6.041 Fall 2002 Final Exam
3 Hours

DO NOT TURN THIS EXAM OVER UNTIL YOU ARE TOLD TO DO SO

- You have 3 hours to complete the exam.
- Write your solutions in the exam booklet. We will not consider any work not in the exam booklet.
- This exam has 2 problems, each with multiple parts, that are not necessarily in order of difficulty.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of numbers that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. <u>You should concisely indicate your reasoning and show all relevant work</u>. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- This is a closed-book exam except for three handwritten, 2-sided 8.5x11 formula sheets plus a calculator.
- Be neat! If we can't read it, we can't grade it.
- At the end of the exam, turn in your solutions along with this exam (this piece of paper).

Problem 0. (2 points) Write your name on the front of your answer booklet.

Problem 1. (44 points) An arrival process that begins at time $t = 0$ is characterized by independent and identically-distributed interarrival times X described by the PDF

$$f_X(x) = \frac{1}{2}\left(\lambda e^{-\lambda x} + \lambda^2 x e^{-\lambda x}\right) \quad , \quad x \geq 0 \qquad ,$$

where parameter λ denotes a positive-valued constant.

(a) (4 pts) Determine $\mathbf{E}[X]$.

(b) (4 pts) Determine $\text{var}(X)$.

(c) (5 pts) Let random variable N_τ be the number of arrivals during the time interval $[0, \tau)$. Determine the probability $\mathbf{P}(N_{10} = 0)$. You may find the following equality useful:

$$\int_a^b \lambda^2 z e^{-\lambda z} dz = (-\lambda z - 1)e^{-\lambda z}\Big|_a^b$$

(d) Answers to the following should be consistent:

- (i) (6 pts) Is this arrival process a memoryless process? Explain.
- (ii) (5 pts) Let random variable W denote the interarrival time of an interval selected by random incidence. Determine $\mathbf{E}[W]$.

(e) Let random variable K denote the total number of heads occurring in six independent flips of a fair coin and random variable T_K denote the time of the Kth arrival. Assume these coin flips are performed before the arrival process begins and also that the coin-flipping process is independent of the arrival process.

- (i) (4 pts) Determine the expectation and variance of K.
- (ii) (4 pts) Determine the expectation and variance of T_K.

(f) Define events A_n and B_n, both in terms of the first n interarrival times $X_1, X_2, \ldots, X_n$, by

$$A_n : \qquad \left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mathbf{E}[X]\right| - 10^{-6} \geq 0$$

$$B_n : \qquad \frac{X_1 + X_2 + \ldots + X_n}{n} - \mathbf{E}[X] \neq 0$$

- (i) (6 pts) What is the value of $\lim_{n\to\infty} \mathbf{P}(A_n)$? Explain.
- (ii) (6 pts) What is the value of $\mathbf{P}\left(\lim_{n\to\infty} B_n\right)$? Explain.

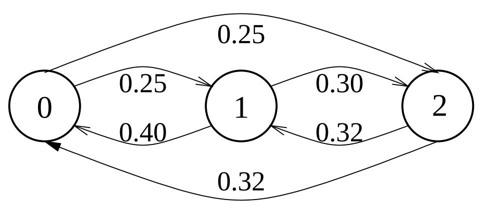
Problem 2. (54 points) The Waste-of-Time Internet Cafe stays open 24 hours on every day of every week. All of its customers arrive (and depart) via an extremely reliable shuttle service that arrives at (and departs from) the cafe's entrance every hour, and precisely on the hour. The number of passengers arriving on each hourly shuttle is a random variable K, independent of the number of passengers arriving (or departing) on previous shuttles, with PMF

$$p_K(k) = p(1-p)^k \,, \quad k = 0, 1, \ldots \qquad . \tag{1}$$

Assume the shuttle service has the resources to always accommodate any number of passengers.

Unfortunately, the cafe offers only two computer terminals, each of which can be used at most by a single customer in any hour. Each hour just before the instant a shuttle arrives, a customer using a terminal will end the session with probability q, independently of how many hours the customer has been at the cafe and the decisions of any other customer or passenger, in order to exit the cafe and board the shuttle. Arriving passengers will immediately enter the cafe and start using any available terminal but, because a passenger is never interested in waiting, arriving passengers who do not find an available terminal immediately reboard the shuttle before it departs.

Letting the state $i \in \{0, 1, 2\}$ be the number of customers in the cafe during any particular hour, the following transition probability graph applies:



(a) (6 pts) Determine the numerical values for parameters p and q implied by the graph above.

(b) (6 pts) Calculate the steady-state probabilities, π_i for each i, associated with the graph above.

In case you are stumped by parts (a) or (b) or simply when you consider it more convenient, it is sufficient to express the answers to questions (c) and (d) in terms of parameters p, q and π_i.

(c) (8 pts) On a particular day, months after the cafe opens for business, it is known that the number of customers during 5am-6am differs from the number of customers during 6am-7am. Conditioned on this fact, determine the probability that there are more customers during 6am-7am than during the previous hour.

(d) Any one customer using a terminal generates outgoing email traffic in a Poisson manner at an average rate of $\lambda > 0$ messages per hour, independently of messages generated by any other customer. All generated emails are instantaneously transmitted from the cafe.

(i) (8 pts) On a particular day, months after the cafe opens for business, determine the PMF for N, the number of total emails tansmitted from the cafe during 10:15am-10:45am.

(ii) (10 pts) On a particular day, it is known that there are exactly two customers in the cafe during 12am-1am. Conditioned on the fact that *each* customer generates at least one message during this hour, derive the PDF for Y, the time (in hours) from 12am until the instant *both* customers will have generated their respective first messages.

(e) The cafe owner purchases 150 prizes in order to run a promotion that will last at most 200 weeks. In each week, starting with the shuttle that arrives Monday at 6am, the number of shuttle passengers are counted and the nth passenger is awarded a prize; however, no prize gets awarded in a week during which the total passengers counted is less than n. Whether or not a prize gets awarded in any one week, the count always restarts on the following Monday at 6am. The owner requires at least an 80% chance that all prizes are awarded during the promotion.

(i) (8 pts) Based on the owner's requirement, determine a lower bound on α, the probability that a prize gets awarded in any particular week.

(ii) (8 pts) Determine an approximation to the allowable range of values for n, expressing it in terms of parameter p in equation 1 and parameter α as defined in part (i).