}$$

and solving these with the constraint: $\sum_i \pi_i = 1$ gives:

$$\pi_1 = \frac{1}{5}, \pi_2 = \pi_3 = \frac{2}{5}.$$

3. **Practice Problem:**

   (a) (proof)

(b) For $0 \le i, j \le k$, we have the following:

$$p_{ij} = \mathbf{P}(i - j \text{ green fish are caught}) = \begin{cases} \frac{n-i}{n} & j = i \\ \frac{i}{n} & j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) The class $\{G_0\}$ is recurrent (since G_0 is an absorbing state), and all of the other states are transient.